Continuum Mechanics and Transport Phenomena
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Lecture – 75
Hooke’s Law: Examples

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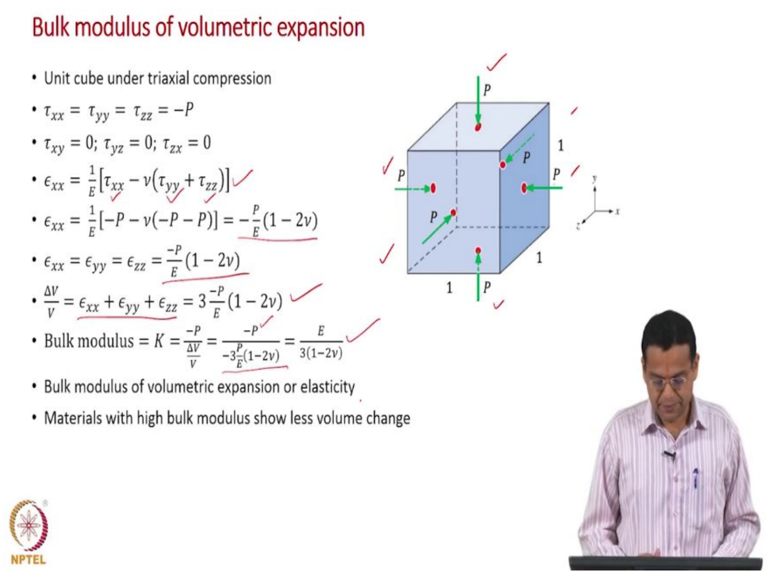

Before we look at some applications like to discuss one more material property; obviously, this cannot be independent material property, it should depend on the previous material properties only. What is that material property? Bulk modulus of volumetric expansion. In one way analogues to what I discussed strain, we had a discussed normal strain, shear strain and then volumetric strain and analogously here we have discussed Young’s modulus, shear modulus and now Bulk modulus. So, in that way they are parallel so, normal strain, shear strain and then volumetric strain, analogously here we discussed Young’s modulus, shear modulus and now Bulk modulus.

How do we derive that? We consider the unit cube under triaxial compression. What is triaxial compression? Unit cube meaning cubes of unit side and triaxial compression, we have the cube is subjected to compressive stresses, normal stresses and all of them are equal magnitude ok. And that is what is shown here we have the normal stresses in x, y and z direction which are all of same magnitude. So,

$$\tau_{xx} = \tau_{yy} + \tau_{zz} = -P$$

$-P$ because it is a compressive stress, all of them same magnitude and compressive as well and there is no shear stress. So,

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Now, let us see how the equations get simplified, we will start the expression for a normal strain in terms of the normal stresses these are the equation which you have seen

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})\right]$$

Let us substitute,

$$\tau_{xx} = \tau_{yy} + \tau_{zz} = -P$$

So, if you simplify and you get

$$\varepsilon_{xx} = -\frac{P}{E}(1 - 2\nu)$$

Same would result for other directions as well because the stresses are same.

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = -\frac{P}{E}(1 - 2\nu)$$

So, the normal strain in all the directions are same. Now how did we define volumetric strain, fractional change in volume, change in volume by original volume and that we related to the some of the normal strains

$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 3\frac{-P}{E}(1 - 2\nu)$$

Now, how do you define Bulk modulus? Bulk modulus is denoted by capital K is the applied stress which is $-P$ divided by the fractional change in volume. So, applies the stress is $-P$ and a fractional change in volume is $3\frac{-P}{E}(1 - 2\nu)$ if you simplify you get

$$Bulk\ modulus = K = -\frac{P}{\frac{\Delta V}{V}} = -\frac{P}{3\frac{-P}{E}(1-2\nu)} = \frac{E}{3(1-2\nu)}$$

That is the expression for Bulk modulus as I told you it cannot be independent property, the Bulk modulus is depending on E and ν there can be only 2 independent properties.

And more formal name Bulk modulus of volumetric expansion or elasticity if you want more formal name for K and what are the significance if K is very large for the same stress the volume change will be very less same like our Young's modulus and shear modulus also. What did we say if our Young's modulus is very large for the same normal stress, the normal strain is very less.

Similarly what do we say for G for the same shear stress material with the larger G will have a very small shear strain, similarly here for the same applied stress the material with higher Bulk modulus will have very small volume change. So, for the case of Young's modulus higher Young's modulus lower normal strain, higher shear modulus lower shear strain and higher Bulk modulus lower volumetric strain.

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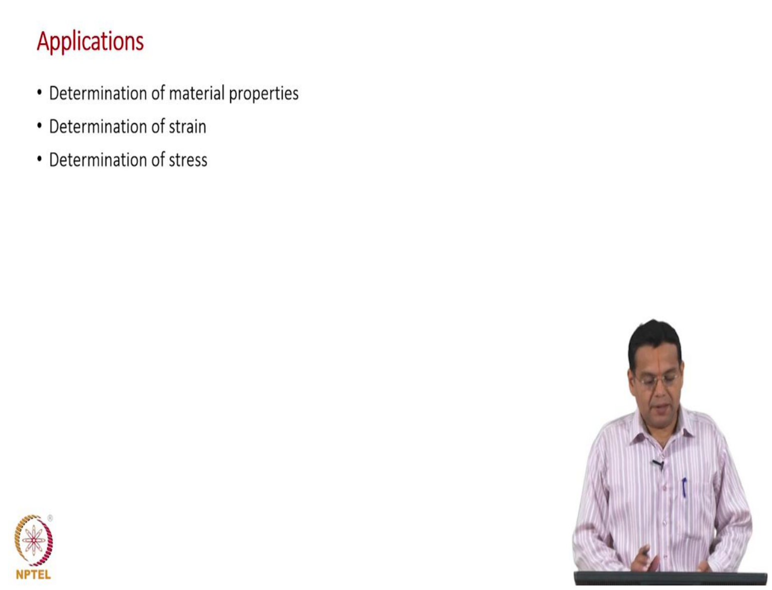
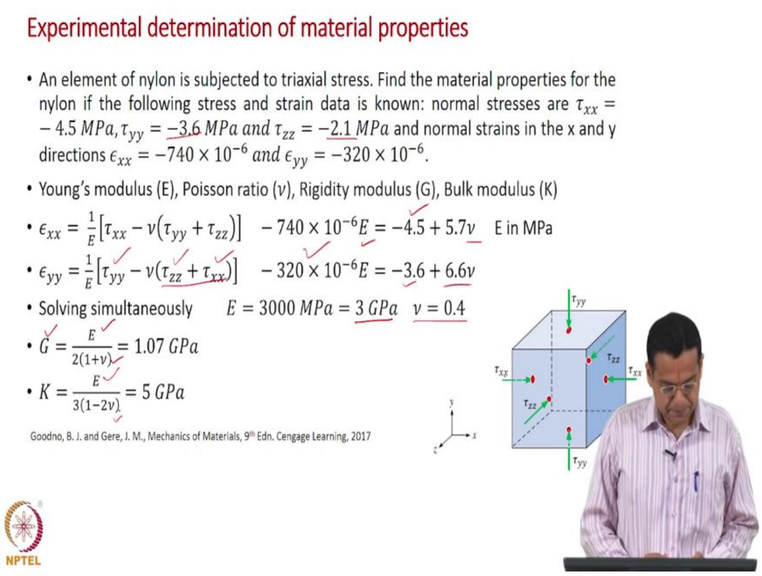


What we will do now is, look at some applications of the equations which we have discussed and derived some numerical simple numerical applications, they are not conceptually difficult I would say just simple substitution examples only, but they will help us to get a feel for what the equations are how are they used etcetera.

And some range some idea of the range of numbers some idea of the range of the values for the different variables which have which we have come across. So, what are the applications? The same equations will be used in different ways. So, first application is to use these equations to determine material properties, once again same set of equations to find strain, same set of equations to find stress.

Example: (Refer Slide Time: 06:43)



So, now let us read the example an element of a nylon is subjected to triaxial stress I think we have come across this triaxial stress few times now which means only normal stresses are applied. Find the material properties for the nylon if the following stress and strain data is known.

The normal stresses are a given $\tau_{xx} = -4.5\ MPa$, $\tau_{yy} = -3.6\ MPa$, $\tau_{zz} = -2.1\ MPa$ all of them are compressive and normal strains in the x and y directions are given $\varepsilon_{xx} = -740x10^{-6}$ and $\varepsilon_{yy} = -320x10^{-6}$ are given to us.

Solution:

So, what are the properties we are going to determine Young's modulus, Poisson ratio, Rigidity modulus, Bulk modulus in fact, all the properties which I have discussed can be determined.

So, let us write the equation for the normal strain in the x direction in terms of the three normal stresses

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right]$$

$$-740x10^{-6}E = -4.5 + 5.7\nu$$

Let us write down the equation for the normal strain in the y direction

$$\varepsilon_{yy} = \frac{1}{E}\left[\tau_{yy} - \nu\left(\tau_{xx} + \tau_{zz}\right)\right]$$

$$-320x10^{-6}E = -3.6 + 6.6\nu$$

So, now, we have got 2 equations in 2 unknowns E and ν the two properties Young's modulus and Poisson ratio. So, solving simultaneously we get

$$E = 3000\ MPa = 3\ GPa; \qquad \nu = 0.4$$

And once you get two properties we know that we can get all other properties we have only 2 independent properties you have solve further two independent properties. So, we know the relationship

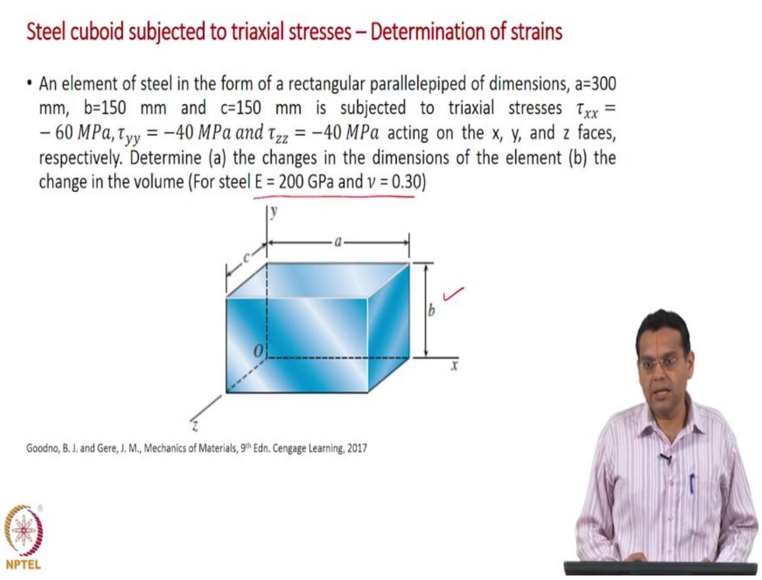
$$G = \frac{E}{2(1+\nu)} = 1.07\ GPa$$

We can also find out the Bulk modulus, we know

$$K = \frac{E}{3(1-2\nu)} = 5\ GPa$$

A very simple example based on experimentally applied forces expressed as per unit area which are the stresses here and the experimentally measured strains we are able to determine the material properties in fact, all the material properties Young's modulus, Poisson ratio, Rigidity modulus and Bulk modulus.

Example: (Refer Slide Time: 11:23)



So, next example is for determination of strains. So, we have an element of steel the form of a rectangular parallelepiped of dimensions 300, 150, 150 mm that is what is shown here a is 300 and b and c are 150 mm. It is subjected to triaxial stress as we had discussed earlier and here again all the stresses are compressive, $\tau_{xx} = -60\ MPa$, $\tau_{yy} = -40\ MPa$, $\tau_{zz} = -40\ MPa$ and they are acting on the x, y and the z faces and what does it we have to determine?

Determine the changes in the dimensions of the element once we find a strain then you can find the change in dimension and also find the change in volume. To proceed further we need to be given the properties, two properties are given any two properties can be given what is usually measured and reported is the Young's modulus very well known property to all of us and Poisson ratio. That is why typically you will see Young's modulus and Poisson ratio being given as the data to us so $E = 200\ GPa$ and $\nu = 0.30$

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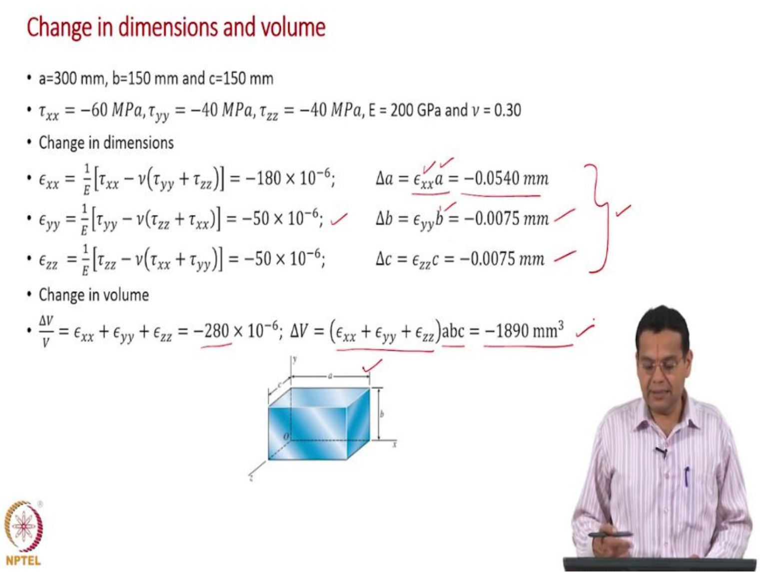


So, let us write down all the data given in the question first the dimensions of the block,

$$a = 300\ mm,\ b = 150\ mm,\ and\ c = 150\ mm$$

Next, the normal stresses and the material properties,

$$\tau_{xx} = -60\ MPa,\ \tau_{yy} = -40\ MPa,\ \tau_{zz} = -40\ MPa$$

$$E = 200\ GPa; \qquad \nu = 0.30$$

To find out the change in dimensions, we will have to find out the normal strains. So,

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right]$$

So, all the values there available as I told you these examples are not conceptually difficult simple substitution examples. So, simply substitute we will get

$$\varepsilon_{xx} = -180x10^{-6}$$

Of course, all the stresses are compressive we have a contraction in the dimension and to find out the decrease in the length we will have to multiply ε_{xx} with the dimension. So, the dimension along x direction is a so,

$$\Delta a = a\varepsilon_{xx} = -0.0540\ mm$$

So, similarly let us do for the y. So, write the expression for the normal strain along y direction simple substitution will give you the normal strain along y direction

$$\varepsilon_{yy} = \frac{1}{E}\left[\tau_{yy} - \nu\left(\tau_{xx} + \tau_{zz}\right)\right] = -50x10^{-6}$$

And,

$$\Delta b = b\varepsilon_{yy} = -0.0075\ mm$$

And similarly,

$$\varepsilon_{zz} = \frac{1}{E}\left[\tau_{zz} - \nu\left(\tau_{yy} + \tau_{xx}\right)\right] = -50x10^{-6}$$

$$\Delta c = c\varepsilon_{zz} = -0.0075\ mm$$

So, these are much smaller than our usual scale dimension the least count of a usual ruler scale to find out the change in volume we will have to find out the fractional change in volume, which is some of all the normal strain.

$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

So, we have found out all the three normal strains when you sum up of course,

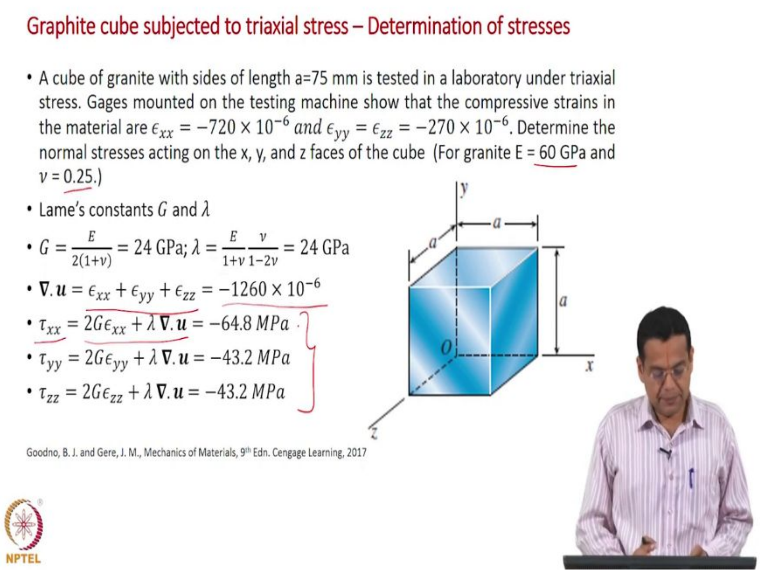
$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -280x10^{-6}$$

Now, to find out the change in volume we will have to multiply this fractional change in volume or the volumetric strain with the original volume which is

$$\Delta V = (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\, V = -1890\ mm^3$$

So, what does it mean? If you subject this cuboid to these normal stresses they are compressive these are the decrease in dimensions along the x, y, z direction and this is the decrease in volume. Remember when we started off we said we are going to look at solids which are not perfectly rigid they are deformable solids and we have an example here when it is subject to this body to these forces a force per area usually you may not observe any change, but there is decreased in dimensions along the 3 direction and a corresponding decrease in volume of the whole object. That is why we have discussed deformable solid bodies not if you are discussing a rigid body, then these dimensions will not at all change that of course, is an ideal condition more realistic condition is where there is change in length and volume as well.

Example: (Refer Slide Time: 17:47)



The last example where we are going to do, remind the stresses. So, a cube of granite, but sides of length 75 mm is tested in a laboratory under triaxial stress gages mounted on the testing machine show that the compressive strains in the material are $\varepsilon_{xx} = -720x10^{-6}$ and $\varepsilon_{yy} = \varepsilon_{zz} = -270x10^{-6}$ the normal strain along the x, y, z axis are shown. These gages are called strain gages they can give us the strain they can be used to measure this strain that is why we said strain is measurable ok, determine the normal stresses on the x, y and z faces of the cube. So, once again we need to be given the material properties $E = 60\ GPa$ and $\nu = 0.25$.

Solution:

So let us find out the Lame's constants G and λ. So,

$$G = \frac{E}{2(1+\nu)} = 24\ GPa$$

$$\lambda = \frac{E}{1+\nu}\frac{\nu}{(1-2\nu)} = 24\ GPa$$

So, G is 24 GPa, λ is also 24 GPa. The expression for stresses involved the divergence of displacement field which is some of the three normal strains and so,

$$\Delta.u = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -1260x10^{-6}$$

We are given the values of the normal strains. So, simple substitution we will give you this value for the sum of normal strain which is same as divergence of displacement field.

So, once I got these values simple substitution we expressed the normal stress in terms of the normal strains. So, simple substitution you found out the value of G found out the value of λ so, we will get

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda\nabla.u = -64.8\ MPa$$

And similarly for y,

$$\tau_{yy} = 2G\varepsilon_{yy} + \lambda\nabla.u = -43.2\ MPa$$

And similarly for z,

$$\tau_{zz} = 2G\varepsilon_{zz} + \lambda\nabla.u = -43.2\ MPa$$

Of course, all of them are compressive and remember the unit is mega Pascal. Usually the stresses as we have seen in all the cases are in terms of mega Pascal the property E, G etcetera are in terms of giga Pascal.

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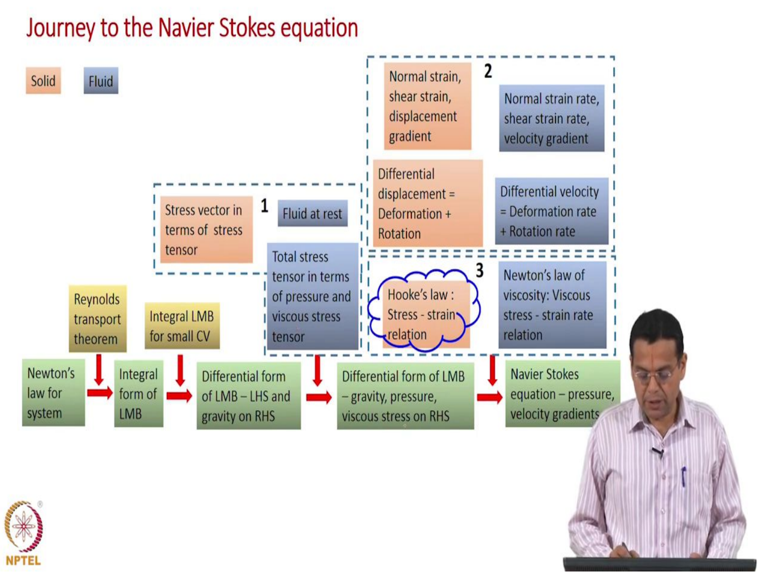

To summarize what we have done is discussed Hooke's law which is the stress strain relationship. Why did we discuss that? We want to express this viscous stress in terms of the velocity gradients or the strain rate, which is the Newton's law of viscosity. But to understand that we took a diversion to solid mechanics the third time in the last time and discussed the stress strain relationship so that you can extend that to stress strain rate relationship.

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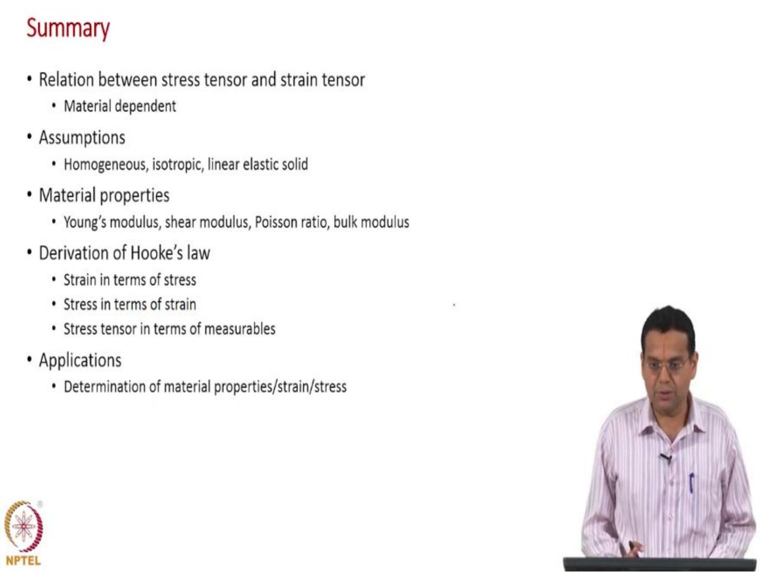

And in terms of summary we discussed the relationship between stress tensor and strain tensor that was an objective or the component of stress tensor and components of strain tensor now we can understand that statement much better. But that relationship is material dependent that is where we started off ok, because it is material dependent been had to discuss the assumptions.

The assumptions are homogeneous, isotropic, linear elastic solid and we also discuss different material properties namely Young's modulus, shear modulus, Poisson ratio, Bulk modulus. Then we derived the Hooke's law, strain in terms of stress, then stress in terms of strain and what is it we achieved expressed stress tensor are the components of stress tensor in terms of measurable which are the strain or still more in terms of the displacement gradients ok. That is a major I would say objective and achievement what we have done here expressing immeasurable in terms of measurable quantities.

Also finally, looked at some applications very simple applications in terms of determination of material properties, strain, stress same sort of relationship, something is given, something is not given and we found out what is not known to us.