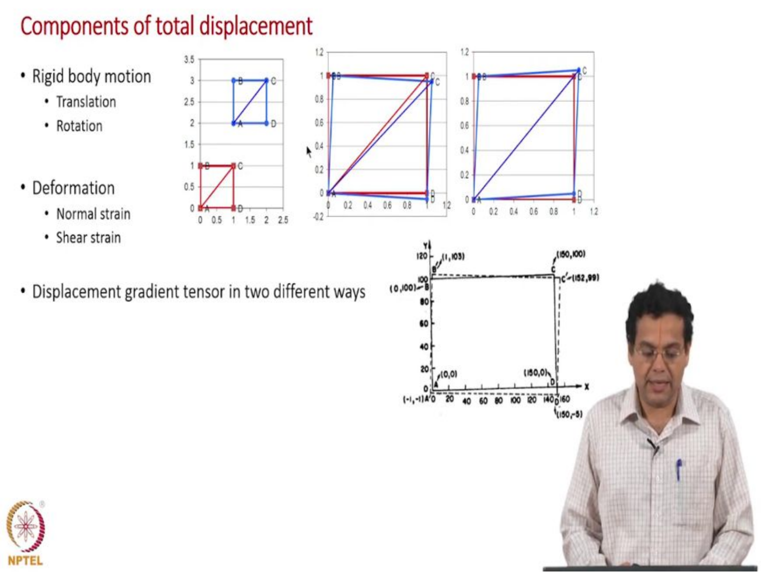
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 62
Strain Tensor and Rotation Tensor - Part 1



We came across this few slides back and discuss this. When we discussed we said displacement refers to change in coordinates and that could be because of a rigid body motion, which has two components namely translation, rotation; and translation is just the whole body moving in any direction; and rotation is just whole body rotating;

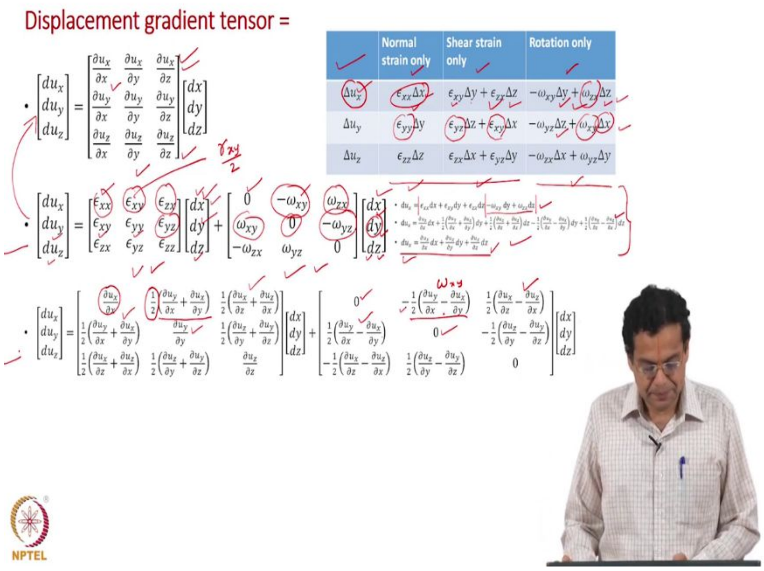
And the change in coordinate displacement could also be because of deformation which is change in length or change in angle, which we call as normal strain and shear strain and

We said that we are going to separate these two components these two contributions namely rigid body motion and deformation by writing the displacement gradient tensor in two different ways and we said that translation can be easily taken care of by writing difference and displacement.

Then we moved on and then arrived at the displacement gradient tensor in two different ways, one mathematical other by analyzing the individual contributions namely normal strain, shear strain and rotation.

So, having done that now we are ready to look at the split between these two the rigid body motion and the deformation. In particular, we ready to separate out rotation from normal strain, shear strain, already retranslation has been taken care because we are working in terms of difference in displacement.

(Refer Slide Time: 02:01)



So, let us do that. So, let us write down the displacement gradient tensor using the first way which was the mathematical way, where we expressed total differential in terms of partial differential, so that is the first expression.

$$\left[du_x \; du_y \; du_z\right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_x}{\partial z} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y} \; \frac{\partial u_y}{\partial z} \; \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \; \frac{\partial u_z}{\partial z}\right] [dx \; dy \; dz]$$

Second what we do was, we analyze the individual component, individual contributions to the displacement in x-direction, difference in displacement due to normal strain, shear strain and rotation;

	Normal strain only	**Shear strain only**	**Rotation only**
Δu_x	$\varepsilon_{xx}\Delta x$	$\varepsilon_{xy}\Delta y + \varepsilon_{zx}\Delta z$	$-\omega_{xy}\Delta y + \omega_{zx}\Delta z$
Δu_y	$\varepsilon_{yy}\Delta y$	$\varepsilon_{yz}\Delta z + \varepsilon_{xy}\Delta x$	$-\omega_{yz}\Delta z + \omega_{xy}\Delta x$
Δu_z	$\varepsilon_{zz}\Delta z$	$\varepsilon_{zx}\Delta x + \varepsilon_{yz}\Delta y$	$-\omega_{zx}\Delta x + \omega_{yz}\Delta y$

And then these are the components and then we summed up all in the previous slide that is what is shown here, and then proved that it is same as the first row for the x displacement and hence got an expression for displacement gradient in the second way also.

Now, we are not going to sum up all the components as shown here, we are going to separately sum the normal strain, shear strain contributions, and keep rotation separately, and that we are going to do in the form of a matrix. Let us do that. The left side is the column vector of difference in displacement. As I told you here we have written difference in displacement to represent similar to this we have writing in terms of differentials of displacement.

$$\left[du_x \; du_y \; du_z \right] = \left[\varepsilon_{xx} \; \varepsilon_{xy} \; \varepsilon_{xz} \; \varepsilon_{xy} \; \varepsilon_{yy} \; \varepsilon_{yz} \; \varepsilon_{zx} \; \varepsilon_{yz} \; \varepsilon_{zz} \right] [dx \; dy \; dz \,] + \left[0 - \omega_{xy} \; \omega_{xz} \; \omega_{xy} \; 0 - \omega_{yz} - \omega_{zx} \; \omega_{yz} \; 0 \right] [dx \; dy$$

So, how do I write this expression. So, left hand side we have a column vector differences of displacement, right hand side we have column vector of dx, dy, dz.

So, we have to represent this table in the form of two mattresses or two tenses that is what we going to do now. The normal strain shear strain have been put together and represented as a one row in this particular tensor or matrix.

So, what is that we have done now, earlier we summed up all the contributions and proved that this expression is a what we obtain through the mathematical way.

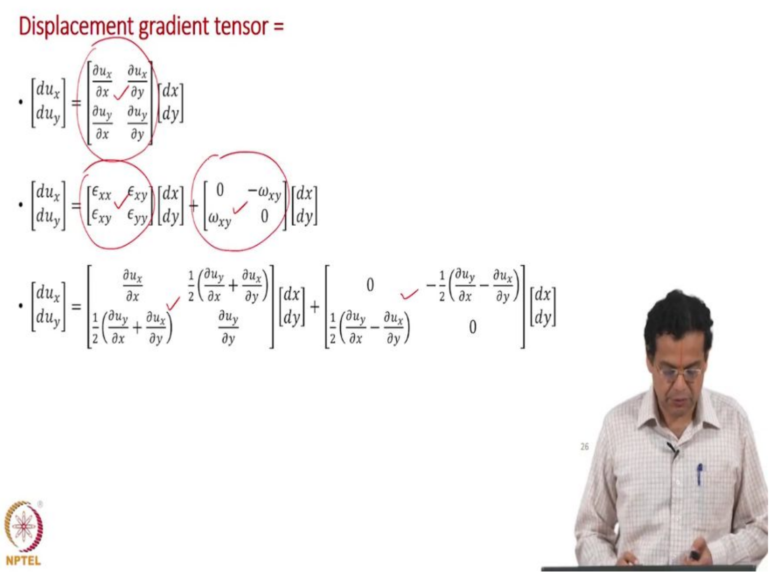
Now, what we have done we have split this as one matrix representation, and these two terms as or represent rotation as another matrix representation. Now, we have shown that the displacement gradient tensor is equal to the tensor obtained by splitting into components which means that the displacement gradient tensor is equal to sum of these two tensors or matrices.

Now, we will write in terms of the displacement gradients

$$\left[du_x \; du_y \; du_z \right] = \left[\frac{\partial u_x}{\partial x} \; \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \; \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \; \frac{\partial u_y}{\partial y} \; \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \; \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \right.$$

So, what we have done is the first one represents the tensor from the mathematical way; the equation (table entries) represents the same tensor when we derive it in terms of components, but represent as two different tenses or two different matrices. What we are done here is represented those two different tensors in terms of displacement gradient ε_{xx}, ε_{xy} are variables; unless we expressed in terms of displacement gradients, they are not useful, they are just expressions, or variables introduced for our convenience.

(Refer Slide Time: 09:39)



This three-dimensional representation of course, looks little more complex. What we will just show in this slide is what is the 2D version of that looks little easier to follow, easier to understand, even for your understanding you will understand most of the concepts by looking at the 2D version of the equation.

$$\left[du_x \; du_y\right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y}\right][dx \; dy]$$

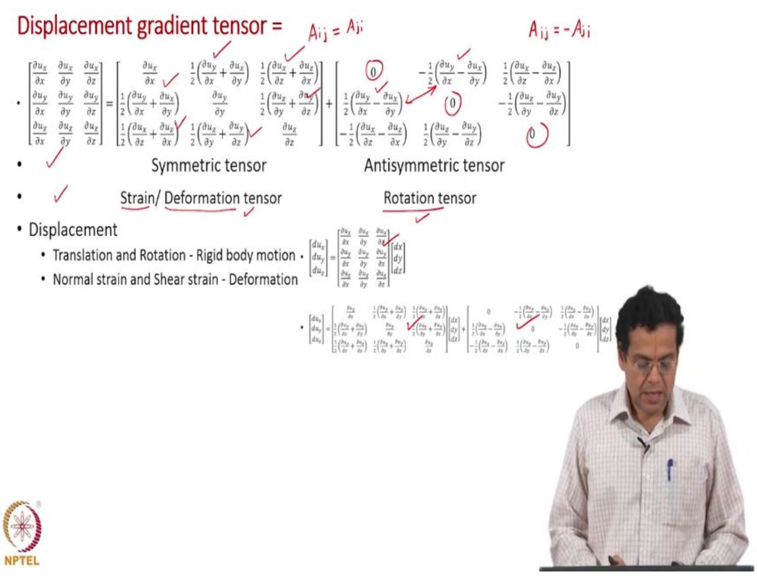
$$\left[du_x \; du_y\right] = \left[\varepsilon_{xx} \; \varepsilon_{xy} \; \varepsilon_{xy} \; \varepsilon_{yy}\right][dx \; dy] + \left[0 - \omega_{xy} \; \omega_{xy} \; 0\right][dx \; dy]$$

$$\left[du_x \; du_y\right] =$$

$$\left[\frac{\partial u_x}{\partial x} \; \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \; \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \; \frac{\partial u_y}{\partial y}\right][dx \; dy] + \left[0 \; -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \; \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \; 0\right][dx \; dy]$$

So, this slide is same as the previous slide those equation where for 3D. corresponding equations are shown here in this slide for two-dimensional case, all have the same significance as the previous slide just 2D version of the previous slide of a little looks little simpler to look at it and understand as well.

(Refer Slide Time: 10:39)



Now, we said that the displacement gradient tensor is equal to the sum of the two tensors and that is what we are going to see the next slide for a 3D version,

$$\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z}\ \right] = \left[\frac{\partial u_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{\partial u_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\ \frac{1}{2}\left(\right.\right.$$

The left hand side is contains all the nine combinations of displacement gradients; and right hand side you have the normal strain along the diagonal and then you have the shear strain along the non-diagonal elements and when you come to the second tensor, it is all zero along the diagonal elements and of diagonal elements have the rotation in the different planes.

Now, how do you name them? first of all these are in terms of characteristics this is a symmetric tensor; you can just understand a symmetric matrix. What is symmetric matrix? The off diagonal components are same that is why it is symmetric tensor, also called as a symmetric matrix for simple understanding.

If you come to the second matrix or tensor, now the off diagonal components have same magnitude, but opposite sign and that is why it is called a antisymmetric matrix or antisymmetric tensor. So if you want to write formally let us say A is a matrix and when you right for symmetric matrix you write as

$$A_{ij} = A_{ji}$$

And for anti-symmetric you write as

$$A_{ij} =- A_{ji}$$

That is why when $i = j$ you get 0 as the diagonal elements off diagonal elements are opposite to each other.

Now, coming to the nomenclature, in terms of nomenclature the first tensor is called strain tensor, very easy to understand. It contains contribution of normal strain and shear strain. It represents deformation. So, it is called as strain tensor or deformation tensor.

$$Strainor\ deformation\ tensor = \Big[\frac{\partial u_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{\partial u_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \right.$$

The second one has all its component as rotation, in the different planes hence called as rotation tensor.

$$Rotation\ tensor = \Big[0\ -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\ 0\ -\frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\right.$$

So, first contains information about normal strain and shear strain and hence called strain, they represent nothing but deformation, so deformation tensor. The right hand side is the rotation tensor for our understanding you can understand as just matrices.

And so, we have achieved our objective that is objective with which we started this lecture. What is that; left hand side has all components of difference in displacements of course, of varying different directions, which means translation has been taken into account. Now, difference in displacement have two components, one was the rigid body motion because of rotation, other was for deformation because of normal and shear strain and we want to separate the rotation part from the deformation part and that is what we have achieved.

Once again why did we all this will see later the last slide after this we will have numeric example now, let us put this in words displacement we said because of translation and rotation which we called as rigid body motion and then normal strain and shear strain were called as deformation.

Now, what we will see is a numerical example to illustrate whatever we have discussed so far. We have so far proved that

$$The\ displacement\ gradient\ tensor = Strain\ tensor + Rotation\ tensor$$

We have proved that more analytically. We will show now in terms of a numerical example.