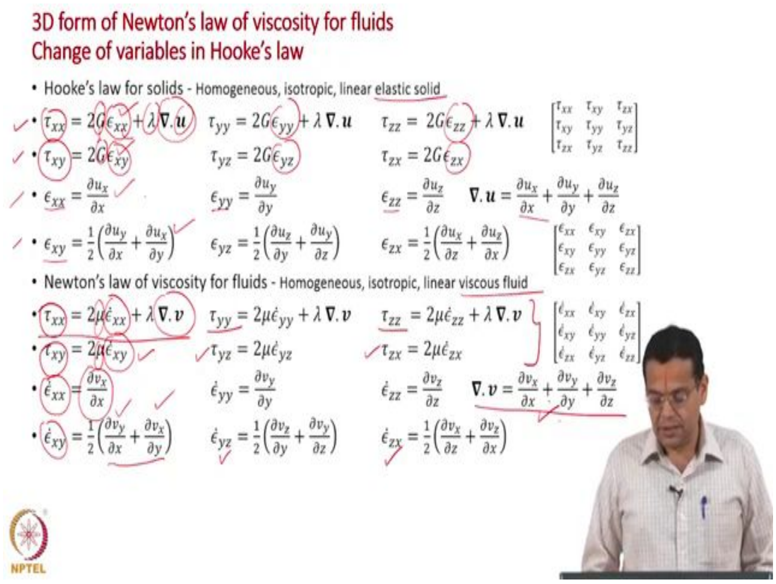
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 78
Newton's Law of Viscosity: 3D Form

(Refer Slide Time: 00:13)



We have recalled the difference between solids and fluids, strain for solids strain rate for fluids. And, we listed the assumptions for the stress strain rate relationship and we looked at the one dimensional form of Newton's law of viscosity and simple application. With all this background, now we can almost write down the 3D form of Newton's law of viscosity for fluids.

Especially, having understood the difference between solids and fluids in terms of strain, strain rate we will be able to do that very easily. So, let us do that. So, let us start with writing the Hook's law for solids moment we say Hook's law, the assumptions follow immediately homogeneous, isotropic, linear elastic solid.

So, let us write down all the six relationships expressing stress in terms of strain

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda \nabla .u \qquad \tau_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yy} = 2G\varepsilon_{yy} + \lambda \nabla .u \qquad \tau_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{zz} = 2G\varepsilon_{zz} + \lambda \nabla . u \qquad \tau_{zx} = 2G\varepsilon_{zx}$$

Remember, we expressed Hook's law both in terms of strain in terms of stress and stress in terms of strain, what is relevant to us and what is important to us is stress in terms of strain. Because, the Newton's law of viscosity we want is stress in terms of strain rate. So, that we can substitute in the linear momentum balance, so, that is why we take the Hook's law which expresses stress in terms of strain.

Now, let us write the other expressions also for the components of strain tensor. The expressions for the components of strain tensor in terms of displacement gradients

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}; \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right); \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right); \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

And,

$$\nabla . u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

The first two lines expresses in terms stress in terms of the strain, but we have to express finally, in terms of the displacement gradients, because they are the measurable ones.

Now, let us write down the Newton's law of viscosity. We did a small derivation to arrive at the Hook's law of solids, but for the case of Newton's law of viscosity having understood the difference between solids and fluids I would say analogy between solids and fluids, we know what variables to be replaced. So, let us write down the Newton's law of viscosity for fluids.

So, moment we say Newton's law of viscosity assumptions follow, homogeneous, isotropic, linear viscous fluid. In the case of solids it is elastic solid, in the case of fluids; it is viscous fluid. So, let us write down the expression analogously for Newton's law of viscosity.

$$\tau_{xx} = 2\mu\dot{\varepsilon}_{xx} + \lambda \nabla . v \qquad \tau_{xy} = 2\mu\dot{\varepsilon}_{xy}$$

$$\tau_{yy} = 2\mu\dot{\varepsilon}_{yy} + \lambda \nabla . v \qquad \tau_{yz} = 2\mu\dot{\varepsilon}_{yz}$$

$$\tau_{zz} = 2\mu\dot{\varepsilon}_{zz} + \lambda \nabla . v \qquad \tau_{zx} = 2\mu\dot{\varepsilon}_{zx}$$

So, these expressions are for the viscous stress, total stress if we recall it has two components hydrostatic stress and viscous stress. These are the expressions for the viscous stress; we will have to just add the pressure part as hydrostatic part.

So, now left hand side is viscous stress components or viscous stress tensor and right hand side what are the replacements we have done, we know that for solids it is strain for fluids it is strain rate, because strain rate plays the same role for fluids as strain plays for solids. So, replace the normal strain with the normal strain rate and then what else I have replaced we know that so, the displacement field, we have the velocity field here.

So, this term represents some of the normal strains instead of that here we have $\nabla .v$ which is divergence of velocity and it represents some of the normal strain rates, or some of the diagonal components of the strain rate tensor. What other changes are made? The property G is for solids the analogous property for fluids is viscosity, μ.

So, G relates shear stress and shear strain, μ relates shear stress and shear strain rate. So, that is why we replace G with μ. So, if you make these replacements you get the equations for the Newton's law of viscosity for fluids. So, now, what we have done is expressed the normal viscous stresses and viscous shear stresses in terms of the strain rates.

Now, these set of six equations constitute the Newton's law of viscosity for fluids in the three dimensional case. For one dimensional case, we looked at only the one equation. Now, we got a three dimensional case so, we got six equations relating six components of viscous stress tensor to the six components of strain rate tensor.

So, now as I told you have to express the components of strain tensor in terms of displacement gradient that is what we have done for solids. Similarly, here we have to express the components of strain rate tensor, in terms of the velocity gradient.

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x}; \qquad \dot{\varepsilon}_{yy} = \frac{\partial v_y}{\partial y}; \qquad \dot{\varepsilon}_{zz} = \frac{\partial v_z}{\partial z}$$

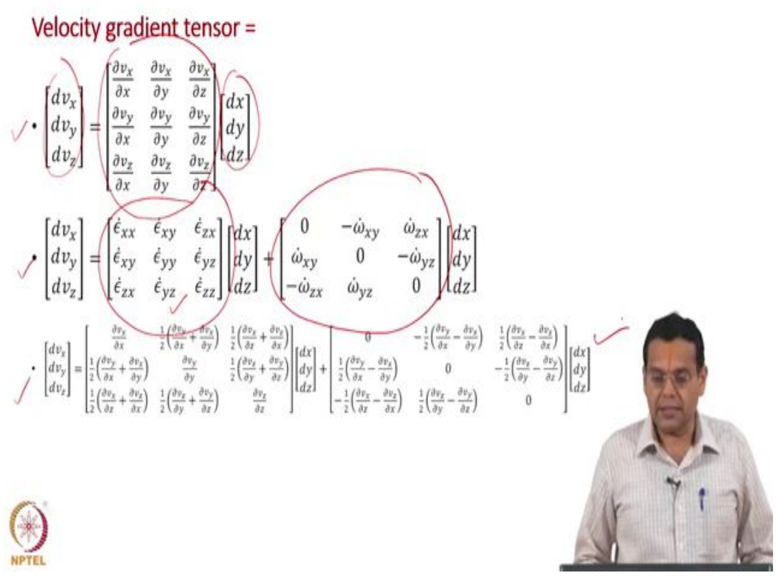
$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \qquad \dot{\varepsilon}_{yz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \qquad \dot{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

And,

$$\nabla .v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Now, like we did for solids, let us connect this with what we have discussed earlier.

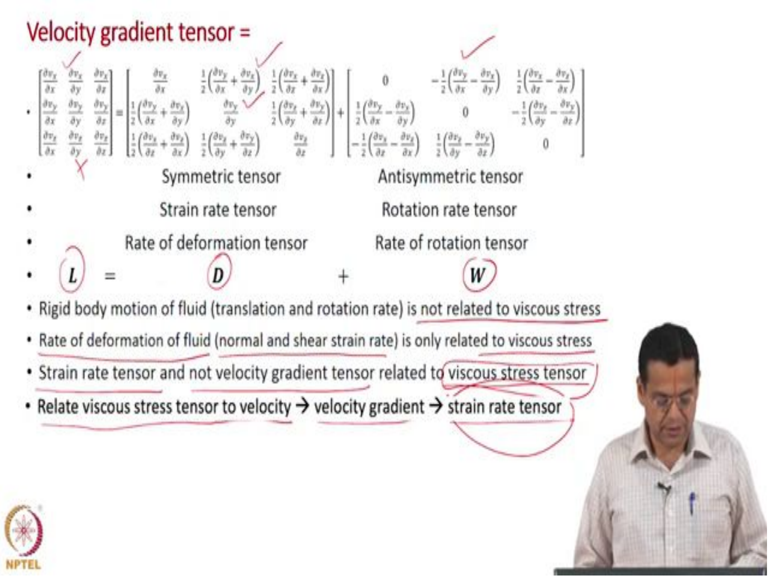
(Refer Slide Time: 10:06)



So, we have a recall slide here, what we discussed earlier was we derived or we I would say wrote down the expression relating d_{v_x}, d_{v_y}, d_{v_z} and dx, dy, dz in 2 different ways; one by the first equation. Secondly, either this or this equation.

First was in terms of one velocity gradient tensor, second case we wrote it as some of in second case we wrote in terms of the strain rate tensor and the rotation rate tensor ok. So, based on these two different ways of expressing the relationship between d_{v_x}, d_{v_y}, d_{v_z} and dx, dy, dz. We concluded that this velocity gradient tensor is a sum of strain rate tensor and rotation rate tensor. Of course, here we have expressed in terms of the velocity gradients.

(Refer Slide Time: 11:12)

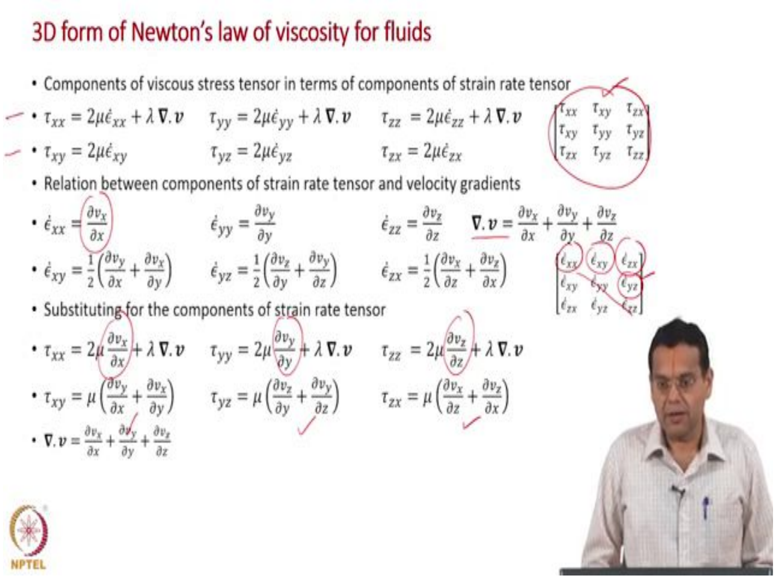


So, once again I recall slide expressing the velocity gradient tensor, in terms of a strain rate tensor and rotation rate tensor. And, in terms of a nomenclature L (Velocity gradient tensor) and then D (Strain rate tensor), and then W (Rotation rate tensor).

Now, we also discussed that and this was in fact, once again the last slide, when we discussed strain rate for fluids. We said rigid body motion of fluid namely translation of fluid, rigid body rotation of fluid is not related to viscous stress, rate of deformation of fluid which includes normal and shear strain rate, that only is related to the viscous stress.

And, strain rate tensor not velocity gradient tensor is related to viscous stress tensor. So, you said relate viscous stress tensor to velocity more precisely velocity gradient still more precisely components of strain rate tensor, and that is what we have done today. We have related the components of viscous stress tensor, to the components of strain rate tensor.

(Refer Slide Time: 12:39)



So, let us look at it again in a more little more detailed way components of viscous stress tensor in terms of components of strain rate tensor. So, we are writing expression for the six independent components of stress tensor here. And, we are writing that in terms of the six independence components of strain rate tensor, that is what we set forth as our objective and that is what we are doing in today's class.

So, let us write down the expression these expressions are same as what we have seen two slides back just rewriting them. So, that now you can look at them as relationships connecting stress viscous stress tensor and strain rate tensor.

$$\tau_{xx} = 2\mu\dot{\varepsilon}_{xx} + \lambda\nabla.v \qquad \tau_{xy} = 2\mu\dot{\varepsilon}_{xy}$$

$$\tau_{yy} = 2\mu\dot{\varepsilon}_{yy} + \lambda\nabla.v \qquad \tau_{yz} = 2\mu\dot{\varepsilon}_{yz}$$

$$\tau_{zz} = 2\mu\dot{\varepsilon}_{zz} + \lambda\nabla.v \qquad \tau_{zx} = 2\mu\dot{\varepsilon}_{zx}$$

They are all in terms of the components of strain rate tensor. So, now, we will have to express the components of strain rate tensor in terms of velocity gradient.

$$\dot{\varepsilon}_{xx} = \frac{\partial v_x}{\partial x}; \qquad \dot{\varepsilon}_{yy} = \frac{\partial v_y}{\partial y}; \qquad \dot{\varepsilon}_{zz} = \frac{\partial v_z}{\partial z}$$

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad \dot{\varepsilon}_{yz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \quad \dot{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

And,

$$\nabla . v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

So, now we will substitute these expressions in the six equations ok. And, express the viscous stresses in terms of the velocity gradients.

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} + \lambda \nabla . v \qquad \tau_{xy} = \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)$$

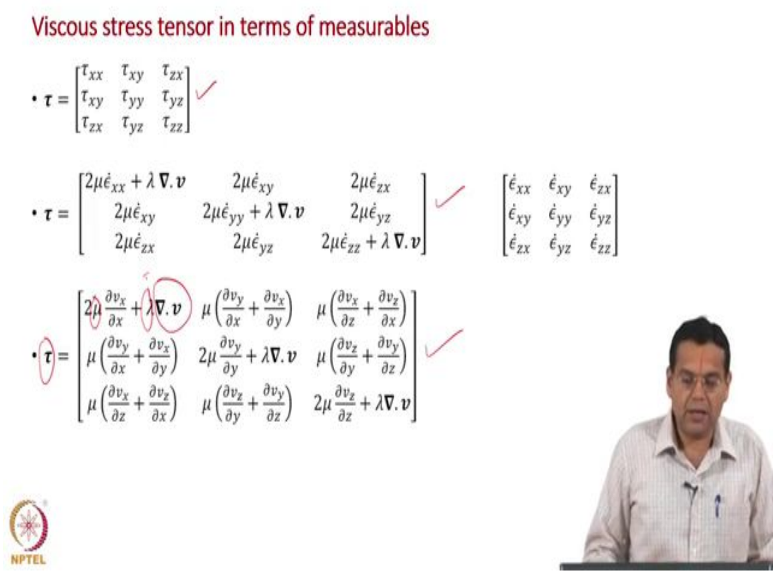
$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} + \lambda \nabla . v \qquad \tau_{yz} = \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} + \lambda \nabla . v \qquad \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

Of course, divergence of v is also expressed in terms of the velocity gradients.

$$\nabla . v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

(Refer Slide Time: 15:36)



So let us put them all compactly in form of a tensor notation or a matrix notation

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{zx}\ \tau_{xy}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{yz}\ \tau_{zz} \right]$$

This is a viscous stress tensor, has good physical significance, but when we introduced it was a not measurable. Now, moment we expressed in terms of the components of strain rate tensor,

$$\tau = \left[2\mu\dot{\varepsilon}_{xx} + \lambda \nabla . v\ 2\mu\dot{\varepsilon}_{xy}\ 2\mu\dot{\varepsilon}_{zx}\ 2\mu\dot{\varepsilon}_{xy}\ 2\mu\dot{\varepsilon}_{yy} + \lambda \nabla . v\ 2\mu\dot{\varepsilon}_{yz}\ 2\mu\dot{\varepsilon}_{zx}\ 2\mu\dot{\varepsilon}_{yz}\ 2\mu\dot{\varepsilon}_{zz} + \lambda \nabla . v \right]$$

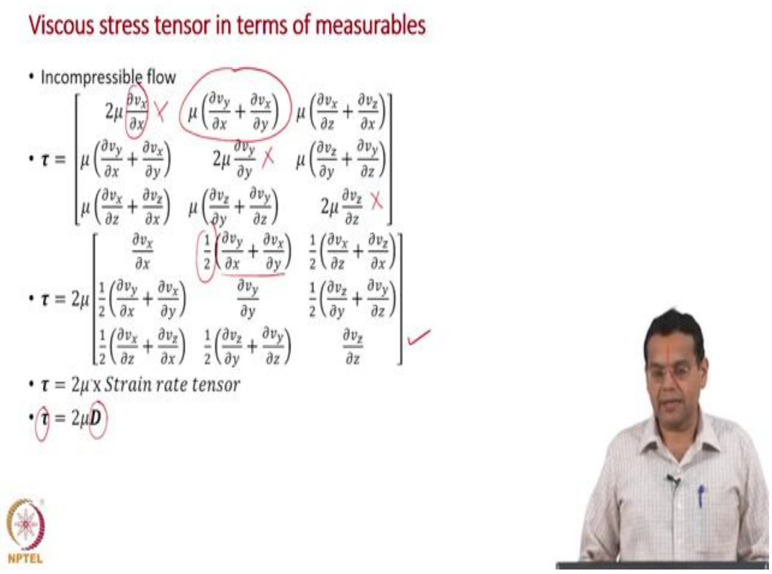
And further expressed in terms of velocity gradients it becomes measurable. Because, velocity is measurable velocity gradients are measurable.

$$\tau = \Big[2\mu \frac{\partial v_x}{\partial x} + \lambda \nabla . v \;\; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \;\; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; 2\mu \frac{\partial v_y}{\partial y} + \lambda \nabla . v \;\; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \;\; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

And, hence this equation, which expresses τ in terms of velocity gradients makes τ indirectly measurable not directly measurable, but now we can calculate we can calculate the viscous stresses. Of course, it has 2 properties namely viscosity and then λ.

Now, of course, nothing more discussed in this slide, because whatever we discussed in terms of components has been put here in terms of some tensor notation. The viscous stress tensor in terms of components of strain rate tensor, in terms of velocity gradients or in fact, in terms of components of the strain rate tensor, expressed in terms of velocity gradients ok. Now, as I told you there are 2 constants mu and lambda. Now, what we will see in the next slide is how does this get simplified for the case of incompressible flow? We have seen that for the case of incompressible flow $\nabla . v = 0$.

(Refer Slide Time: 17:34)



So, you what you see here? In the case of incompressible flow tau is equal to of course, there is no change in the off diagonal elements, we had $\lambda \nabla . v$ in the diagonal elements. So, that does not appear here.

$$\tau = \Big[2\mu \frac{\partial v_x}{\partial x} \;\; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \;\; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; 2\mu \frac{\partial v_y}{\partial y} \;\; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \;\; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \;\; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

Also like to emphasize that, when the flow is incompressible, there is only one material property namely viscosity. We set for a homogeneous, isotropic, linear viscous fluid there are two material properties, which were μ and λ.

Now, if you make one more assumption incompressible flow, then λ also is not required you require only one material property namely viscosity. And, why are we simplifying for incompressible flow, most of the flows are within our engineering curriculum can be simplify to incompressible flow. So, more practical flow I would say.

Let us take out 2 μ and you get as this tensor.

$$\tau = 2\mu \left[\frac{\partial v_x}{\partial x} \;\; \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \;\; \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) \;\; \frac{\partial v_y}{\partial y} \;\; \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \;\; \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \;\; \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \;\; \frac{\partial v_z}{\partial z} \right]$$

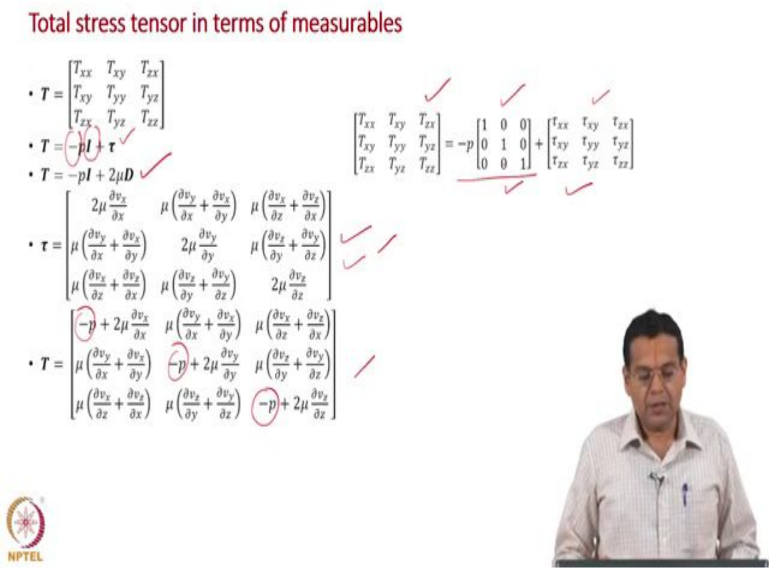
Now this is the strain rate tensor. And, which we denoted as capital D, that is why I also recall the nomenclature.

$$\tau = 2\mu \, X \, \textit{Strain rate tensor}$$

$$\tau = 2\mu D$$

So, viscous stress tensor expressed in terms of the strain rate tensor. And, this is what we exactly we wanted to do express viscous stress tensor in terms of strain rate tensor.

(Refer Slide Time: 20:15)



Now, having expressed the viscous stress tensor in terms of measurables, we can also extend that for total stress tensor, total stress tensor in terms of components is given by

$$T = \left[T_{xx} \; T_{xy} \; T_{zx} \; T_{xy} \; T_{yy} \; T_{yz} \; T_{zx} \; T_{yz} \; T_{zz} \right]$$

And, of course, once again at this point it is not measurable, then we express the total stress tensor in terms of a hydrostatic stress tensor and viscous stress tensor. In terms of components this is how it looks like.

$$T =- pI + \tau$$

$$\left[T_{xx} \; T_{xy} \; T_{zx} \; T_{xy} \; T_{yy} \; T_{yz} \; T_{zx} \; T_{yz} \; T_{zz} \right] =- p \left[1 \; 0 \; 0 \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 \right] + \left[\tau_{xx} \; \tau_{xy} \; \tau_{zx} \; \tau_{xy} \; \tau_{yy} \; \tau_{yz} \; \tau_{zx} \; \tau_{yz} \; \tau_{zz} \right]$$

Left hand side you have components of total stress tensor, the hydrostatic stress how do you express you have minus p into identity tensor or unit matrix in simple language and that is what is written here.

And, then this tau is represented here as components of the viscous stress tensor at this moment you express this way the first term is measurable, because pressure is measurable, still second term is not measurable and that is what we have done in today's class.

$$T =- pI + 2\mu D$$

So, moment you express in terms of $2\mu D$, D is the strain rate tensor that is measurable. So, at moment you write in this way now total stress tensor has been expressed in terms of measurables that is the key.

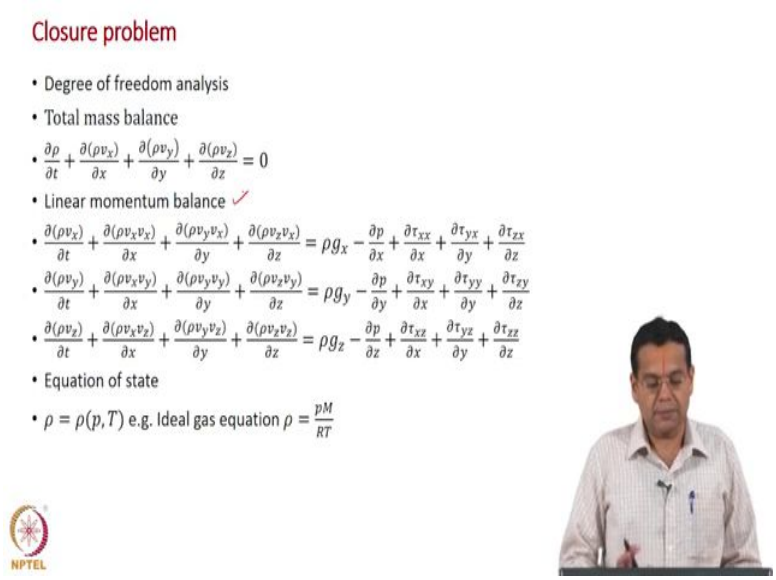
$$\tau = \Big[2\mu \frac{\partial v_x}{\partial x} \; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \; 2\mu \frac{\partial v_y}{\partial y} \; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

This expression we have seen for tau in terms of the velocity gradients and so, let us substitute very simple just $-$p gets added to the diagonal elements that all happens. Because, hydrostatic stress, remember we discussed for a fluid under static condition, there are no shear stresses, there are only normal stresses and that normal stresses pressure and is compress compressive in nature hence -p.

$$T = \Big[-p + 2\mu \frac{\partial v_x}{\partial x} \; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \; \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \; -p + 2\mu \frac{\partial v_y}{\partial y} \; \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \; \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \; \mu$$

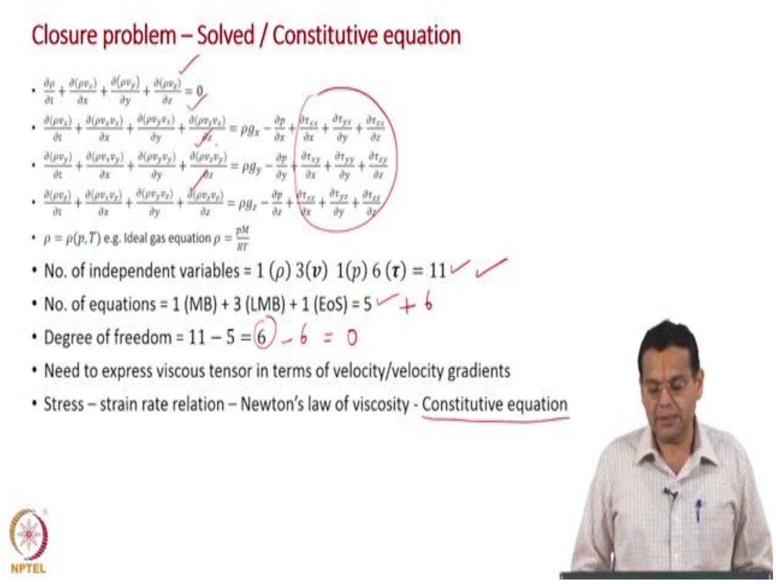
When you add, $-$p gets added only to the diagonal elements, because you have only normal stresses under fluids under static condition for a fluid under static condition. So, which is the pressure and of course, $-p$, because it is compressive. So, that is the expression for total stress tensor in terms of measurables.

(Refer Slide Time: 22:55)



Now, once again to relate what we have discussed long time back, we have found the answer for a long standing problem namely the closure problem. Let me recall after deriving the linear momentum balance with the viscous stresses on the right hand side, we did a degree of freedom analysis. What did we do? We write down the differential form of total mass balance and of course, the 3 linear momentum balance equations and the equation of state.

(Refer Slide Time: 23:30)



And, then we counted

the number of independent variables = 11

the number of equations = 5 and we said

Degree of freedom = 11 – 5 = 6

So, 6 variables are unknown, and what are the 6 variables they are the components of the viscous stress tensor. And, then we said to close the problem, what does closing the problem means expressing these unknown variables in terms of the known variables, which are existing in the equation already.

We said we need to express viscous stress tensor in terms of velocity gradients and that is what exactly we have done today, which means that we have closed the problem we have solved the closure problem. We have closed the set of equations.

Now, if you do a degree of freedom analysis. You will have,

The number of independent variables = 11 and then

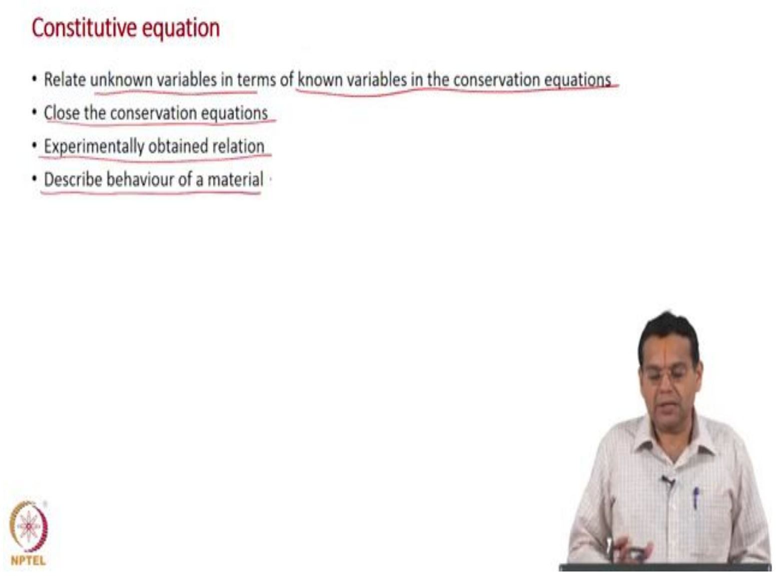
the number of equations = 5 + 6 = 11 (6 more equations for every component of τ)

Degree of freedom = 11 – 11 = 0

That is what we mean by closing a set of equations. So, we have solved the closure problem, which has been therefore, several classes.

Now, like to introduce one nomenclature here called constitutive equation. The stress strain rate relationship, which in our case is the Newton's law of viscosity is a example for a constitutive equation. What we have derived here are all conservation equations. We conservation of mass, conservation of momentum etcetera, but when you write there are some variables, there are some let us say unknowns and we have to give expressions for those unknowns, in terms of already known variables and that relationship is the constitutive relationship.

(Refer Slide Time: 25:48)



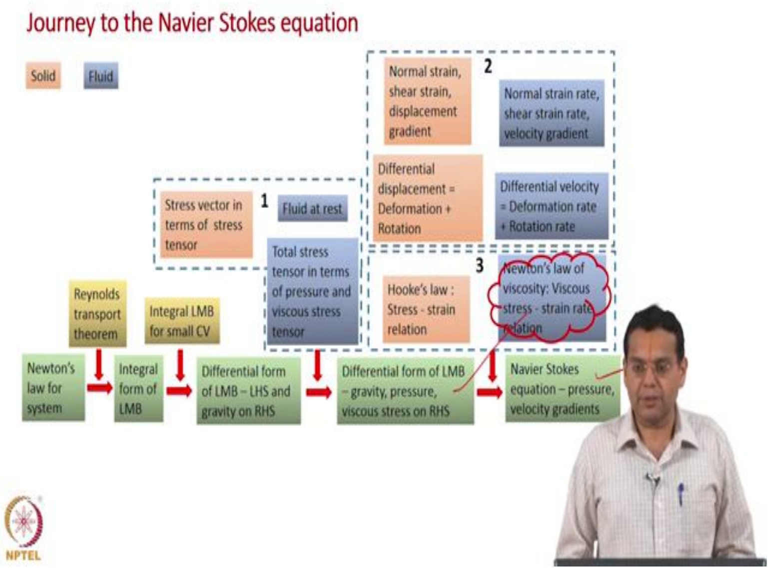
So, the constitutive equation relate unknown variables in terms of known variables in the conservation equations. In this particular case τ or the unknown's variables, the velocities are the known variables, we express τ in terms of velocity.

What do they do? They help to close the conservation equations; otherwise they are not closed. Close meaning we have seen reducing the degree of freedom to 0. Otherwise, we have degree of freedom more than 0. And, these constitutive equations are mostly or always experimentally obtained, we have seen that a Newton's law of viscosity is experimentally observed relationship. And, then μ is a proportionally factor etcetera.

And, correspondingly they describe behavior of a material. In this particular case, what behavior in this particular case they describe the mechanical behavior of fluids, stress strain rate. Later on we will see that we will come across few more constitutive equations, one in energy balance, one in species balance, they help to describe the thermal behavior or diffusion behavior etcetera.

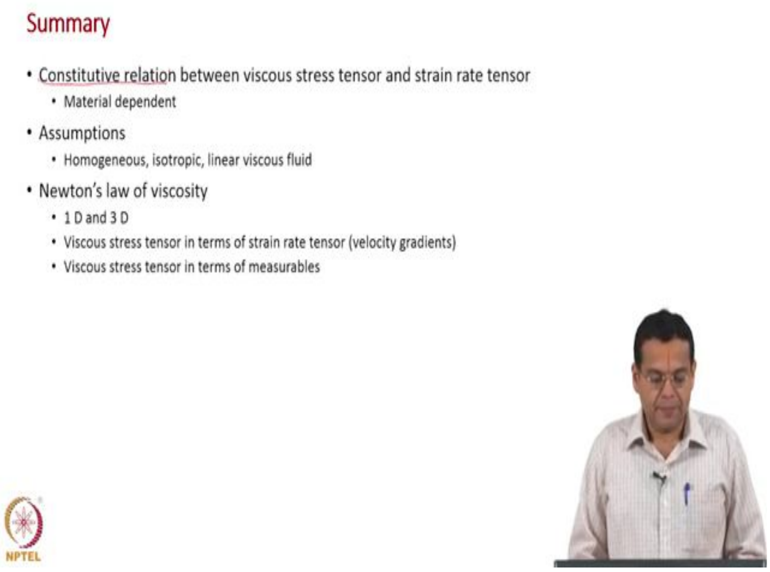
So, constitutive equation is a very general name. They are empirical; they help to close the set of equations. Mostly they describe a behavior, even when you will see in species balance that the rate equation the kinetic equation is also a constitutive equation. So, mostly they represent material behavior, but that could be other constitutive equations also.

(Refer Slide Time: 27:40)



So, in terms of summary of course, we have discussed the Newton's law of viscosity. Almost, we wrote down the Newton's law of viscosity I would say, because we derived the Hook's law relating stress and strain, we will have to just do a few replacement of variables we could get the Newton's law of viscosity relating viscous stress and strain rate. And, so, now, we are ready to substitute this viscous stress strain relationship here and get the Navier strokes equation.

(Refer Slide Time: 28:17)



So, in terms of summary we discussed a constitutive relation. I used a more formal terminology a constitutive relation between viscous stress tensor and strain rate tensor and we emphasize that it is material dependent. And, so, we discussed the assumptions just like for Hook's law; namely, homogeneous, isotropic, linear viscous fluid.

And, we discussed the Newton's law of viscosity for 1 dimensional case and 3 dimensional case, we express the viscous stress tensor in terms of strain rate tensor or velocity gradients that is a Newton's law of viscosity. And, finally we expressed viscous stress tensor in terms of measurables and that is a big thing we have done.