Continuum Mechanics And Transport Phenomena
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Lecture – 59
Displacement Gradient Tensor

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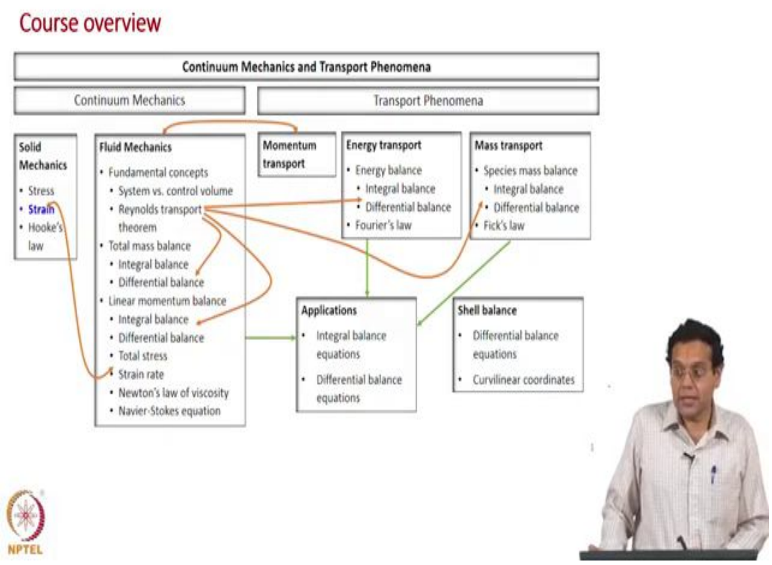

We have derived the linear momentum balance and then we are proceeding towards the Navier-Stokes equation. And we found that the linear momentum balance the viscous stresses were unknown and we need to related to the velocity; velocity gradient. And to understand velocity gradient we have come to solid mechanics and we are discussing strain.

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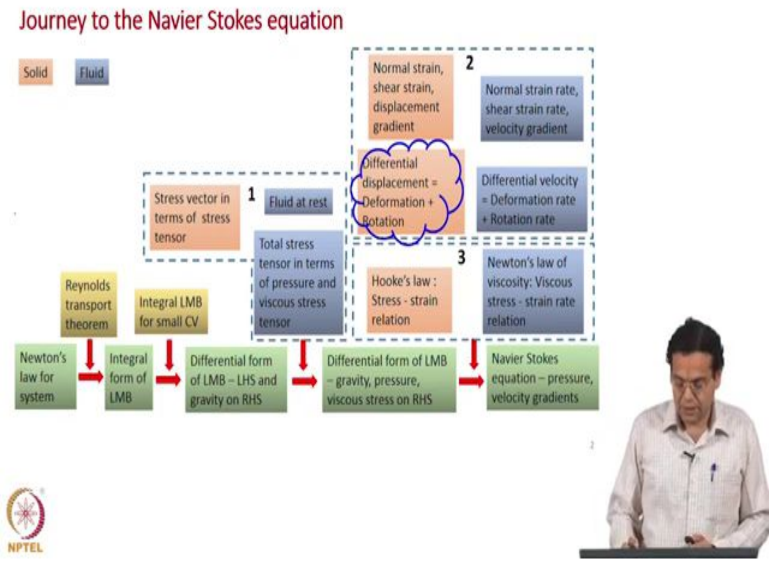

And with reference to this journey to the Navier-Stokes. The having derived the differential form of linear momentum balance the viscous stresses on the right hand side were unknown.

We need to relate it to the velocity gradient, I think now we can have some field for velocity gradient having discussed displacement gradient. So, to understand velocity gradient we came to solid mechanics and so, that you can understand and displacement gradient. We introduced normal strain, shear strain also discussed about rotation, we also discussed about displacement field and displacement gradient that is where we stand. Now, we are going to discuss about this block, which says differential displacement is equal to deformation plus rotation let us see what it means.

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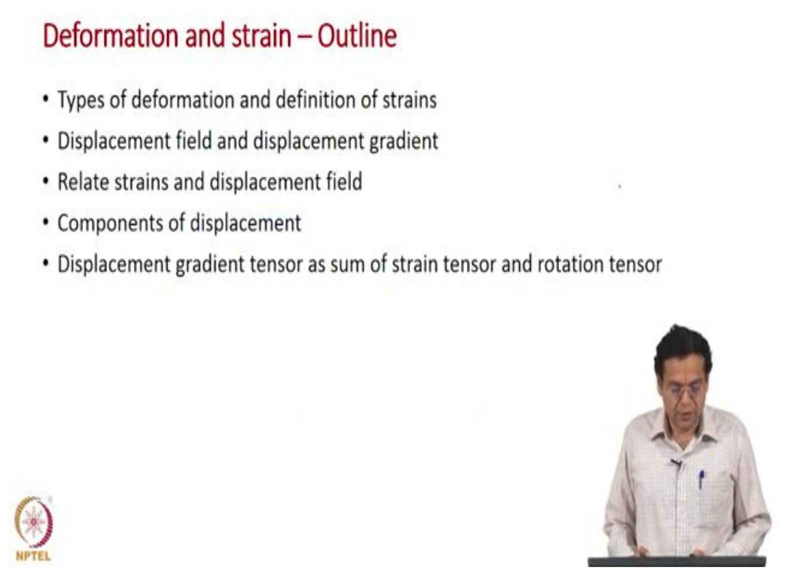

In terms of outline, this is the outline for the our second visit to solid mechanics title deformation and strain. We defined the types of deformation either change in length or change in angle and their measures are called as strains namely normal or shear strain. And we found that they are related to the displacement field. So, we discussed displacement field displacement gradient in 1 D, 2 D and then 3 D. Then we related the strains to the displacement field in terms of displacement gradient.

Now, we are going to look at the last two bullets namely components of displacement and then as a result of that we will show that the displacement gradient tensor is sum of two other tenses namely strain tensor and rotation tensor. The motivation for splitting this displacement gradient into strain and rotation we will discuss later towards end of this lecture.

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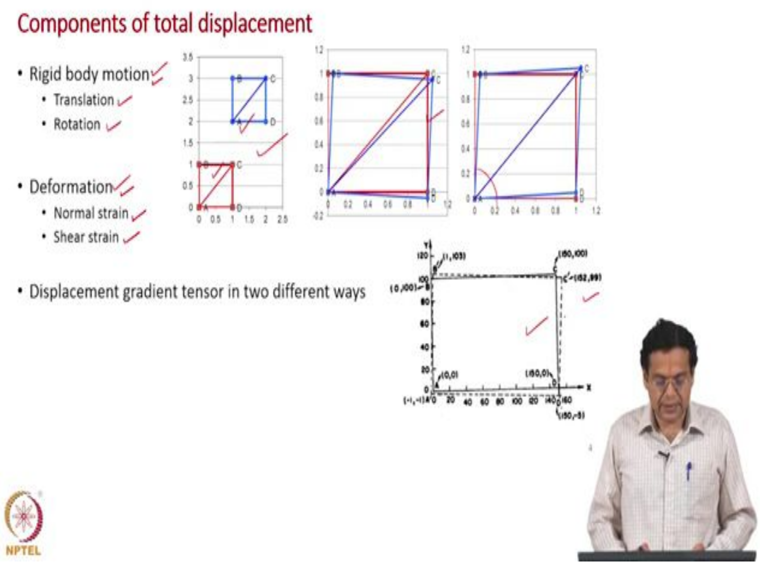

Now, we look at the title it says total displacement. Now when I say displacement it means change in coordinates that is all its means nothing more than that. If you have a plate and then you apply a force it undergoes deformation, there is some change in coordinates that change in coordinates could be because of several reasons or several components. First if you just move the plate it is translation that is what we have seen. So, that can result in change in coordinate, just moves.

So, remember you have the same graph sheet. So, your axis is same and your object moves or the plate moves there is change in coordinate and that is what is we have seen earlier, we discussed in example where when we changed α, the initial state translated to the final state, but there is a change in coordinate which means it is got displaced.

Now, what is the second case? If it just rotates the entire body rotates once again there is change in coordinate and that is what we have seen in this case. We took the same displacement field and this time we kept $\alpha = 0$ and gave a non 0 value for β and we found that it rotates, but what we are discussing now is by rotation there can be change in coordinate.

We took another case third case where there was shear strain, there is change in angle and once again if you look at this, there is change in coordinate. And of course, we have not discussed a normal strain here this particular example did not have normal strain, but we took another example where we had normal strain, shear strain and rotation as well of course, this example does not have translation, but this also we concluded that when there is normal strain change in the length of the plate let us say then there is change in coordinate or displacement.

So, now, if you want to put them all together what are the components of total displacement. The first 2 are called rigid body motion that is translation and then rotation why is it so? The entire body just moves something like I moving in this direction and I am just rotating. These 2 are called rigid body motion there is no relative displacement between different points in the body something like you what you did in a rigid body mechanics that is why it is called rigid body motion.

So, rigid body motion could be

- Translation just moving in a way way or
- Rotation or moving this way and then rotation.

Deformation; so when we say a deformation strictly, it means

- Change in length namely normal strain and then
- Change in angle namely shear strain.

Now, we are slowly trying to distinguish these 2 very clearly and that is in fact, the objective of the today's lecture also. Total displacement could be because of rigid body motion or deformation. So, far we could not have been very formal in using this terminology, but this lecture we have to be very precise and clearly mention what is rigid body motion, what is deformation we are going to distinguish these 2 or separate these 2.

Or in other ways dissect the total displacement into its components what are the components? Rigid body motion and then deformation. Rigid body motion could be because of translation and then rotation and deformation is because of normal strain and shear strain. What we are going to see is if you have a displacement how do you factor that into that due to translation, that due to rotation and that due to normal strain and that due to shear strain, we are going to separate into we are going to separate deformation part from the rigid body motion part.

As I told you the motivation we will see towards the end of the lecture why do we really do this is will be discussed later. Now, how are we going to do this? The way in which we are going to do this is express displacement gradient tensor in 2 different ways we will take displacement gradient express the displacement gradient tensor in 2 different ways and then achieve our objective of splitting the displacement into rigid body motion and deformation that is what we were going to do.

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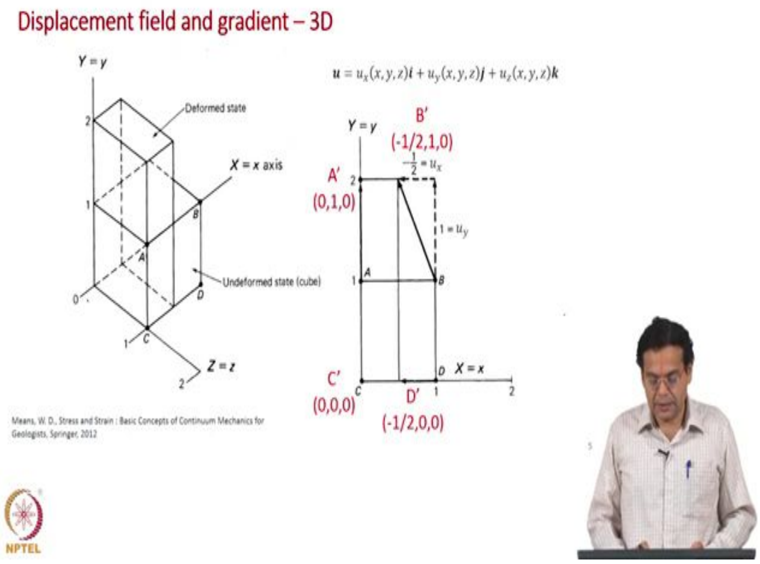

Now, as a recall to displacement gradient tensor, let us look at the example which we have discussed in the previous lectures this example was taken to illustrate the displacement gradient tensor for a 3 dimensional case. We said we had a cube it become a cuboid, then we noted the initial configurations, the position at the initial state and then the final positions in terms of A'B'C', we have also noted the displacement in all the directions for these points.

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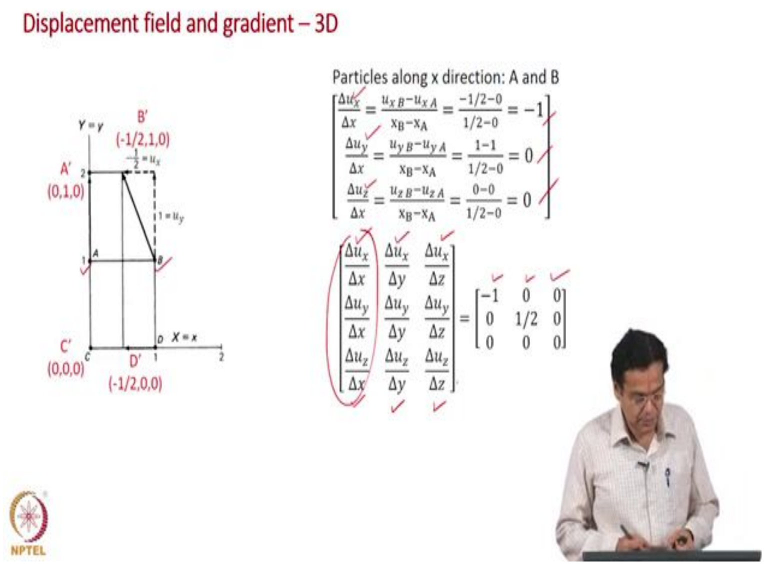

And then we wrote the displacement gradient tensor what we did was, we took particles along one direction let us say A and B, found out the difference in x displacement, y displacement, z displacement they varied by the distance between them and then we got values minus 1, 0, 0. What we did was we found out these values and then arranged all those values along a column we said we will justify it to later.

$$\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z}\right] = \left[-1\ 0\ 0\ 0\ \frac{1}{2}\ 0\ 0\ 0\ 0\right]$$

And then of course, similarly for the y direction z direction. Now this particular example shows the displacement gradient tensor for a specific example. Now, what we will do is make this more generalized this looks as it for a referred particular example we will make it more generalized and also we will you know why we would collected all these values and arranged as a column rather than row.

So, two things we will do, first instead of attaching this displacement gradient tensor to a particular example, we will derive more general expression which expression is going to be same, but in a more general sense and also you will automatically understand why we arranged these values along a column rather than a row.

What it means is if you look at a column then it corresponds to difference in displacement x and y direction for two particles along x direction and second column tells same thing for two particles along y direction third column tells about particles along z direction. So, if you take one row it tells about change in x displacement for particles along x direction, y directions, z direction or tells about change in x displacement along x direction, y direction, z directions. Why did we arrange like this?

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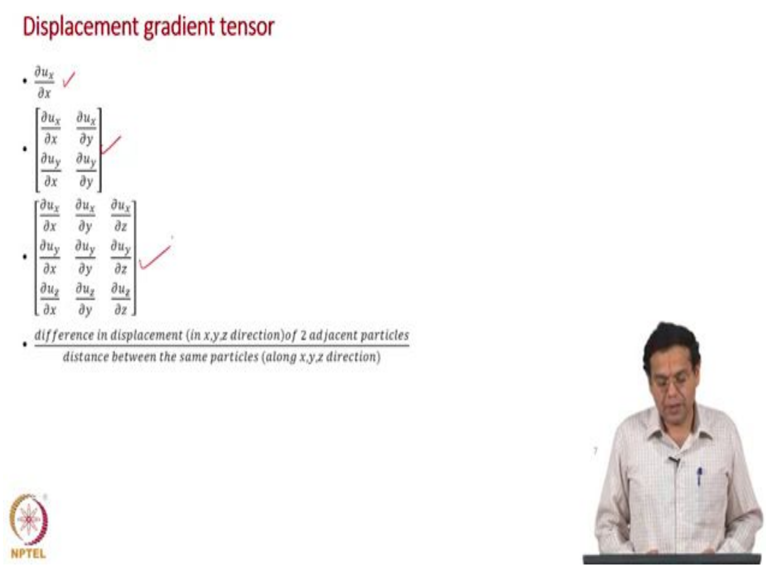

Of course, then we saw this displacement gradient tensor same slide as what we have seen earlier for 1 dimensional case, the 2 dimensional case and the 3 dimensional case . So, it is just to recall what a displacement gradient tensor is.

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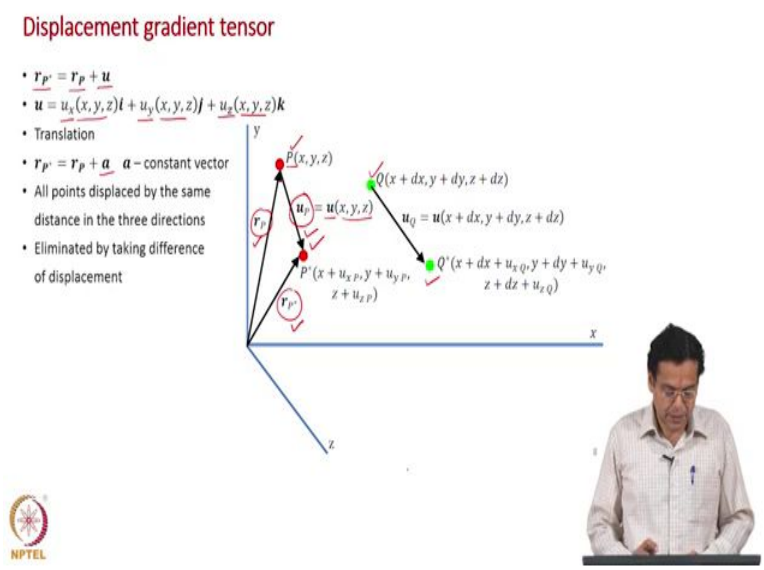

Now, we will proceed with our objective of deriving the displacement gradient tensor in a more generic way in two different ways and we will discuss now the first way can be called as a more mathematical way. Before going ahead with that what is shown in this figure are points P and P*, Q and Q*. So, what is shown is you take a plate identify point P identify point Q and it undergoes change in coordinate, it could be rigid body motion, it could be deformation. Because of that, because of any of these reasons it could be rigid body motion and it could be deformation P moves to P*, and Q moves to Q*. As I told you P has got displaced to P* and Q has got displaced to Q* that displacement could be because of rigid body motion which is translation and rotation or because of deformation which is normal strain and shear strain.

Now, objective is to spilt rigid body motion or take away rigid body motion from the deformation first thing what we will takeout is translation. What is translation? If you take let us say a plate and moves along one direction; every point, let us say if you focus on two points both the points move by the same distance. Let us say the plate's moves by 5 centimeter this and you focus on these two points both the points move by 5 centimeter and that is what is shown here

$$r_{p^*} = r_p + u$$

Here, r_{p^*} and r_p star are position vectors. What do you mean by position vectors? The vector joining the origin and the point of interest so, the vector joining origin and point P is r_p the vector joining origin and P* it is its r_{p^*} they are position vectors and u is the displacement of particle P and because it is displacement for a point at P it is denoted as u vector as a function of x, y, z. So, r_p is a position vector for p, r_{p^*} position vector for P*, $u_P = u$ and that is the displacement of point P which is at a position x, y, z. And we will come to Q little later right now focusing on P.

Of course, u is a displacement field which has 3 components u_x, u_y, u_z which we have already discussed and all three can be functions of x, y and z.

$$u = u_x\,(x,y,z)\,i + u_y\,(x,y,z)\,j + u_z(x,y,z)k$$

Now, what is translation and our objection is to remove translation and rotation from deformation is it very easy to remove translation and that is what we are going to do first. As I told you translation how do you understand translation? You have a plate and the plate moves what is implication? If you mark any 2 points they move by the same distance how do you represent

$$r_{p^*} = r_p + a$$

Here, a is a constant vector why is a vector?. If it is the plate is moving in one direction it is a scalar, but suppose if it is the same plane but moves in an angle then it is a vector of two components. But suppose the plate moves in 3D direction then it is constant vector, but 3 dimensional vector all these are translations because the entire body moves.

I will repeat again the plate let us say moves along a straight line these two points also have the same displacement, both are the same displacement. And in this case a is a constant vector constant, a is a scalar now all the case a is constant in this case it is a scalar. But suppose the plate gets translated in 2D then a is 2D vector, but now the plate can also move or translate in 3D, which case a is 3 dimensional vector. So, in a general sense a is 3 dimensional vector, every point gets displaced by that either scalar or vector.

So, now, how to eliminate or how to remove translation? Instead of talking displacement of every points separately if you talk about difference in displacement they will just cancel out. Let us say you are at P and then another point slightly away let us say P and then Q and then if the plate moves by 5 centimeter, P also moves 5 centimeter, Q also moves 5 centimeter, difference in displacement is 0.

So, to remove translation, all points displaced by the same distance in the 3 directions. And eliminate by taking difference of displacement. So, what we will now on work is, in terms difference in displacement. Reason for a working in terms of difference in displacement is its takes care of translation has been eliminated.

For work in terms of displacement it includes translation. If a if there is change in coordinate and if you are working terms of displacement alone of every coordinate, then it can include translation we want to avoid that how to avoid is just by taking the difference in displacement between two points on the plate ok. So, next slide onwards in fact, throughout the further discussion, we will work in terms of difference in displacement.

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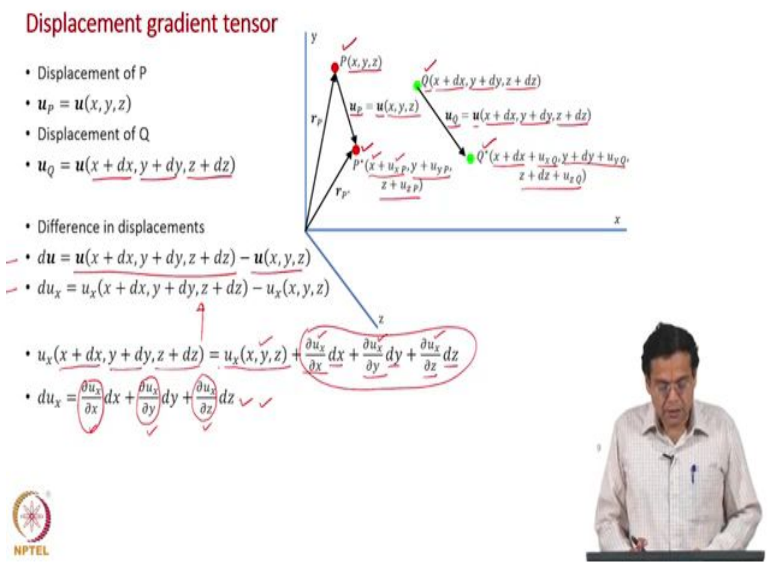

Remember we are proceeding towards deriving expression for displacement gradient tensor, we are going to do it in 2 difference ways and we are discussing the first way now. Before that we said that instead of working in displacement working in terms of difference in displacement, we have taken care of translation by taking difference in displacement.

Now, we will focus on the point Q as well. Now, P is at a point x, y, z, we have another point neighboring point Q which is at a another location $x+dx$, $y+dy$, $z+dz$. So, the x, y, z distance between P and Q are dx, dy, dz. Now, the plate undergoes may be rigid body motion and deformation P had moved to P* and Q has moved to Q*. Now, the respective displacements are denoted by u_P, u_Q alternatively in terms of displacement field because P is at x, y, z it is displacement field at x, y, z.

The displacement of Q is displacement vector at $x+dx$, $y+dy$, $z+dz$. Now what is P*? P* is the new position, as we have seen earlier the new position can be find out as the old coordinate plus the displacement similar for x direction, y direction, z direction.

So, the old coordinate is x; and the displacement in x direction is u_{xP} similarly y direction, z direction.

$$P\ (x,y,z) \rightarrow P^*(x+u_{xP}, y+u_{yP}, z+u_{zP})$$

Now, coming to Q*, the new coordinates are the old coordinate $x + dx$ plus the displacement along the x direction for point Q which u_{xQ}. So, similarly y direction and then z direction.

$$Q\,(x + dx,\ y + dy,\ z + dz) \rightarrow Q^{*}(x + dx + u_{xQ}, y + dy + u_{yQ}, z + dz + u_{zQ})$$

So, displacement of P is

$$u_p = u(x, y, z)$$

And the displacement of Q is

$$u_Q = u(x + dx,\ y + dy,\ z + dz)$$

Now, we said that we look at difference in displacement right. Yeah, let us write expression for difference in displacements as I told you the reason is it takes care of translation, translation gets eliminated. Now, how do you represent difference in displacement is

$$du = u_Q - u_P = u\,(x + dx,\ y + dy,\ z + dz) - u(x, y, z)$$

What are this line tell you? The vectorial difference in displacement of Q and P. So, we are looking at 2 points of course, remember these points are in a plate which has under gone rigid body motion and deformation and remember the points here are shown far away, but they are very near neighboring points. Now, instead of working in terms of the vectorial difference which is little difficult let us work in terms of component.

So, let us now focus on the x component of the previous equation. So,

$$du_x = u_x\,(x + dx,\ y + dy,\ z + dz) - u_x(x, y, z)$$

This equation tells you the difference in x displacement between P and Q, this is the vectorial equation and written a x component of that. So, this tells you the difference in the x displacements of points Q and P. Now, we will take this displacement $u_x\,(x + dx,\ y + dy,\ z + dz)$ and we will write in terms of a Taylor series,

$$u_x\,(x + dx,\ y + dy,\ z + dz) = u_x\,(x, y, z) + \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz$$

Now, let us substitute this equation in the above $u_x\,(x, y, z)$ cancels out,

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz$$

What is the other way of looking at it? u_x is a function of x, y, z and du_x is total derivative of u_x you are expressing that in terms of partial derivative. Two different way of interpreting this. What you have done is a little more geometrical physical way of understanding du_x little more mathematical way is u_x is a function of x, y, z, u_x is x displacement all the components are functions of x, y, z. So, u_x also a function of x, y, z. So, expressing total derivative in terms of partial derivative gives you this expression straight away fine.

Now, what is the significance of this expression? The left hand side what do you have is let us tentatively call as difference, the what we have left hand side is the total difference in x displacement between Q and P. Right hand side that has been expressed a three different terms. What do they represent? This x displacement is a function of x, y and z, u_x varies with x, y, z. The first partial derivative takes into account variation of u_x in the x direction, second partial derivative takes into account variation of u_x in the y direction, third derivative takes into account variation of u_x in the z direction. So, left hand side tells you the total difference in displacement and that has been apportioned into 3 parts because of the variation of x displacement along x direction, y direction, z direction. So, mathematical statement total derivative relating partial derivative.

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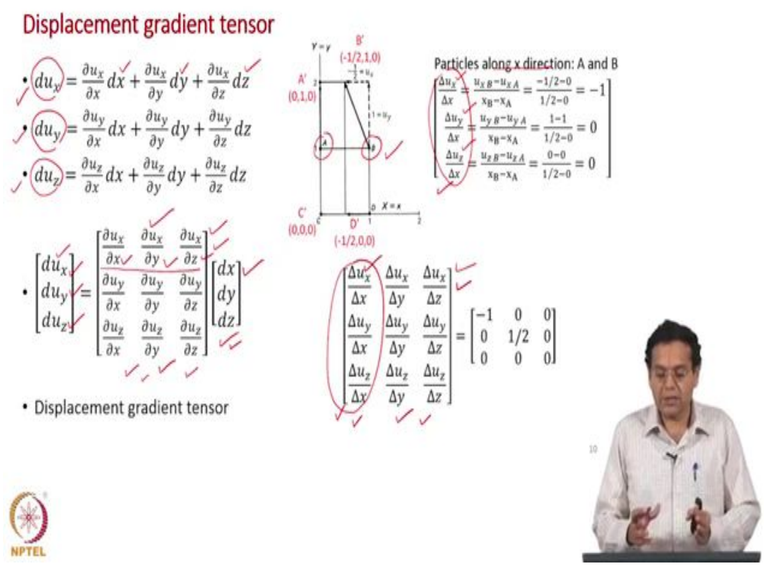

Now, let us do this. For the other two direction as well.

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz$$

Similarly, you can do for the y direction

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_y}{\partial z}dz$$

And similarly for the z direction

$$du_z = \frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz$$

The first tells you about the difference in x displacement between P and Q, the second tells you difference in the y displacement between P and Q and then similarly last one tells a z displacement between P and Q. The difference in x displacement, difference in y displacement, difference in z displacement. Now, let us arrange this in a form of a matrix equation

$$\left[du_x \; du_y \; du_z\right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_x}{\partial z} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y} \; \frac{\partial u_y}{\partial z} \; \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \; \frac{\partial u_z}{\partial z}\right] [dx \; dy \; dz]$$

Now, look at the example which you have seen earlier (right hand side in the above slide image) we took this example and then found out we took two particles along x direction, we found out these three derivatives in terms of difference in x, y, z displacement and then we wrote this as a in a form of a column not in a form of a row.

Now, if you compare these two displacement gradient tensors, they are both same. That is why we collected all those values and wrote as a column rather than a row entry in this particular matrix or tensor.

Now, what does this tensor tell you? We have already discussed that the left hand side we have the displacements and the displacements are in x, y, z directions and each of them can vary in x, y, z direction which are here and hence resulting in nine components of the displacement gradient tensor; u_x, u_y, u_z themselves have directions x displacement, y displacement, z displacement. All of them can each of them can vary in x, y, z direction that is what is given here.

So, the tensor connecting these two has two directions attached to it. And if you write in this way one row corresponds to the x displacement along x direction y direction and z direction and when we wrote here we took particles along one direction and considered their x displacement, y displacement, z displacement.

So, in the displacement gradient what happens is along the row you have only x displacement, y displacement, z displacement, but when we considered here the particles we considered two practices along x direction and considered their x displacement, y displacement, z displacement difference in x, y and z displacement that is why we wrote them collected them and wrote as a column rather than a row. So, one row belongs to a displacement in one direction, but when we wrote from here we took difference in displacement along three different directions considering particles along one direction that is why we collected and wrote them as a column rather than a row.

So, of course, that time it was not possible to explain the reason, now we have a very good reason to explain why we wrote those element as a column rather than a row. And also we said that this displacement gradient tensor was looked like for a specific example, it is not the case, we have derived the displacement gradient tensor in a more generic way and so, what is derived is more general form of the this displacement gradient tensor. Same form, but we did not restrict to a particular geometry, we did not restrict to a particular configuration we have derived in a more general sense.

What we have done now is derived a relationship between the total difference in displacement or differential in displacement, we have almost use this were synonymously throughout the lecture differential and difference. So, we have a related their differential in displacements to the differential in the directions x, y, z in terms of a displacement gradient tensor. Now, this is more mathematical we took two points looked at the difference in displacement etcetera. We are now going to arrive at the same expression more geometrically and that is what we are going to do now.