Continuum Mechanics And Transport Phenomena
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Lecture - 65
Strain Rate Velocity Gradient Relation

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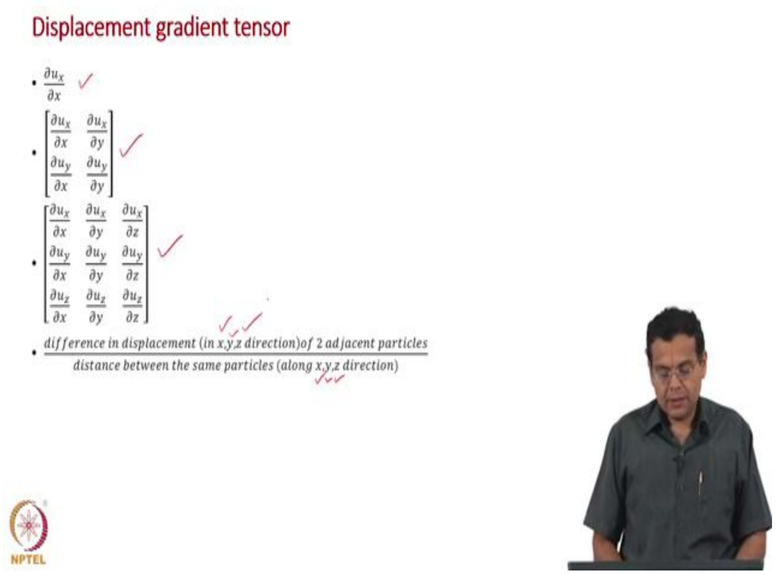

Then we introduced displacement field and then displacement gradient tensor for solids first for 1D, 2D and then 3D. And what was the definition of displacement gradient? We consider two adjacent particles and then take the difference in displacement of two adjacent particles, divide by the distance between the same particles. We have also said that the two particles can be considered along x, y, z directions and displacements could be in x, y, z directions and hence the displacement gradient as a tensor.

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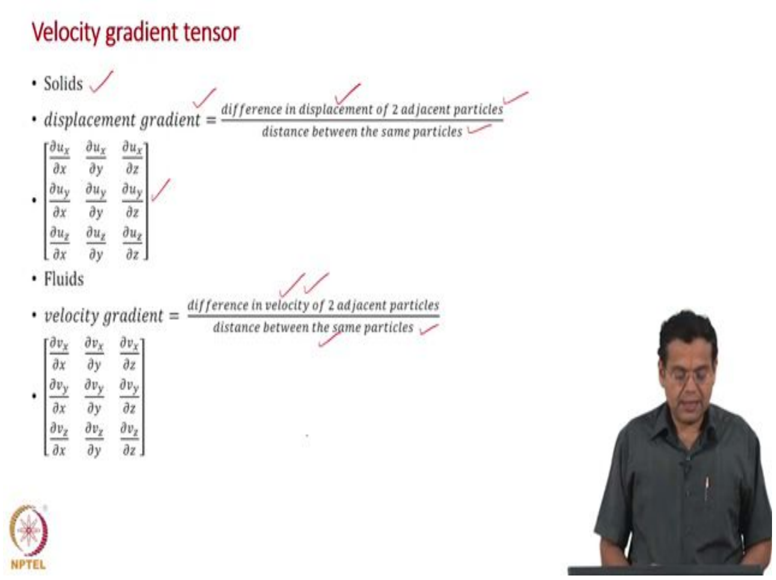

And now let us see how do we extend that for fluids. This part is same as what we have seen in the last slide; for solids we have

$$Displacement\ gradient = \frac{Difference\ in\ displacement\ of\ two\ adjacent\ particles}{Distance\ between\ the\ same\ particles}$$

Now for the case of fluids, let us define the velocity gradient. How do we define velocity gradient? Instead of displacement now we take velocity. So, the definition is

$$V\,elocity\ gradient = \frac{Difference\ in\ velocity\ of\ two\ adjacent\ particles}{Distance\ between\ the\ same\ particles}$$

Now, can analogously write the velocity gradient tensor; once again it is a tensor the reason is we can consider the particles along the x, y, z directions and the difference in velocity can be along once again x, y, z directions resulting in a velocity gradient tensor.

$$V\,elocity\ gradient\ tensor = \left[\frac{\partial v_x}{\partial x}\ \frac{\partial v_x}{\partial y}\ \frac{\partial v_x}{\partial z}\ \frac{\partial v_y}{\partial x}\ \frac{\partial v_y}{\partial y}\ \frac{\partial v_z}{\partial z}\ \frac{\partial v_z}{\partial x}\ \frac{\partial v_z}{\partial y}\ \frac{\partial v_z}{\partial y} \right]$$

Other way of explaining is that velocity itself is a vector, there are three components. Now, you are looking at the gradient of that velocity component that could be in three directions resulting in a tensor; velocity in three directions, the gradient in three directions, resulting in 9 combinations of directions, resulting in a velocity gradient tensor.

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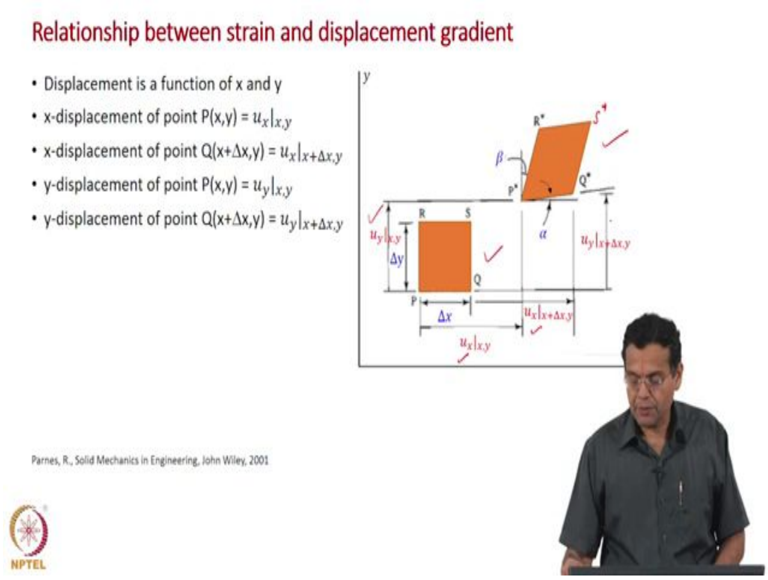

Then what we did in solid mechanics was having discussed strain and discussed displacement field, displacement gradient. We expressed strain in terms of displacement gradient and we are going to repeat the same exercise here. But now find relationship between strain rate and then velocity gradient. This is the diagram which we have discussed to relate strain and then displacement gradient where we initially took a two dimensional plate or a two dimensional region of length Δx Δy and then the subjected to a force.

And so, there was translation, rotation, normal strain, shear strain meaning change in length and change in angle as well. So, PQRS became P*Q*R* S*. And, you also noted the displacements of all the points. For example, $u_x|_{x,y}$ is the x displacement of P and $u_x|_{x+\Delta x,\, y}$ is a x displacement of Q and $u_y|_{x,y}$ is a y displacement of P, $u_y|_{x+\Delta x,y}$ is a y displacement of Q.

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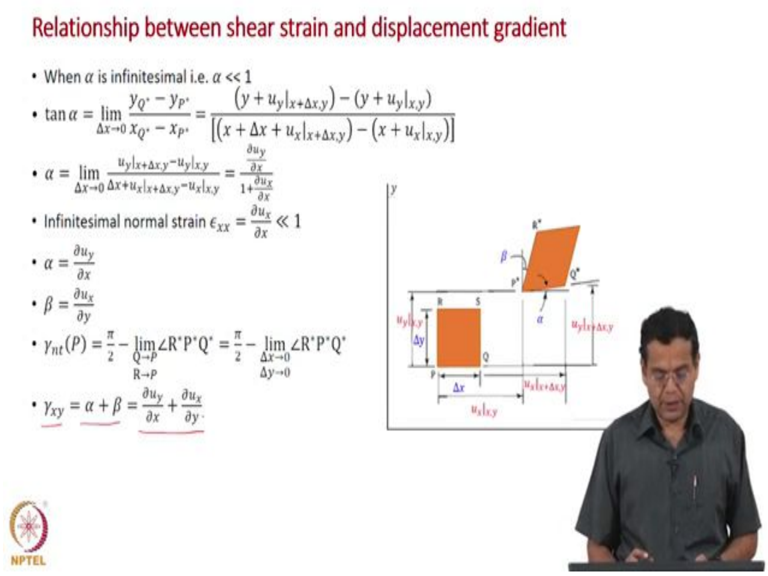

And, we derive the relationship between normal strain displacement gradient. So, let us quickly recall that so that we can easily understand the relationship between normal strain rate and velocity gradient which we are going to derive in the next slide. Normal strain is defined as change in length by original length we are considering a line element along x axis and hence it is

$$\varepsilon_{xx} = \frac{|P^*Q^*| - |PQ|}{|PQ|}$$

Then we said we are going to assume infinitesimal rotation.

$$\varepsilon_{xx} = \frac{[(x+\Delta x+u_x|_{x+\Delta x,y}) - (x+u_x|_{x,y})] - \Delta x}{\Delta x}$$

So, we said the length of P*Q* is same as the projected length of P*Q*. So, length of P*Q* becomes the difference in the x coordinate of Q* and then P*.

Now, Δx cancels out and we are left with

$$\varepsilon_{xx} = \frac{[u_x|_{x+\Delta x,y} - u_x|_{x,y}]}{\Delta x}$$

When you take limit $\Delta x \to 0$ we get

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

So, what we have done is related normal strain to the displacement gradient. Now, see analogously we derived relationship between shear strain and displacement gradient where we derived

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

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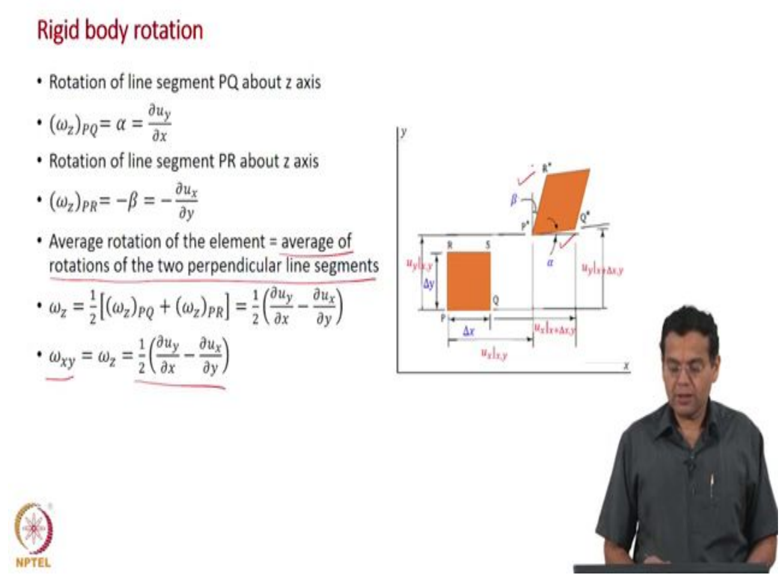

We also derived relationship for rigid body rotation and we defined that as average of rotations of two perpendicular line segments. We took into account the sign as well and we got the expression

$$\omega_{xy} = \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

So, one is anti-clockwise other is clockwise. So, we take in account the sense of rotation.

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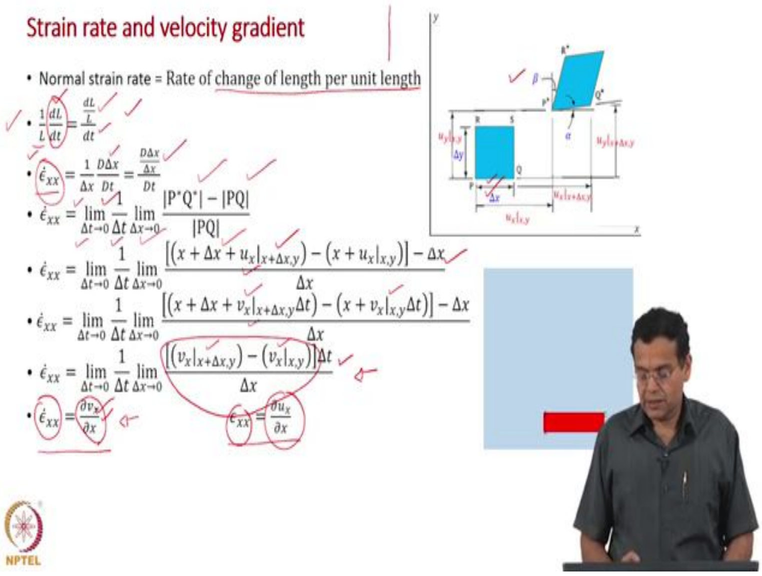

So, now we are ready to derive the relationship between strain rate and velocity gradient. The same figure has shown here just colored in blue to get a feel that we are looking at a fluid element. Let us see how do we derive

The normal strain rate is rate of change of length per unit length. Change of length per unit length this normal strain; rate of change of length per unit length is the normal strain rate. How do we understand this normal strain rate so, that it will be easy to derive.

$$\frac{1}{L}\frac{dL}{dt} = \frac{\frac{dL}{L}}{dt}$$

Suppose if you have a rod and just imagine that the length keeps increasing, then at any instant of time the rate of change of length divide by the instantaneous length is the normal strain rate. Or let us say there is a thread whose length keeps increasing with time, then how do you define normal strain rate? The rate at which that length changes which is $\frac{dL}{dt}$.

The nomenclature is shown here divide by the instantaneous length, length at any instant of time that is the normal strain rate. Now, we are going to apply this for this small line segment PQ; let us do that. So,

$$\dot{\varepsilon}_{xx} = \frac{1}{\Delta x}\frac{D\Delta x}{Dt} = \frac{\frac{D\Delta x}{\Delta x}}{Dt}$$

Now instead of L, I have written this expression in terms of Δx because Δx equivalent to our L. Now the D which is used here is capital $\frac{D}{Dt}$, the reason is this derivatives taken following the fluid motion. Remember the this line PQ is in a fluid and the rate of change of its length is determined by following its motion along with the fluid that is why we take a substantial derivative $\frac{D}{Dt}$.

Now, you can express this as,

$$\dot{\varepsilon}_{xx} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{|P^*Q^*| - |PQ|}{|PQ|}$$

So, this is the expression for the normal strain rate for a line element along the x direction. So, now, let us express this change in length by original length exactly as we have done earlier. So, this expression is exactly same as what we have done earlier in terms of displacements.

$$\dot{\varepsilon}_{xx} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\left[\left(x+\Delta x+u_x|_{x+\Delta x,y}\right)-\left(x+u_x|_{x,y}\right)\right]-\Delta x}{\Delta x}$$

Now the difference comes, remember these are in terms of displacement that is why we had a quick recall of derivation of normal strain. Now these expressions are in terms of displacement, we will have to express in terms of velocities. So, we will express the displacements in terms of velocity into Δt, we consider small time interval Δt over the time interval, we express the displacement as the velocity into Δt that velocity is some average value over the Δt.

$$\dot{\varepsilon}_{xx} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\left[\left(x+\Delta x+v_x|_{x+\Delta x,y}\Delta t\right)-\left(x+v_x|_{x,y}\Delta t\right)\right]-\Delta x}{\Delta x}$$

So, $u_x|_{x+\Delta x,y}$ is expressed as $v_x|_{x+\Delta x,y}\Delta t$. Now, x cancels out, Δx cancels out and then we have

$$\dot{\varepsilon}_{xx} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\left[v_x|_{x+\Delta x,y} - v_x|_{x,y}\right]\Delta t}{\Delta x}$$

Now of course, Δt cancels out and this becomes

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x}$$

Now, you are taking limit $\Delta t \to 0$ which means that the velocities become instantaneous values that is the difference between the two steps, when you cancel out Δt and when you are taking $\Delta t \to 0$. These two velocities were average velocities over some time interval Δt, but now we are taking $\Delta t \to 0$ so, they become instantaneous values. So now, this gradient of velocity is gradient of instantaneous velocities and as we expect this relationship is valid at every instant of time and also at every special location.

So, the relationship between the normal strain rate and velocity gradient is given by

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x}$$

If you compare with the corresponding relationship for solids which is normal strain is equal to the displacement gradient ok. So, if you want to let us say vaguely say, it is just time derivative on both sides that is a little crude way of saying the time derivatives of course, substantial time derivative.

So, one way it is just time derivative on both sides this I would say crude view point will help us to extend analogously all the relationship. Though we have done the derivation in detail, this will help us to understand and write the expressions analogously. Of course, this animation has been going on which shows which I have seen already shows the that the length of line segment AD keeps increasing and length of line segment AB keeps decreasing.

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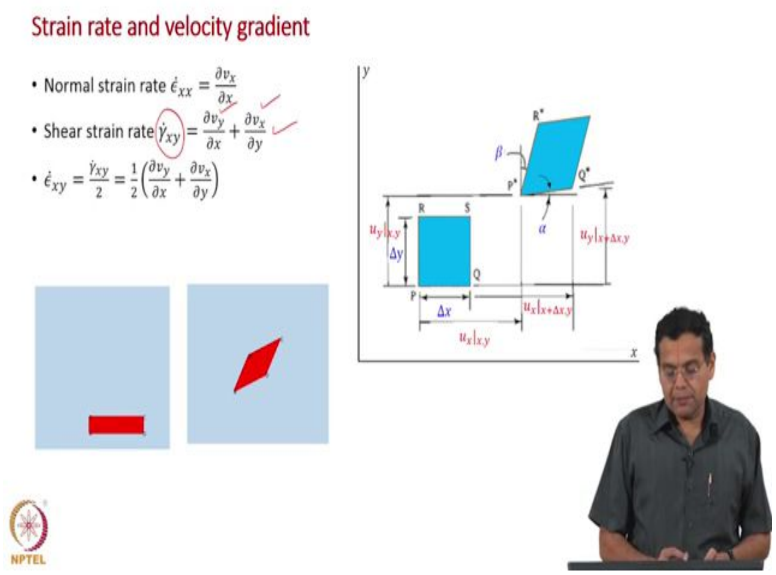

This rewritten from the last slide normal strain rate

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x}$$

Now, as we have discussed we can now quickly write the expression for shear strain rate which is $\dot{\gamma}_{xy}$ shear strain rate.

$$\dot{\gamma}_{xy} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}$$

So, once we understand that is just time derivative for quick understanding, then we can analogously write all the expressions. You can also introduce

$$\dot{\varepsilon}_{xy} = \frac{\dot{\gamma}_{xy}}{2} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)$$

And, of course, we have this animation. So, one for normal strain rate the left hand side and the right hand side is for shear strain rate; one for normal strain rate, the left hand side. The right hand side is to illustrate shear strain rate and one in which the length keeps increasing or decreasing other in which the angle decreases with respect to time.

Once again want to mention that the shear strain rate is always with reference to two line elements which are perpendicular to each other. So, only the initial a time let us say whatever time t where we have two perpendicular line segments and the angle decreased between the two line elements that rate of change of angle is a shear strain rate. It is not applicable throughout the motion of the fluid element.

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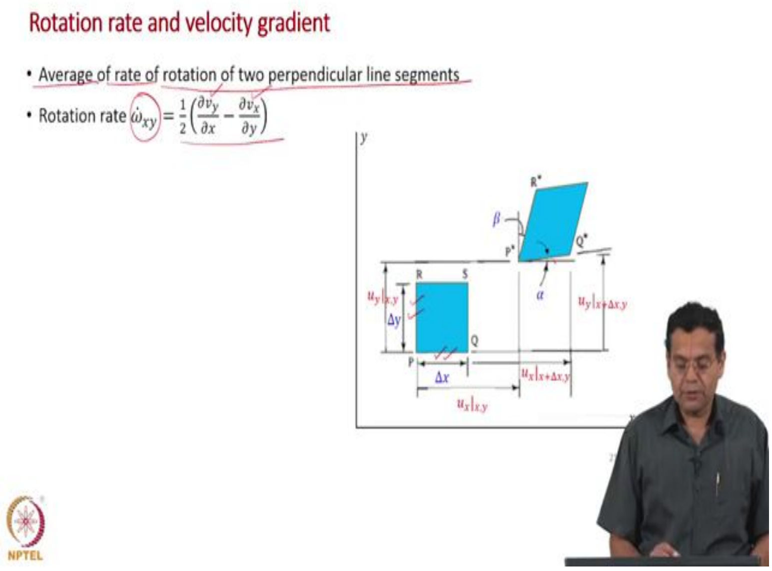

We can also relate rotation rate and then velocity gradient. We said for the case of rotation, we define it as average of rotation of two perpendicular line segments. Now, it is very simple average of rate of rotation of two perpendicular line segments. What does it mean? lets say if you consider two line segments P Q and R S which are perpendicular to each other, rate at which P Q rotates plus rate at which P R rotates average of these two is the rotation rate. Of course, taking into account the sense of rotation so, rotation rate is

$$\dot{\omega}_{xy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$