Continuum Mechanics And Transport Phenomena
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Lecture - 61
Components of Total Displacement - Part 2

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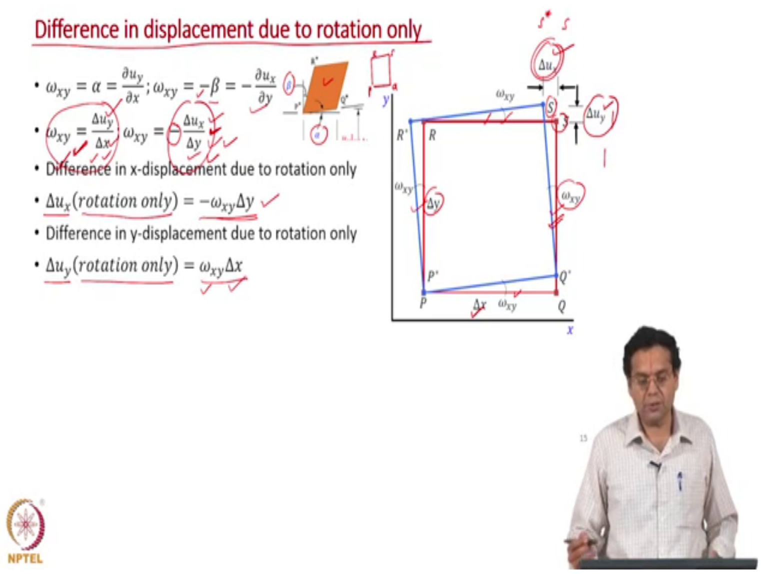

Now, the last part difference in displacement due to rotation only. What is shown here, the same plate with just rotation which you called as a rigid body rotation, the entire body just rotates. PQRS is the initial state P*R*Q*S* is the final state. If you look at this there is no change in length between PR and P*R* and the angle RPQ is equal to angle R*P*Q*; no shear strain also, only rotation and this is the components you want to separate out. Remember we have already separated out translation but difference in displacement still includes rotation we are going to separate it out.

So, let us see What are these components contribute. Look at this, S has moved to S*, there is a difference in displacement. The reason is rotation we have to evaluate that and find expression for that. Now once again, this figure is we had discussed already the context of deriving relationship between normal strain, shear strain and rotational also if you derived in terms of displacement gradient, same figure here as we have seen the previous slide.

Now when we derived that

$$\omega_{xy} = \alpha = \frac{\partial u_y}{\partial x}$$

Then we also derived for rotation for the element PR to P*R* and the relationship was

$$\omega_{xy} = -\beta = -\frac{\partial u_x}{\partial y}$$

But we PR rotated in the clockwise direction because we have taken to account, the direction, clockwise is negative. We added a negative sign. Now, if you write in terms of difference

$$\omega_{xy} = \frac{\Delta u_y}{\Delta x}; \qquad \omega_{xy} = -\frac{\Delta u_x}{\Delta y}$$

ω_{xy} is the rotation and that is related to difference in y displacement for particle separated along x axis by Δx and that is what we have here; P and then S once again there is no movement displacement of P. And So, we can apply this relationship for the present case for difference in displacement between P and S.

Similarly, second expression tells you that relates the rotation omega to the difference in x displacement for two particles which are separated along the y axis by Δy. That is why we have here also P and S separated by Δy along the y-axis.

So, now what we will do is apply the expression for this triangle; what is that triangle? So, the angle between the lines corresponding lines between the initial state and final state, all of them are ω_{xy}.

Now, we have discussed the negative sign; whenever you talk about Δy, it is higher y minus lower y, you are along the y axis, ok. So, there is no ambiguity. When you talk Δu_x, it is always final state minus initial state, that is what Δu_x represents. In this particular example S* is the final state, S is the initial state and that has moved along the negative x axis to account for that we have the negative sign.

So, if it is both along the positive axis respective axis, then negative sign is not required, as we have in this case. In the case of $\frac{\Delta u_y}{\Delta x}$, S moved vertically up. So, it is along the positive y axis and of course, x is along the positive x axis. But when you take the second expression,

Δy is still along the positive way, but S moves towards the negative x axis and so the negative sign takes care of that.

Now, as usual we will use these expressions for writing expressions for Δu_y and Δu_x, what do they represent? They represent difference in y displacement and x displacement because of rotation only. So, you first write for x direction

So, difference in x displacement due to rotation only. So, we use the second equation,

$$\Delta u_x(rotation\ only) = -\omega_{xy}\Delta y$$

Let us use the first equation and write expression for difference in y displacement due to rotation only.

$$\Delta u_y(rotation\ only) = \omega_{xy}\Delta x$$

So, as usual the nomenclature has been made very clear. The first is Δu_x difference in x displacement due to rotation only. Second equation is difference in y displacement due to rotation only. Just as a quick check in this particular case is ω_{xy} positive or negative; anticlockwise rotation so ω_{xy} is positive.

So, now let us let us take this equation, ω_{xy} is positive, Δy is positive, Δu_x is negative,. We should get a negative difference in x displacement. So, it is a only cross check in fact, we written the negative sign taking into account the x displacement just as a quick cross check or ω_{xy} is positive and Δy is anyway positive. So if you substitute delta, the difference will be negative which is in line with our moment of S to S* along the negative x-axis.

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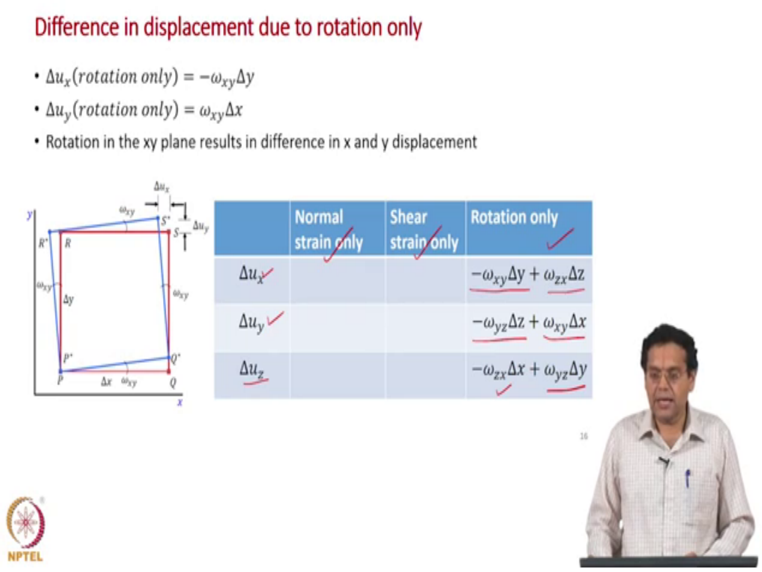

Now, of course for the difference in y displacement that is very clear, ω_{xy} is positive, Δx is positive and S moves along the positive y axis. So, now let us summarize the expressions which we have derived in the previous slide for difference in x displacement and difference in y displacement due to rotation only.

Now, let us fill up this table; we have filled up normal strain, we have filled up shear strain, they are not shown just for clarity. Now we have to fill up this rotation only. Now just like for shear strain, when shear strain happens it contributes to x and y displacement. Similarly one rotation happens in x y plane, it contributes to x displacement and y displacement and so, you will have one entry for Δu_x, one entry for Δu_y and that is what is shown here.

	Normal strain only	**Shear strain only**	**Rotation only**
Δu_x			$-\omega_{xy}\Delta y + \omega_{zx}\Delta z$
Δu_y			$-\omega_{yz}\Delta z + \omega_{xy}\Delta x$
Δu_z			$-\omega_{zx}\Delta x + \omega_{yz}\Delta y$

So, now we have filled the last column also based on rotation in the x y, y z and z x planes. So, rotation in the x y plane results in difference in the x and y displacement.

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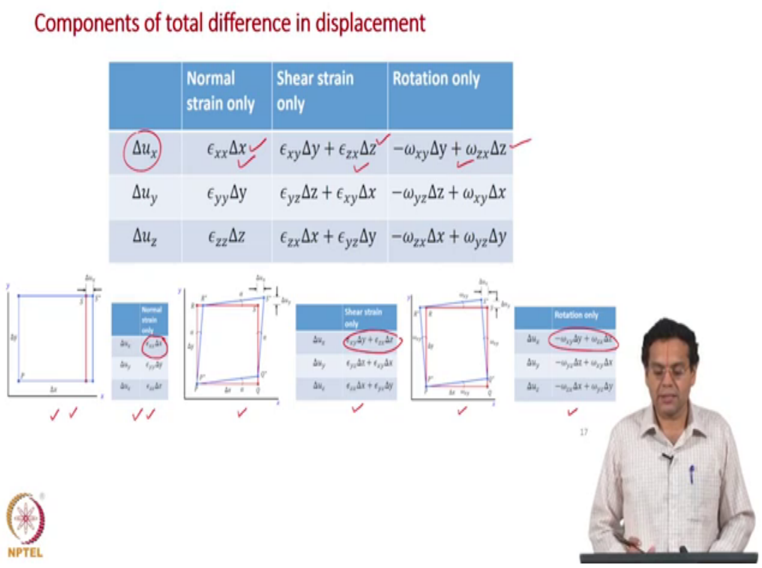

So, put together all of them. The individual figures are shown here, the individual tables are show here, the bottom of the slide. The first figure is where normal strain corresponding table, second figure for shear strain only and the corresponding table, third figure is for rotation only the corresponding table.

Now, we can fill the entire table. We said we are proceeding towards that

	Normal strain only	**Shear strain only**	**Rotation only**
Δu_x	$\epsilon_{xx}\Delta x$	$\epsilon_{xy}\Delta y + \epsilon_{zx}\Delta z$	$-\omega_{xy}\Delta y + \omega_{zx}\Delta z$
Δu_y	$\epsilon_{yy}\Delta y$	$\epsilon_{yz}\Delta z + \epsilon_{xy}\Delta x$	$-\omega_{yz}\Delta z + \omega_{xy}\Delta x$
Δu_z	$\epsilon_{zz}\Delta z$	$\epsilon_{zx}\Delta x + \epsilon_{yz}\Delta y$	$-\omega_{zx}\Delta x + \omega_{yz}\Delta y$

Now, I have written Δu_x, Δu_y, Δu_z the total difference in displacement; total difference in displacement that is a left hand side. Now, in the right hand side we have three columns, let say normal strain only, shear strain only, rotation only.

So, this full table is just combination of the three separate tables what we have shown earlier. And what is the other way of looking at it; if you focus on a particular row what can be physically told, the total difference in x displacement is or has contributions from normal strain, shear strain and rotation, that is what we said. The displacement has an additional contribution translation; we have taken care of that. Difference in displacement does not have that component namely translation. It has rotation, normal strain and shear strain.

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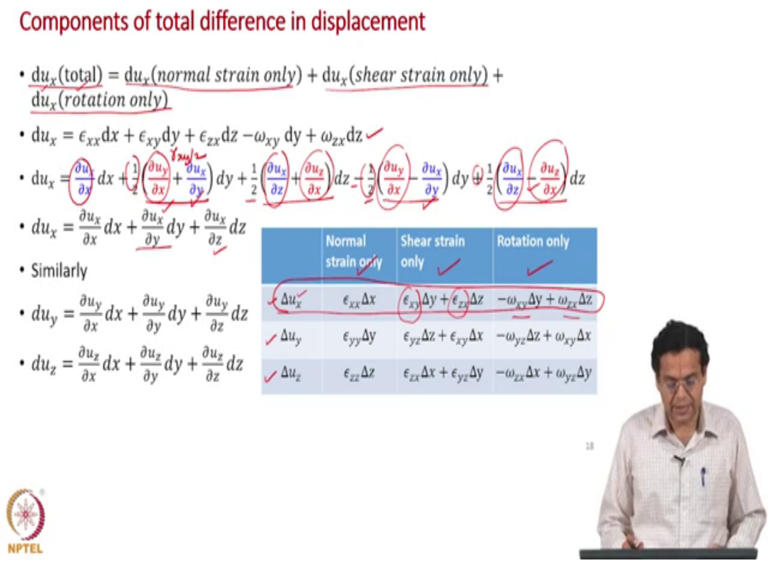

Now, we are proceeding towards getting the relationship between du_x, du_y, du_z and dx, dy, dz in terms of displacement gradient in the second way which is more geometrical way and that is what we will do now, that is a last step. We said we will derive the displacement gradient in two different ways. The first one was more mathematical, the second one is more geometrical and that is what we have done now.

So, now we are at the last step, where we are going to arrive at the expression. So, now what we do, I come back to differential because our expression was in terms of the displacement gradient expression was in terms of differential, though this table is in terms of difference I am writing in terms of differentials.

$$du_x(total) = du_x(normal\ strain\ only) + du_x(shear\ strain\ only) + du_x(rotation\ only)$$

So, look at the nomenclature very clearly written here. The left hand side is difference in displacement and then the bracket says total, and difference in displacement along x direction. So, left hand says difference in displacement or differential in displacement, there is a total difference in displacement, right hand side where difference in displacement into normal strain only, difference in displacement due to shear strain only and difference in displacement due to rotation way.

Now, we are going to do sum up all the components, all the terms in the first row, ok. All of them correspond to difference in x displacement coming from contributions from normal strain, shear strain and rotation, we are going to sum up all of them. Let us do that and see what do we get

$$du_x = \epsilon_{xx} dx + \epsilon_{xy} dy + \epsilon_{zx} dz - \omega_{xy} dy + \omega_{zx} dz$$

So, if you have observed the two shear strain terms, difference in x displacement because of shear strain has further two components; one from shear strain in x y plane, shear strain in z x plane. Similarly for rotation if you say the difference in displacement x direction has two components, one coming from rotation in x y plane, other coming from rotational in z x plane.

So, this equation is just sum of all the terms in the first row. We will express them in terms of displacement gradients. So, let us do that

$$du_x = \frac{\partial u_x}{\partial x} dx + \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) dy + \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) dz - \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) dy + \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) dz$$

If you simplify this

$$du_x = \frac{\partial u_x}{\partial x} dx + \frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial z} dz + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right) dy + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right) dz - \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right) dy - \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right) dz$$

The last terms cancel each other, and we have

$$du_x = \frac{\partial u_x}{\partial x} dx + \frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial z} dz$$

And similarly we can get derive for other directions for du_y and du_z as well.

$$du_y = \frac{\partial u_y}{\partial x} dx + \frac{\partial u_y}{\partial y} dy + \frac{\partial u_y}{\partial z} dz$$

$$du_z = \frac{\partial u_z}{\partial x} dx + \frac{\partial u_z}{\partial y} dy + \frac{\partial u_z}{\partial z} dz$$

So, I think now these expressions are familiar to you. The same expression is what we derived earlier more mathematically. That is what I am going to see in the next slide.

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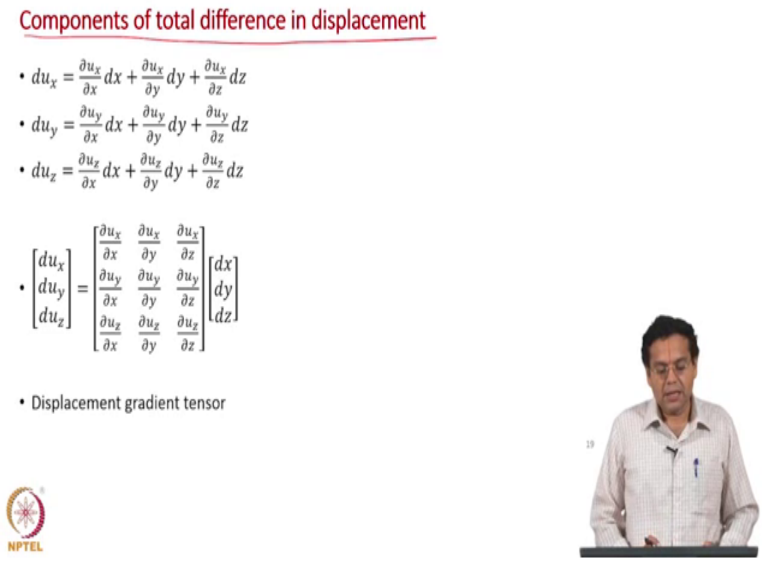

So, if we summarize all this; we will summarize all the total difference in displacement.

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz$$

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_y}{\partial z}dz$$

$$du_z = \frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz$$

These are same as what you have seen in the previous slide. And then we will arrange in a form of a matrix that is what we did earlier also,

$$\left[du_x \, du_y \, du_z\right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_x}{\partial z} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y} \; \frac{\partial u_y}{\partial z} \; \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \; \frac{\partial u_z}{\partial z}\right]\left[dx \, dy \, dz\right]$$

Remember this slide is same as earlier slide, only the title is different. Earlier title was displacement gradient tensor obtained more mathematically, but right now this is obtained more geometrically based on the components.

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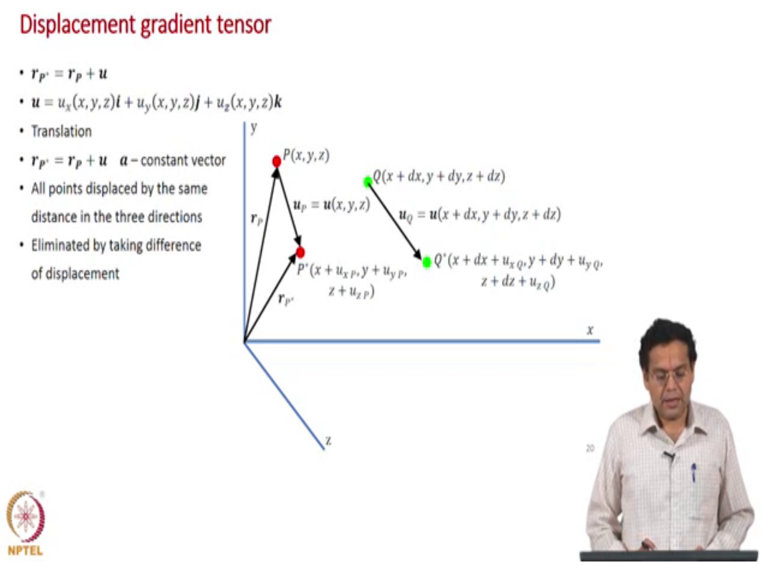

Just to recall these are the slides which we had earlier where we consider two points separated by dx, dy, dz and then, we first took of translation by looking at difference in displacement.

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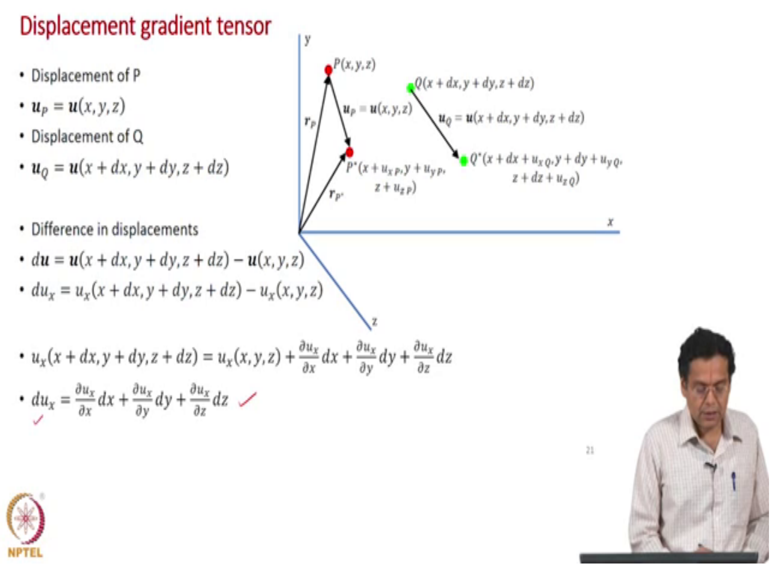

Then we wrote expression for difference and displacement first vectorially and then along x direction and got this relationship by expanding in Taylor series or expanding this total differential in terms of partial derivative.

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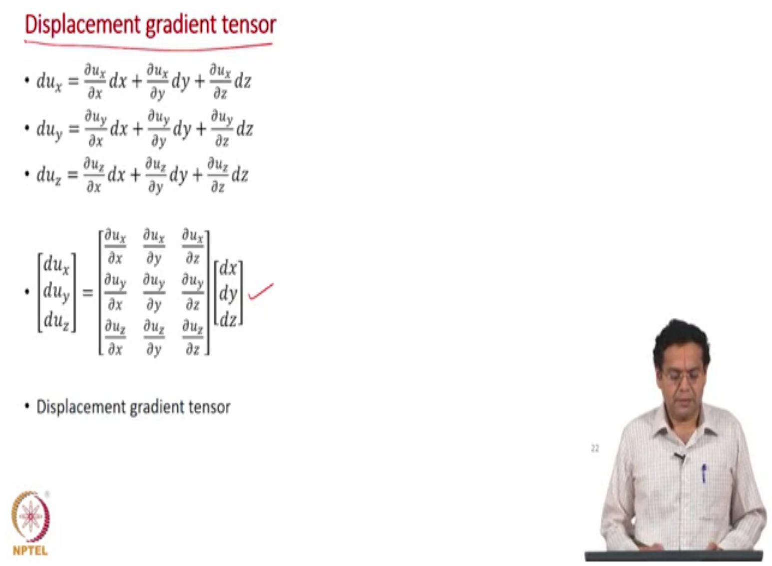

And in the previous slide what we had and this slide are same the title is different, because the way in which we have written this the first way what we are done, right. Now is the second way. Both result in a same expression, same matrix equation or same set of equations in two different ways.

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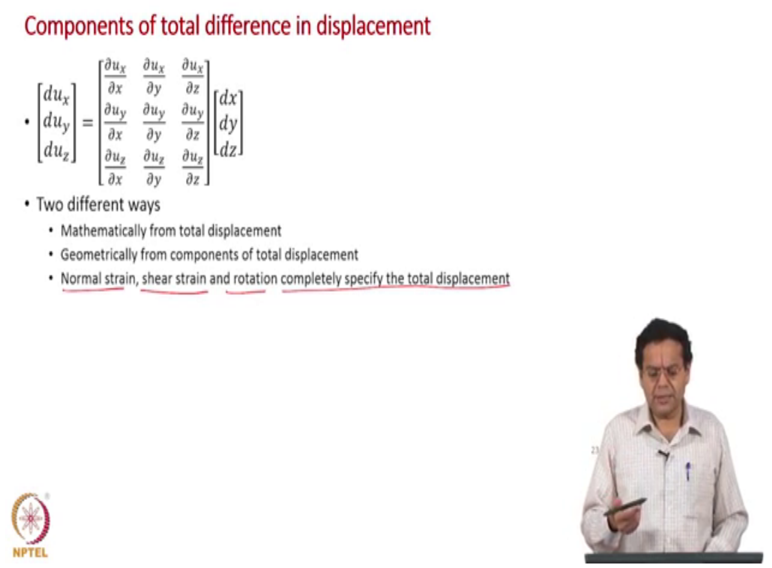

So, let us write down the relationship which we have got into different ways; Physical significance is same. This means if you have got the same expression in two different ways what are the conclusions, that is what we are going to discuss now,

$$\left[du_x\ du_y\ du_z\right] = \left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z}\right][dx\ dy\ dz\,]$$

Two different ways

- First one mathematically from total displacement,
- Second one geometrically from components of total displacement.

That is what we have done. What is implication that is the statement now normal strain, shear strain and rotation completely specify the total displacement, what does it mean? If you are considering the difference and displacement when I say total displacement, total difference in displacement we made a statement that it has three components; normal strain, shear strain, rotation, we are proving it now.

There could be other terms also, other components also, but this proof tells you that we consider; we consider only three ways normal strain, shear strain and rotation. By considering the individual components, we get the same expression what we had taken earlier; by considering total displacement which means that normal strain, shear strain and rotation completely specify the total displacement or difference in displacement. That is an important conclusion.

Initially we made a statement that I am going to consider only normal strain, shear strain only, rotation only, but why should I consider only these three; this is the conclusion. It is enough if you considered those three you can arrive at the same expression which implies that those are the only three, one or two strains and rotation to be considered to arrive at the total displacement; that is the conclusion which we arrived.