Continuum Mechanics And Transport Phenomena
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Lecture – 63
Components of Total Displacement: Example

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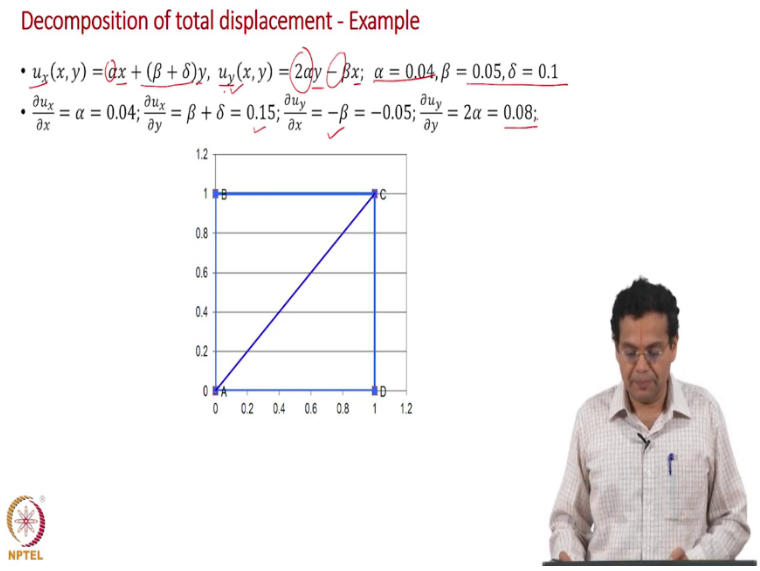

Now, we will start with the displacement field, now this displacement field meaning u_x as a function of x, y, and u_y as a function of x, y.

$$u_x(x, y) = \alpha x + (\beta + \delta)y, \quad u_y(x, y) = 2\alpha y - \beta x$$

$$\alpha = 0.04, \ \beta = 0.05, \ \delta = 0.1$$

This displacement flied looks similar to what we are considered earlier, but of course, slightly different for example, both u_x and u_y are functions of x and y.. These are chosen so, that to demonstrate certain features or demonstrate that the total displacement can be split into its components that is a idea.

So displacement field is chosen to demonstrate that the total displacement as sum of the deformation plus the rigid body rotation. Now let us calculate all the four gradients which will be useful for us. So,

$$\frac{\partial u_x}{\partial x} = \alpha = 0.04; \ \frac{\partial u_x}{\partial y} = \beta + \delta = 0.15; \ \frac{\partial u_y}{\partial x} = - \beta = - 0.05; \ \frac{\partial u_y}{\partial y} = 2\alpha = 0.08$$

So, this we will keep it handy so, that we will you can use it very as we go along. We are given the displacement flied and we have calculated all the 4 displacement gradients once again what does this displacement flied represent? We have a plate and then we it has subjected to same force and this displacement tells you displacement of all the points in the plate. And this displacement includes all the components we are going to separate each of them.

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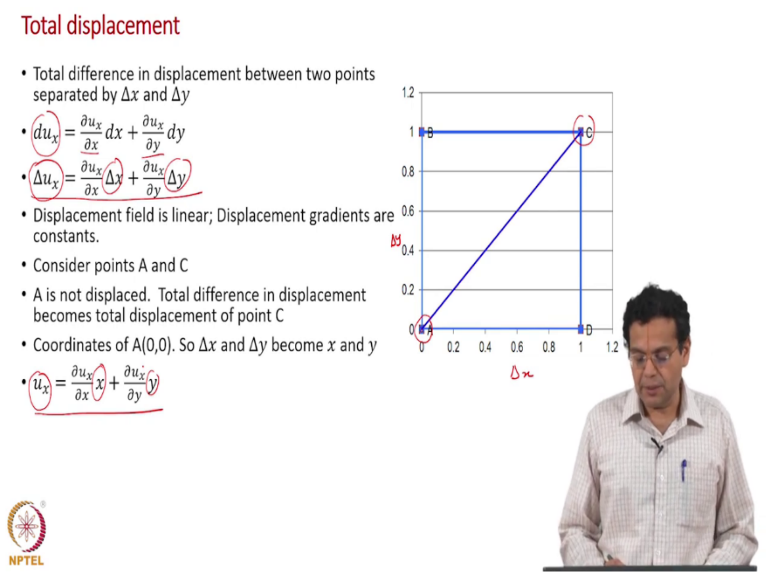

First let us look at the total displacement. We are going to arrive at total displacement in two different ways as we are done in the derivation. First in terms of the partial derivatives more mathematically, second in terms of splitting each of the components.

So, the first method for total displacement. So, we are going to look at difference in displacement. As we are discussed difference in displacement takes care of translation. Now how do you write total difference in displacement between two points

$$du_x = \frac{\partial u_x}{\partial x} dx + \frac{\partial u_x}{\partial y} dy$$

We have written this using the first method where you express the total differential in terms of partial differentials.

Now I will write in terms of Δ for easy understanding.

$$\Delta u_x = \frac{\partial u_x}{\partial x}\Delta x + \frac{\partial u_x}{\partial y}\Delta y$$

Now in the particular case the displacement field is linear they are all functions of x y which means that our displacement gradients are constants remember we had values 0.04, 0.5, 0.8 or whatever value which mean they are constant throughout the plate. So, displacement flied is linear. So, displacement gradients are constants.

So, now, let us consider points A and C, I will demonstrate all for point C then you can extent to other points. Now in the particular case A is not displaced so, the total difference in displacement becomes total displacement of point C. We actually suppose to consider difference in displacement between C and A, but because A is not getting displaced the difference in displacement between C and A becomes displacement of C itself.

In this case total displacement of C. So, A is not displaced. So, total difference in displacement becomes total displacement of point C. So, we will talk in terms of displacement itself. So, coordinate further the coordinates of A are (0, 0). So, Δx, Δy becomes x and then y. So, the second expression just becomes simplified as

$$u_x = \frac{\partial u_x}{\partial x}x + \frac{\partial u_x}{\partial y}y$$

Why is it simplified? Left hand side is difference in displacement we just now said it becomes equivalent to just displacement. So, I writing it as u_x, in the right hand side we have Δx and Δy because coordinates of A are (0, 0), they becomes just x and then y, this is what we use for further similarly we will use later on also. So, what we have done is, started with total difference in displacement in terms of the partial derivatives and then Δx, Δx difference in displacement becomes just becomes displacement because one point does not move.

So, for all the conditions we will take A as our reference point. So, that it becomes easy to understand because A does not get displaced.

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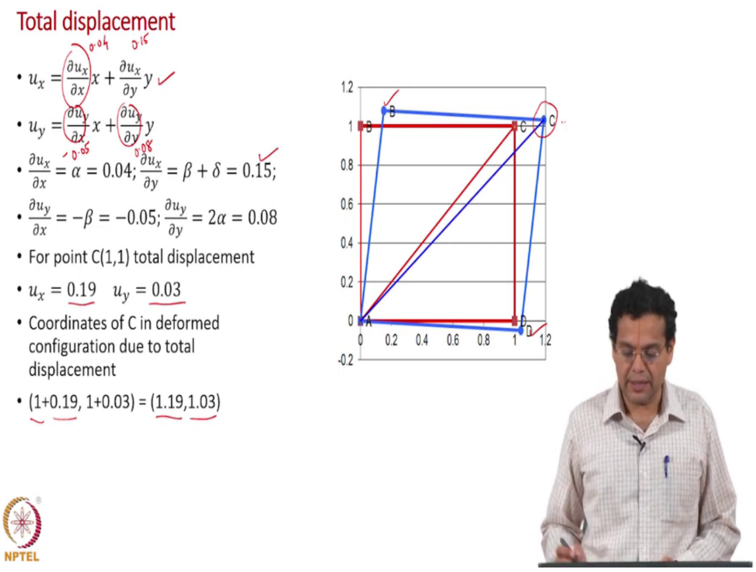

Now, we will calculate

$$u_x = \frac{\partial u_x}{\partial x}x + \frac{\partial u_x}{\partial y}y$$

Similarly we can write for u_y also

$$u_y = \frac{\partial u_y}{\partial x}x + \frac{\partial u_y}{\partial y}y$$

Now, we will take all the values which we calculated for all the gradients. We calculated all the combination of gradients.

$$\frac{\partial u_x}{\partial x} = \alpha = 0.04; \quad \frac{\partial u_x}{\partial y} = \beta + \delta = 0.15; \quad \frac{\partial u_y}{\partial x} = -\beta = -0.05; \quad \frac{\partial u_y}{\partial y} = 2\alpha = 0.08$$

So, we calculated the components of the displacement gradient tensor more formally. So, now, for point C which is (1, 1), total displacement

$$u_x = \frac{\partial u_x}{\partial x}x + \frac{\partial u_x}{\partial y}y = 0.04(1) + 0.15(1) = 0.19;$$

$$u_y = \frac{\partial u_y}{\partial x}x + \frac{\partial u_y}{\partial y}y = -0.05(1) + 0.08(1) = 0.03$$

So, as usual the red boundary shows the initial configuration initial state and the coordinates are (0, 0) etcetera and the blue boundary shows the final state. Remember the final state shows the displacement of all the points we have calculated for C alone, if you repeat for all other points you can get the coordinates of B, coordinates of D, then you will be able to draw the boundary of the plate ABCD in the final state.

Now, remember it shows the total displacement. This displacement could be because of rotation, normal strain, shear strain and this is what we are going to do now separately represent the normal strain, share strain and rotation then sum up and see that we will be able to get back this total displacement and that is what we did analytically, but now we are going to do in terms of a numerical calculation.

So coordinates of C in deformed configuration due to displacement you know the displacement. So, just add to the original coordinate so,

$$Final\ coordinates\ of\ C = (1 + u_x, 1 + u_y) = (1 + 0.19,\ 1 + 0.03) = (1.19, 1.03)$$

So, those are the coordinates similarly, you can calculate for ABCD.

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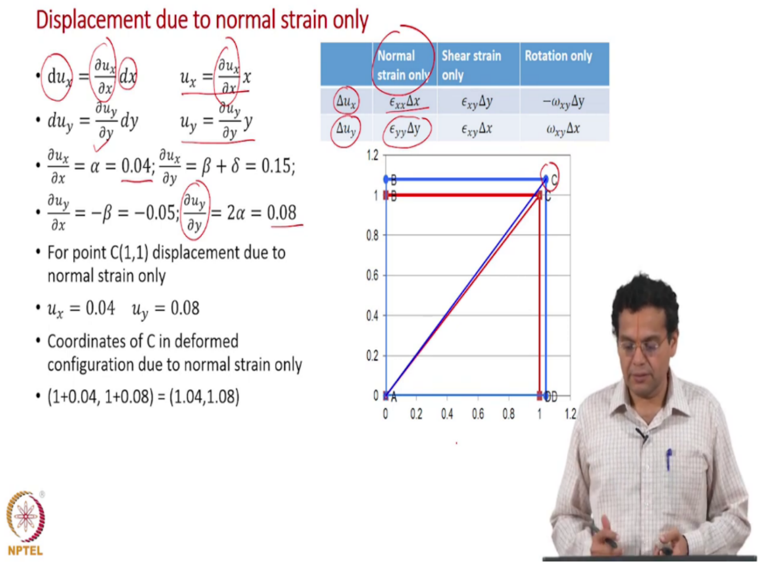

Now, take normal strain alone shear strain alone, rotation alone and see how the displacement happens. When we consider normal strain only, the difference in displacement or the differential displacement is

$$du_x = \frac{\partial u_x}{\partial x} dx \rightarrow \Delta u_x = \frac{\partial u_x}{\partial x} \Delta x$$

And as usual as we are discussed earlier difference in displacement become displacement Δx becomes x and so, we get this simple expression.

$$u_x = \frac{\partial u_x}{\partial x} x$$

Similarly in the y direction we write

$$du_y = \frac{\partial u_y}{\partial y} dy \rightarrow u_y = \frac{\partial u_y}{\partial y} y$$

So, as we have seen earlier we get this expression for displacement. So, we will take all the values of displacement gradients from the first previous slide.

$$\frac{\partial u_x}{\partial x} = \alpha = 0.04; \; \frac{\partial u_x}{\partial y} = \beta + \delta = 0.15; \; \frac{\partial u_y}{\partial x} = -\beta = -0.05; \; \frac{\partial u_y}{\partial y} = 2\alpha = 0.08$$

So, that now, we can calculate for point C, once again I take point C what is it called? Displacement dur to normal strain only so,

$$u_x = \frac{\partial u_x}{\partial x} x = 0.04(1) = 0.04$$

$$u_y = \frac{\partial u_y}{\partial y} y = 0.08(1) = 0.08$$

Now, what are the coordinates of C in deformed configuration due to normal strain only.

$$Final\ coordinates\ of\ C = (1 + u_x, 1 + u_y) = (1 + 0.04,\ 1 + 0.08) = (1.04, 1.08)$$

Similarly can applied for other points and the red boundary becomes the blue boundary this displacement is because of normal strain only.

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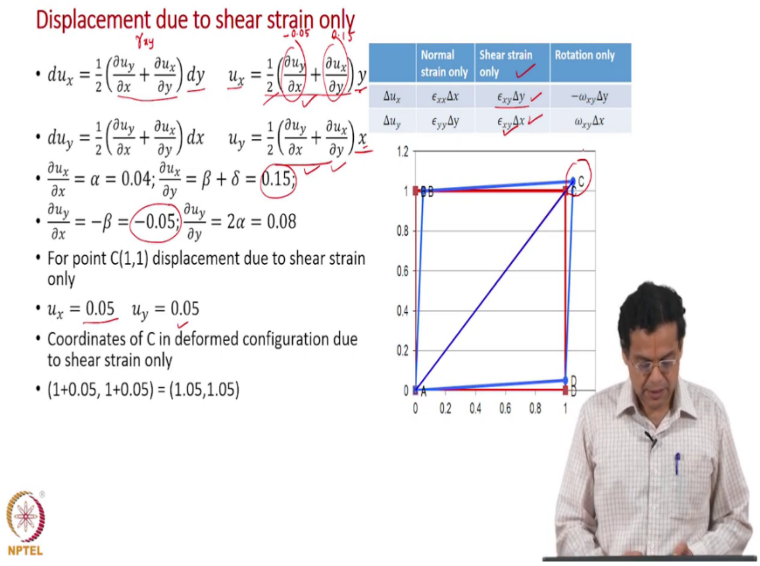

Now, let us proceed to displacement due to shear strain only which mean that we have to consider the second column.

$$du_x = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)dy \rightarrow \Delta u_x = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\Delta y$$

Now as like in the earlier case we write this as

$$u_x = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)y$$

And similarly in the y direction we take this as

$$u_y = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)x$$

Now let us get all the values of the gradients from the previous slides,

$$\frac{\partial u_x}{\partial x} = \alpha = 0.04; \quad \frac{\partial u_x}{\partial y} = \beta + \delta = 0.15; \quad \frac{\partial u_y}{\partial x} =- \beta =- 0.05; \quad \frac{\partial u_y}{\partial y} = 2\alpha = 0.08$$

So, now, for point C which is (1, 1), displacement due to shear strain only

$$u_x = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)y = \frac{1}{2}(-0.05 + 0.15) = 0.05$$

$$u_y = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)x = \frac{1}{2}(-0.05 + 0.150) = 0.05$$

So, now, what are the coordinates of C in deformed configuration due to shear strain only just add to the original coordinate

$$Final\ coordinates\ of\ C = (1 + u_x, 1 + u_y) = (1 + 0.05,\ 1 + 0.05) = (1.05, 1.05)$$

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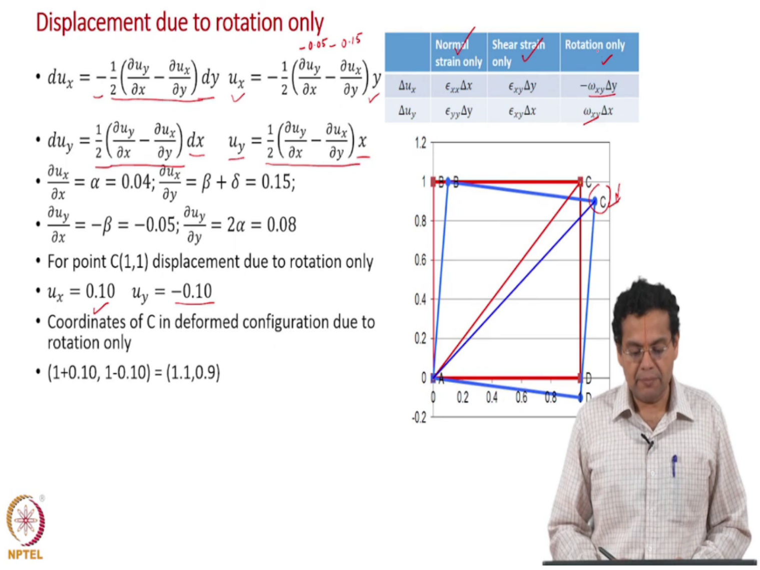

Now, last one displacement due to rotation only. So, we should consider this column the third column, what is expression? The difference in displacement is

$$du_x = -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)dy \rightarrow \Delta u_x = -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\Delta y$$

As we have done earlier lift hand side becomes u_x right hand side becomes just y.

$$u_x = -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)y$$

Let us write down for y

$$du_y = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)dx \rightarrow u_y = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)x$$

So, now, let us look at all the derivatives that is why we calculated all this derivatives to begin with ok. So, we require that several times and so, it becomes handy to calculate and keep it ready.

$$\frac{\partial u_x}{\partial x} = \alpha = 0.04; \quad \frac{\partial u_x}{\partial y} = \beta + \delta = 0.15; \quad \frac{\partial u_y}{\partial x} = -\beta = -0.05; \quad \frac{\partial u_y}{\partial y} = 2\alpha = 0.08$$

So, once again for point C displacement due to rotation only due to rotation only

$$u_x = -\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)y = -\frac{1}{2}(-0.05 - 0.15) = 0.10$$

$$u_y = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)x = \frac{1}{2}(-0.05 - 0.150) = -0.10$$

So, now, let us add to the original coordinates, coordinates of C in deformed configuration due to rotation only.

Just want to mention it says deformed configuration, but says rotation only looks little paradox because we said deformation is normal strain shear strain only, but this is a usual nomenclature used initial configuration, final configuration, initial state, final state. So, that is why this says deformed configuration meaning final state the blue boundary due to rotation only.

$$Final\ coordinates\ of\ C = (1 + u_x, 1 + u_y) = (1 + 0.10,\ 1 - 0.10) = (1.10, 0.90)$$

So, now, what is it we have done? We have separately consider the displacement due to normal strain, shear strain and rotation now put them together and check whether we get the same as we get the same as we are done straight away with total displacement.

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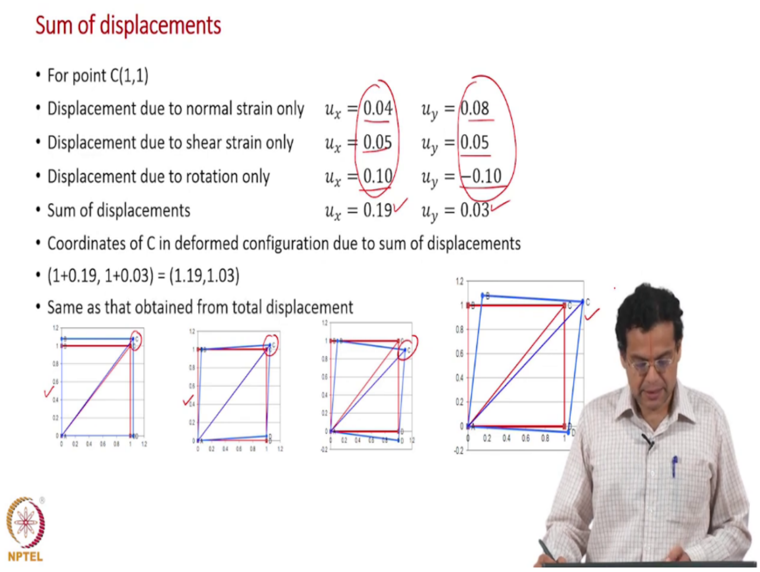

So, let us do that that is what is shown here. Now we will discuss for point C.

Point C displacement due to normal strain only the values were $u_x = 0.04$ and $u_y = 0.08$. The corresponding figures are shown here the bottom of the slide. The first figure shows for normal strain only and the displacements were 0.04 and 0.08 that is also seen here.

Now next we consider shear strain only. So, displacement due to shear strain only were $u_x = 0.05$ and $u_y = 0.05$.

And thirdly we took displacement due to rotation only and the displacements were $u_x = 0.1$ and then $u_y = -0.1$.

Now let us add all the displacements coordinates should not be added anyway that is not meaning full displacement have to be added.

So, sum of displacements are $u_x = 0.04 + 0.05 + 0.1 = 0.19$ and $u_y = 0.08 + 0.05 - 0.1 = 0.03$.

This is same as what we got earlier by straight away writing total displacements and that is what is shown here of cause same as that obtained from total displacement.

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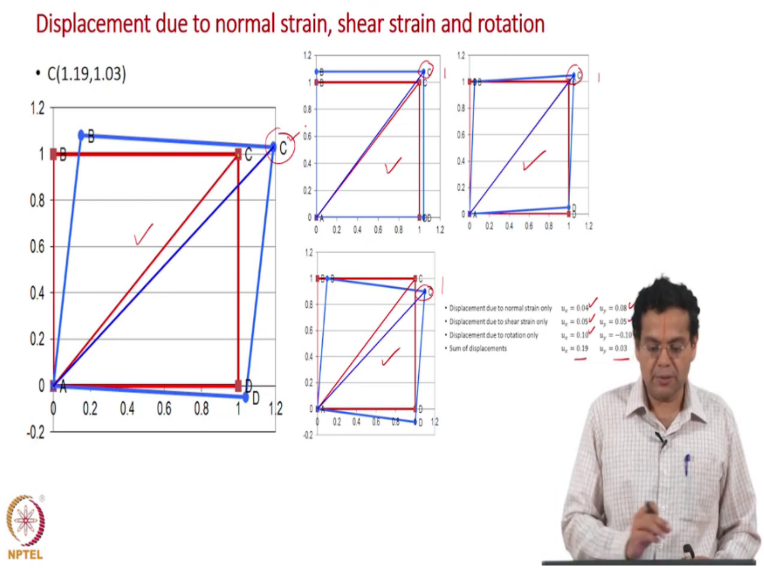

And this is what shown in larger sized figures, this shows a pervious slide were the individual displacements are shown in the x and y direction and the sum is also shown here. This large figure has been obtained two times first by considering only total displacements in teams of partial derivatives which is more mathematical, we got the coordinates of C as 1.19, 1.03.

Now, these individual figures are considering normal strain only, shear strain only, rotation only. When we sum up all the displacements once again you get the left hand side figure, but now by decomposing the total displacement, splitting it that into components because of normal strain, shear strain and rotation only. So, good example to visualise what we mean by representing a displacement gradient tensor as sum of strain tensor plus rotation tensor.

And just to like to mention about this coordinate C of course, obvious if you look at start with normal strain, there is positive x displacement and due to shear strain also there is positive x displacement, due to rotation also positive displacement that is way all of them get added and you have a large positive displacement a total displacement. Now if you look at y displacement it is positive y displacement for normal strain, again a positive y displacement for shear strain but when you came here rotation there is a negative displacement in the y direction.

That is why when you sum up, you get a small positive y displacement that is why if you look at initial C and final C, large x displacement and a small y displacement of course, that explains whatever we discussed earlier little more graphically.

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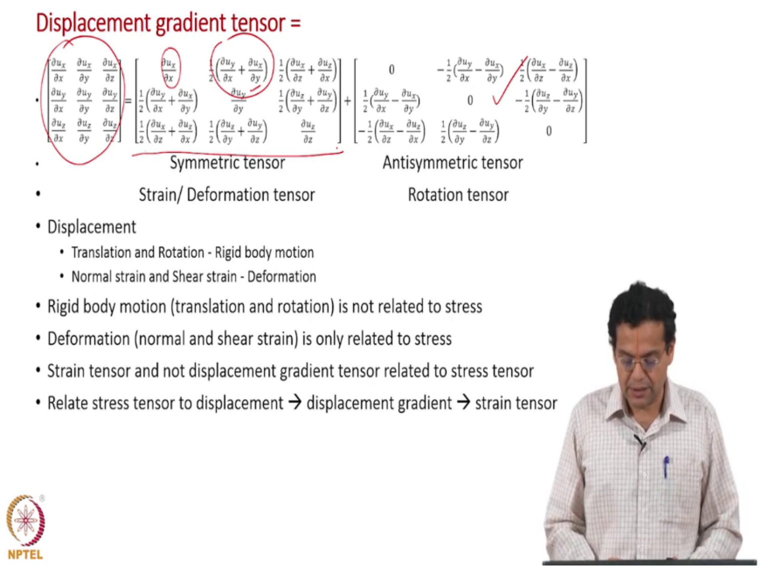

So, now we can write this statement again that

$$The\ displacement\ gradient\ tensor\ =\ Strain\ tensor\ +\ Rotation\ tensor$$

Now I think we can look at it more physically geometrically instead of just looking at terms of calculus or differentials and remember that is why in the end of the pervious discursion, we were looking at the physical significance of these terms. We said $\frac{\partial u_x}{\partial x}$ initially just a mathematical gradient slowly we said see it represent normal strain and then we said the two terms sum together represent shear strain and we said that we will leave it to the present and that is what we have done now.

$$\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z}\right] = \left[\frac{\partial u_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{\partial u_y}{\partial y}\ -$$

So, symmetric tensor, anti symmetric tensor we have seen and strained deformation tensor, rotation tensor, you need not even discuss in terms of tensor just understand in terms of matrix easy for understanding.

Now kept as a kind of suspense why did we really split into two parts and that answer there will be answered now.

Now if you take a plate and then apply a force and just let us a translates; obviously, there will not be any internal stress it just rotates there will not be any internal stress, but now let us say you try to pull then there is a internal stress it means that a change in length the strain and the stress are related you try to squeeze in what happens? The shear strain and stress are related other example would be if I just move I am translating no I do not feel any stress inside.

If I just rotate I do not feel any stress of course, if I keep rotating I may fall feel giddy and leave fall, but that does not mean internal stress are developed. But imagine somebody is trying to pull me this side, pulling me that side really I will be on severe stress and that is what exactly is happening. So, physically intuitively we can understand more mathematically also we can say.

Physically we can see that the stress is related to the strain part only, but not to translation and rotation. Translation we have taken care, but we have take care of rotation that is why we separated, see this tensor on the left hand side includes rotation normal strain shear strain.

Based on the discursion that stress is not to be related to the rotation so, we will have to take out this rotation part and that is what we had done now.

So, the important tensor for us or the important components for us are the components of the strain tensor not there of the rotation tensor. Just one more point we are right now discussing strain for solids we are discussed stress and we are going to relate these two and that we will of course, later.

So, let us complete whatever I said

- Rigid body motion (translation and rotation) is not related to stress,
- Deformation (normal strain shear strain) is only related to stress and hence strain tensor not displacement gradient tensor related to stress tensor.

How are we going to relate that we will see, but we should be ready with the right hand side. Left hand side is stress tensor the right hand side is the not the displacement gradient tensor, but the strain tensor, components of that.

So, relate stress tensor to displacement when I say displacement, it is displacement gradient not even displacement gradient tensor, but the one part of it namely strain tensor that is a biggest take away from this slide. I said I will tell the motivation later now the motivation becomes very clear now, that is why as been reserved till this point that the reason for splitting is the only one part of the displacement gradient tensor is to be related to stress.

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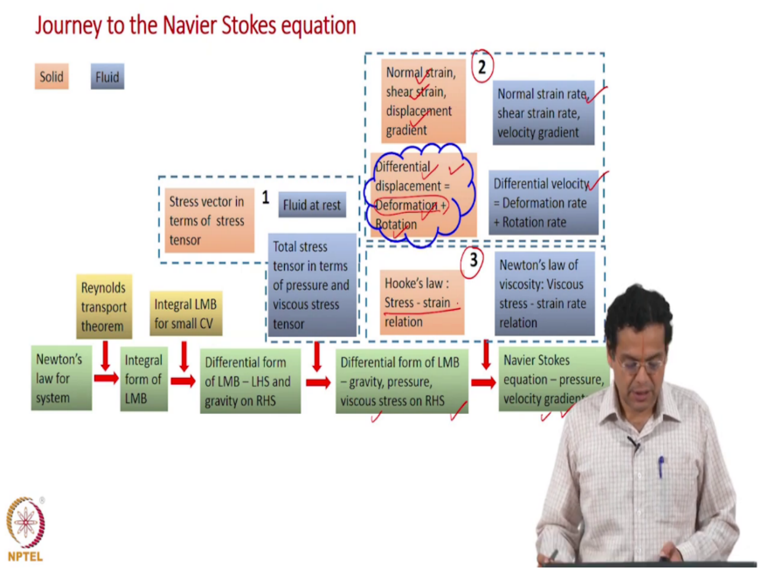

And in terms of our journey to the Navier Stokes, we ready with the linear momentum balance for fluid and we found out that the viscous stress tensor is unknown, we need to relate it to the velocity gradients and we said to understand velocity gradients, we will take a diversion to solid mechanics and first we understood normal strain, shear strain, rotation, displacement gradient and now we understood this particular block which says differential displacement is equal to deformation plus rotation.

Reason is that, this deformation or the strain tensor alone is important for us to relate to stress. So, in terms of the plan, now that we are discussed all the concepts of soiled mechanics, understood the displacement gradients now it's time for us to go fluid mechanics, discuss next two blocks and we will see how all these concepts analogously apply for the case of fluids, what is our objective to relate the viscous stresses to velocity gradients.

Understand velocity gradients we came to solid mechanics displaced discussed displacement gradients we go to fluid mechanics and discuss about velocity gradient. And then we will have to relate these two that is what we do here were we relate stress to the strain and go to fluid mechanics and relate stress to strain rate what is strain rate we will understand later. So, when we say stress strain relationship, it is between stress and the strain tensor. Of course, that we will understand as we go along just to give you a outline at this point.

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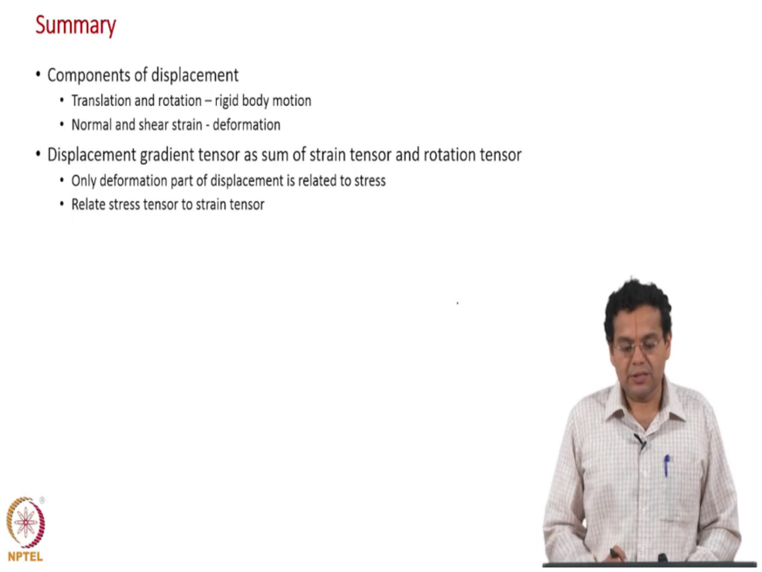

So, let us summarise this part of the lecture alone, the components of displacement or translation and rotation which we call as rigid body motion, normal and shear strain which we call as deformation ok. In the displacement gradient tensor we have seen as a sum of strain tensor and rotation tensor, only the deformation part of displacement is related to stress, relate stress tensor to strain tensor, which we will take little later meaning going back to fluid mechanics and when we come back to solid mechanics third time, we will do this.

But it now we are going back to fluid mechanics in the next lecture to discuss about velocity gradients all the concept discussed was solid mechanics you will see that analogously can be applied for fluid mechanics of course, there small concepts are there that we will discuss.