Continuum Mechanics And Transport Phenomena
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Lecture – 79
Navier Stokes Equation

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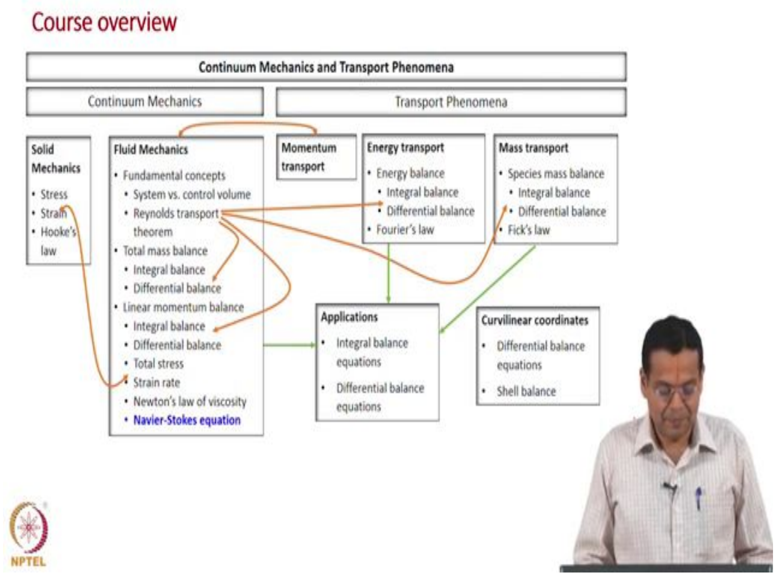

Having understood, total stress, strain rate, Newton's law of viscosity, by going back and forth between solid mechanics and fluid mechanics. Now, we are ready to derive the Navier Stokes equation we have gathered all the background knowledge. So, writing Navier Stokes equation are almost writing the Navier Stokes equation, just 2 steps or 3 steps of derivation and we will be at the Navier Stokes equation.

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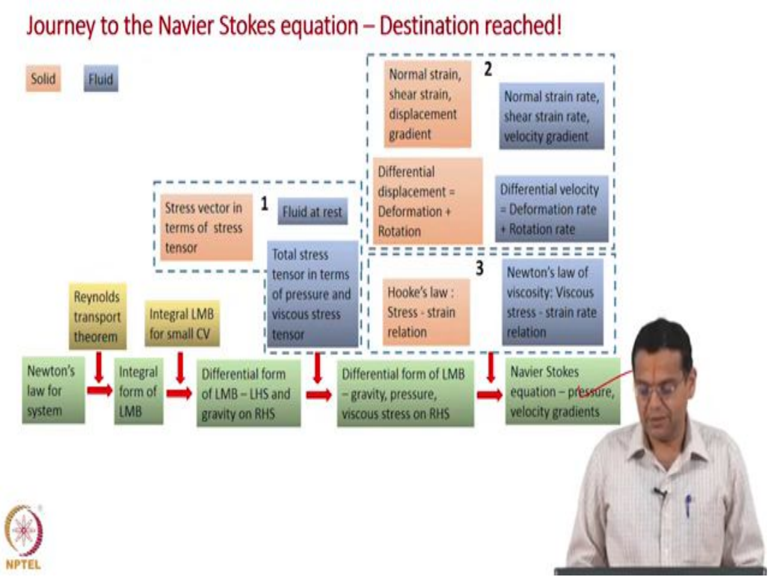

And, and terms of our journey to the Navier Stokes, because very obvious we are at the last station. Namely, writing down the Navier Stokes, deriving the Navier Stokes equation using the stress strain rate relationship just derived. Of course for a change there is no outline reason is there is only one line which is Navier Stokes equation.

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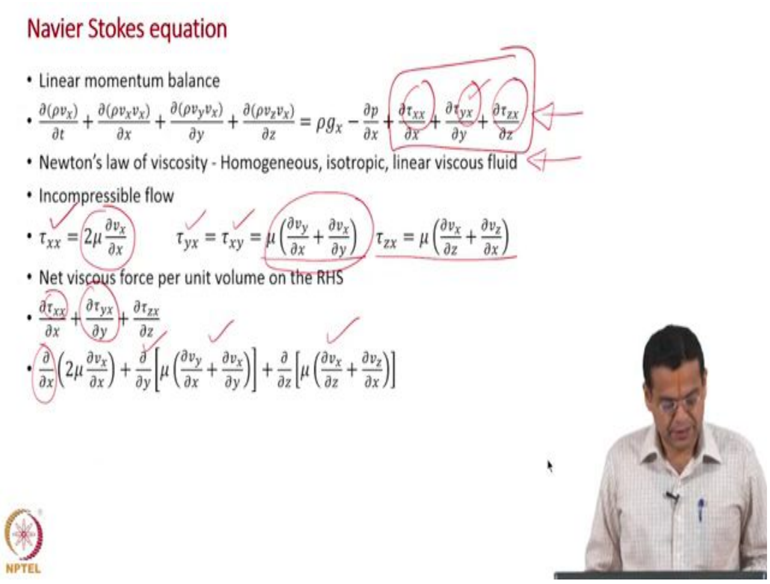

So, let us do that. The starting point is the linear momentum balance equation. So, our starting point is the linear momentum balance equation of course, there are 3 equations one for each direction, let us write down the equation for the x direction.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Just to recall left hand side you have transient term, the convection term, right hand side you have all the forces, the body force because of gravity. And, then you have surface forces namely pressure and then the viscous stresses.

So, now we will have to express this τ_{xx} , τ_{yx} , τ_{zx} in terms of the strain rates or the velocity gradients. So, we will use the Newton's law of viscosity, which is the constitutive equation, when at this stage of linear momentum balance equation, it is an exact theatrical equation. Of course, in terms of body forces we have included only that due to gravity. Other than that there is hardly any assumption of course, the viscous stress tensor is symmetric even that is almost universally valid.

So, in that way, the linear momentum balance equation is the theoretical equation. Moment I go to the next step; so many assumptions have to be made that has to be emphasized. So, moment I substitute Newton's of viscosity, that equation is applicable only for homogeneous, isotropic, linear viscous fluid, that has to be kept in mind.

On top of that we are going to restrict only to incompressible flow, the Newton's law viscosity what we have written there was a term for compressible flow as well, but we are considering only incompressible flow. So, one more assumption added to the above list.

So, now let us write down the Newton's law of viscosity for incompressible flow.

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x}; \quad \tau_{xy} = \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

So, out of six equations we require now only three of them. So, now, what we will do, take the three terms which represent the net viscous force per unit volume in the right hand side. And, then simplify because all other term there is nothing to discuss nothing to simplify also. We will have to take three terms substitute the Newton's law of viscosity and then simplify. Let us do that, they represent net viscous force per unit volume.

So, let us write down them and then substitute.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \frac{\partial}{\partial x}\left(2\mu \frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right]$$

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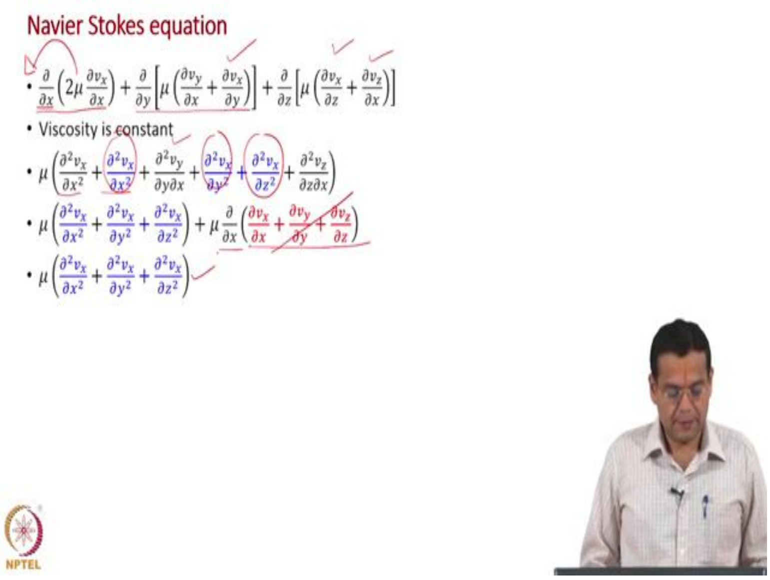

So, let us simplify this equation. So, same equation is being written here.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \frac{\partial}{\partial x}\left(2\mu \frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right]$$

In this equation μ is inside the special derivative. So, this equation can account for spatial variation of viscosity and why should viscosity vary as a function of space, because of probably change in temperature.

But, now we will make one more assumption that viscosity is a constant, so that I can take μ out of the derivative, which means that the equation, which I going to write after this cannot be used, if viscosity is varying as a function of space. So, let us take out μ then let us expand the derivatives.

$$= \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial x}\right)$$

So, now let us collect few terms together and, then from the remaining terms we will take out $\frac{\partial}{\partial x}$ and what is left out are

$$= \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \mu \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

Of course, the second term is the divergence of velocity field and we are considering incompressible flow which mean this is 0.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

So, we are left with only the first three terms alone .

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, the three terms which represented the net viscous force on the right hand side, get simplified to this term.

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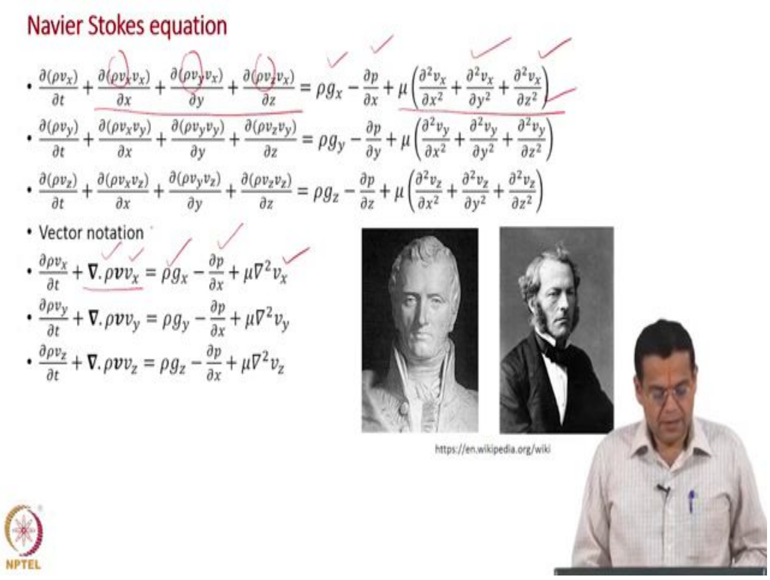

So, this only matter of replacement,

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

The left hand side is same and, right hand side the body force and pressure term is also same, earlier we had the in terms of derivatives of the viscous stresses and now we have simplified them to this term. And, similarly other directions, we know that we had linear momentum balance one for each direction.

$$\frac{\partial(\rho v_y)}{\partial t} + \frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

So, similarly we have the Navier Stokes equation also for one for each direction. And, of course you can put in a vector notation

$$\frac{\partial(\rho v_x)}{\partial t} + \nabla.\rho v v_x = \rho g_x - \frac{\partial p}{\partial x} + \mu \nabla^2 v_x$$

So, that way this equation is more compact, how to understand this? If, we had ρv_x , ρv_y , ρv_z alone, then it would have been divergence of ρv vector, that is what appeared in the continuity equation or differential mass balance equation, we have an additional v_x here. Similarly, we can write in the other directions, y direction and z direction.

$$\frac{\partial(\rho v_y)}{\partial t} + \nabla.\rho v v_y = \rho g_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v_y$$

$$\frac{\partial(\rho v_z)}{\partial t} + \nabla.\rho v v_z = \rho g_z - \frac{\partial p}{\partial z} + \mu \nabla^2 v_z$$

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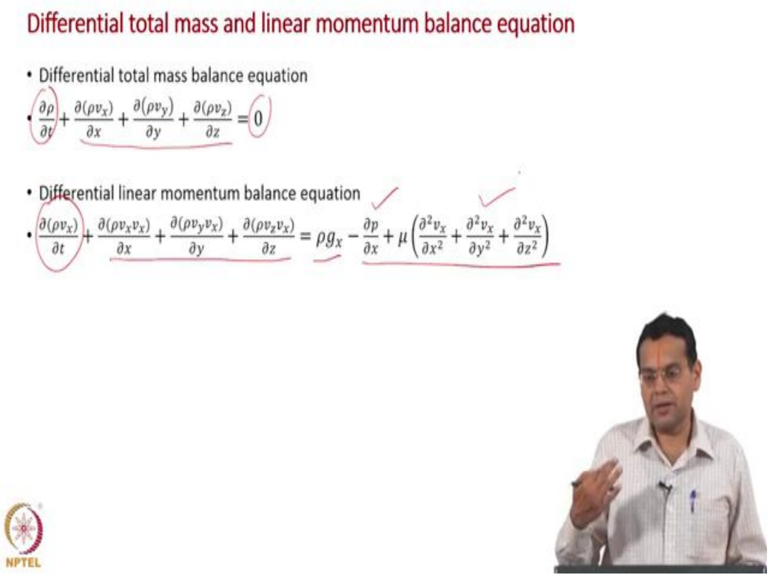

And, now we can compare the differential total mass balance and the linear momentum balance equation,

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

that is a differential total mass balance equation. And, the differential linear momentum balance equation, in terms of the Navier stoke equation is what I have derived just now.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, to compare we have the transient term in both and, we have here the convection term in both. Of course, there the terms where for mass, here it is for momentum, right hand side we had just 0 for the case of mass balance, but for momentum, we have all the forces. The body force and the surface forces; surface force due to pressure and then viscous stresses, and that is the comparison between these two. And, we will build on the slide after we derive differential balance for energy and then species mass also, you know that all equations can be compared.

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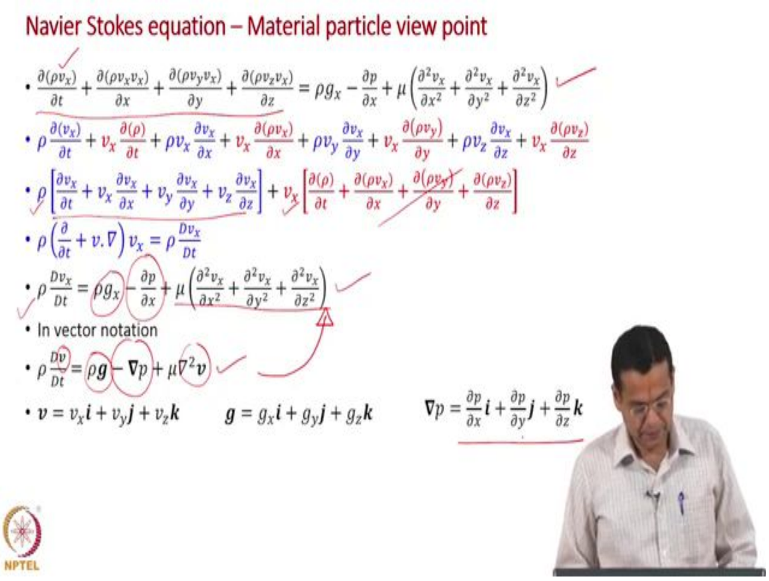

The Navier Stokes equation can also be put in the material particle view point, we have already done this for linear momentum balance,

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

What we had done earlier is. The right hand side when it was in terms of the stresses, for the linear momentum balance we took the left hand side expressed in terms of D/Dt.

We already done that to tell you in short, we just expand we just use product rule and express all the derivatives. And, collect terms in terms of ρ and then v_x. So, left hand side becomes $\rho\frac{Dv_x}{Dt}$ right hand side of course,

$$\rho\frac{Dv_x}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

This is the more popular form of Navier Stokes equation. Though we have derived in this way, the left hand side with $\rho\frac{Dv_x}{Dt}$ is a more popular form. And, in vector notation you can represent still more compactly,

$$\rho \frac{Dv_x}{Dt} = \rho g - \nabla p + \mu \nabla^2 v$$

So, very compact notation of the Navier Stokes equation. Of course, we know

$$v = v_x i + v_y j + v_z k; \qquad g = g_x i + g_y j + g_z k; \qquad \nabla p = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z}$$

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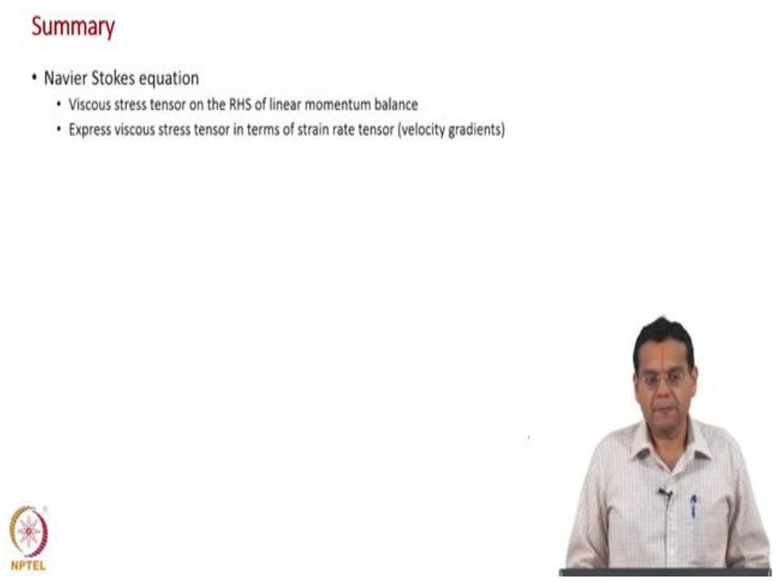

And, to summarize we have derived the Navier stoke equation, as I told you with all the knowledge we have we just few steps to get the Navier Stokes equation, we took the viscous stress tensor terms on the right hand side of the linear momentum balance ok. And, expressed the those components of viscous stress tensor in terms of strain rate tensor or the velocity gradients, moment we do that simple few steps we could get the Navier Stokes equation.

Now, in terms of our journey we have reached the destination and we just discuss the last block here, which says Navier Stokes equation. Now, if you have a long journey after you reach the destination you are happy. And, of course, whomever you people you meet you recall whatever you experienced through the journey ok. It could be several people, we can also see that here we met Reynolds we met somebody called stress, somebody called let us say pressure, Cauchy we had somebody call strain, and then we had Newton of course, and then Navier stokes.

So, we learnt lot of them from each of these people as well or if you make this analogous to different places, may be you went to several places explored several places as you travelled. And, so, you wish to recall all this. And, that is what we will do now, we will also recall quickly our journey, we have repeated this several times I would say almost before every lecture and in between lectures as well.

So, one last time you will go through this slide starting from beginning to the end and at every stage we will have some few recall slides ok, that is how the summary is planned. So, let us start with the Newton's law for system that is where everything began, we applied the Reynolds transport theorem and then we got the integral form of linear momentum balance, and here are the slides.

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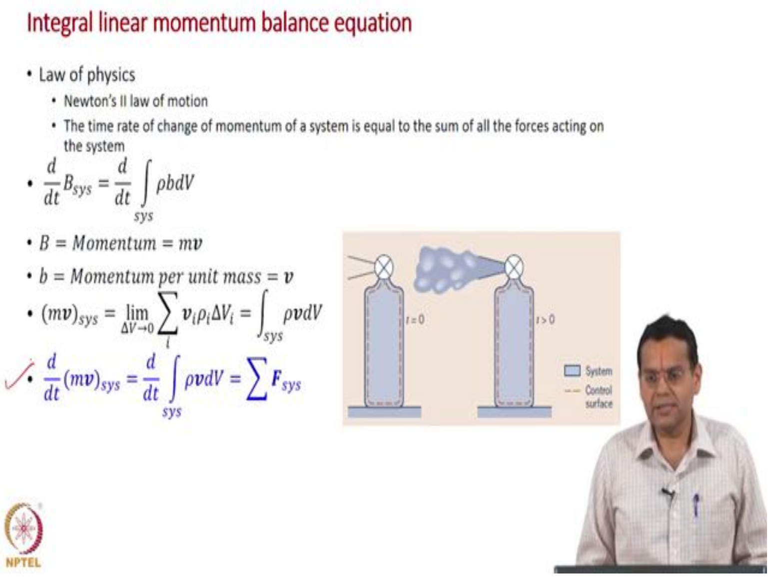

Just only few summary slides from each so, this equation represents the Newton's second law of motion for a system, rate of change of momentum sum of forces acting on the system.

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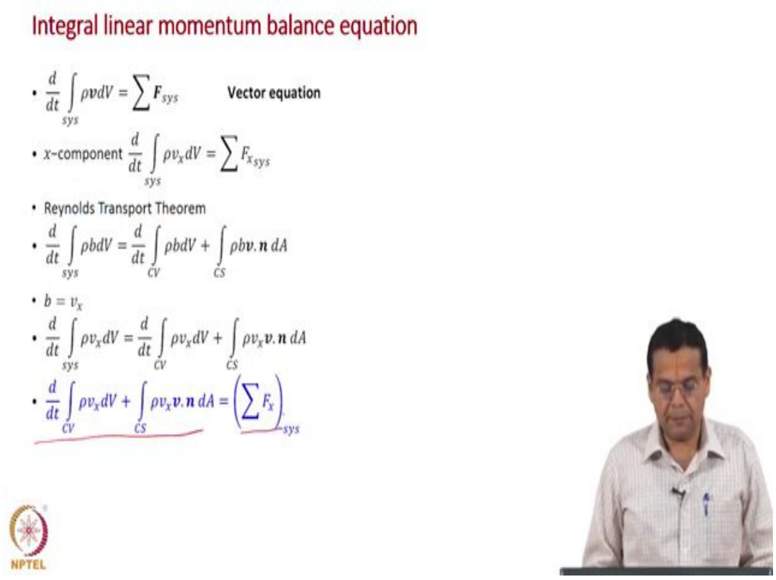

We applied the Reynolds transport theorem. So, the left hand side was expressed in terms of a transient term and a convection term for the control volume, right hand side was still in terms of system we took a coincident system and control volume. So, right hand side became that for the control volume. And, that is the integral form of linear momentum balance equation.

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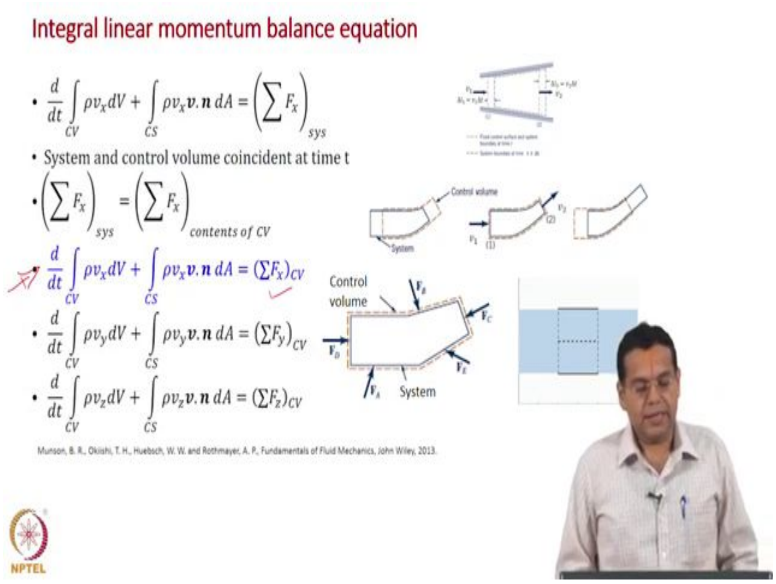

So, we were here. Now, what did we do we took small control volume, applied the integral linear momentum balance for a small control volume. And obtain the differential form of the linear momentum balance, the left hand side and gravity on the right hand side and that is what we will see now.

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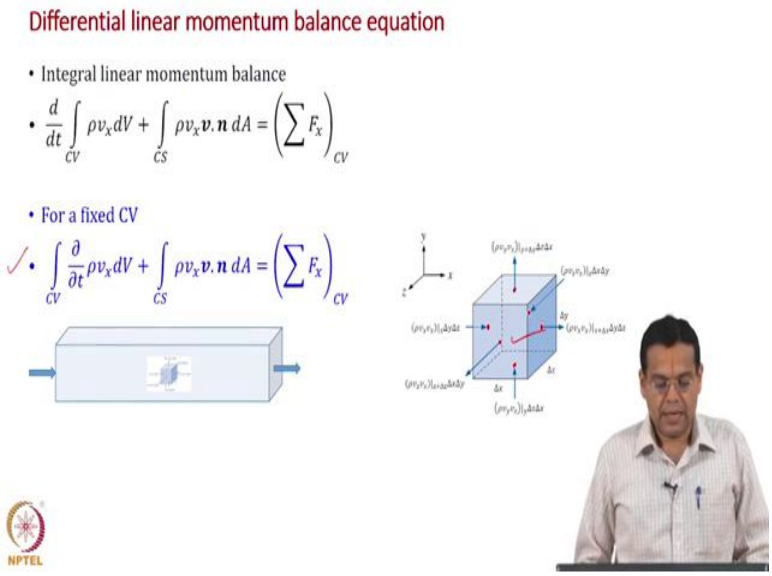

We took the integral form of linear momentum balance for a small fixed control volume.

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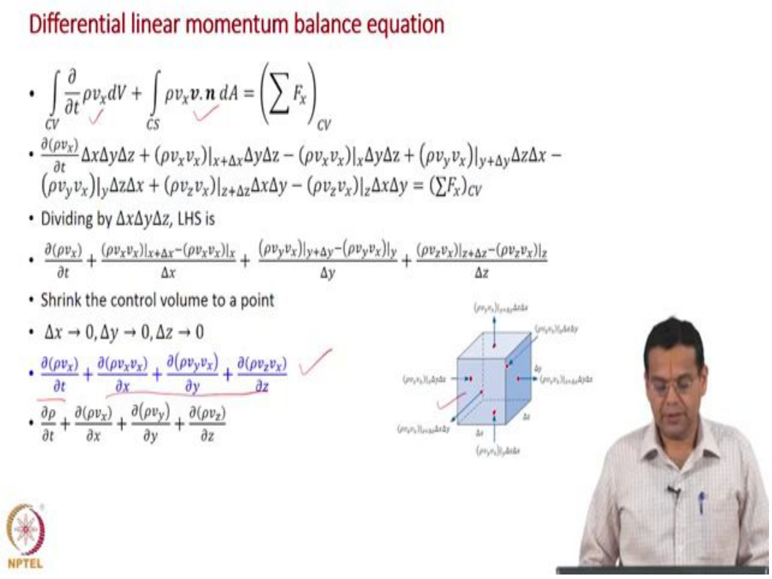

And, then expressed the transient term and the convection term, as applicable to this small control volume, and obtain the left hand side of the linear momentum balance, the transient term and the convection term.

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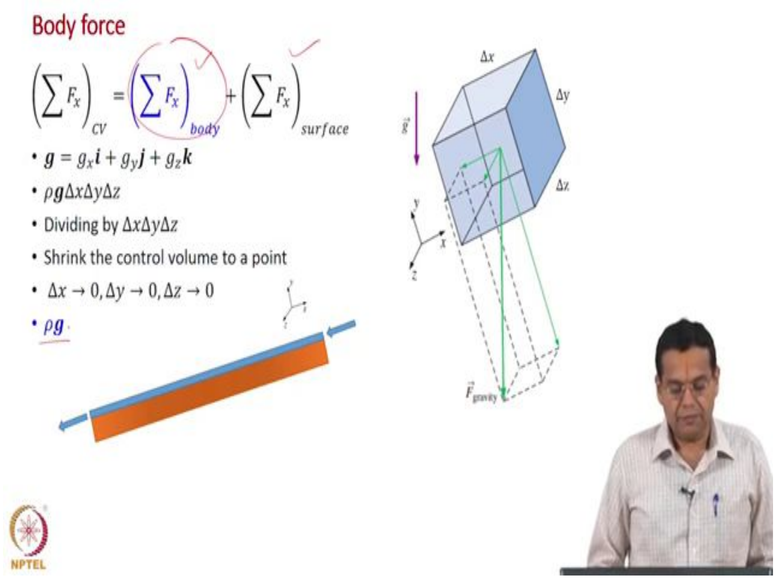

Now, we also on the right hand side we had the body forces and the surface forces at that point of time we discussed only the body forces, that due to only the gravitational effect. And, then we discuss the body forces rho into g vector. So, at end of this we were here, the differential form of linear momentum balance the left hand side and gravity on right hand side. We need to understand the surface forces, we need to express them, understand that we took our first diversion to solid mechanics; we discuss this block stress vector and stress tensor.

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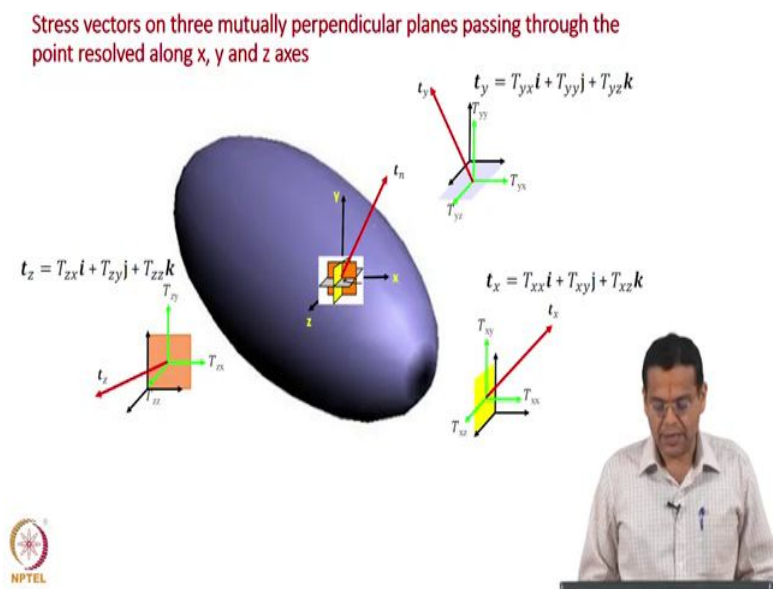

So, few slides from there, so the stress vector and the components of the stress tensor.

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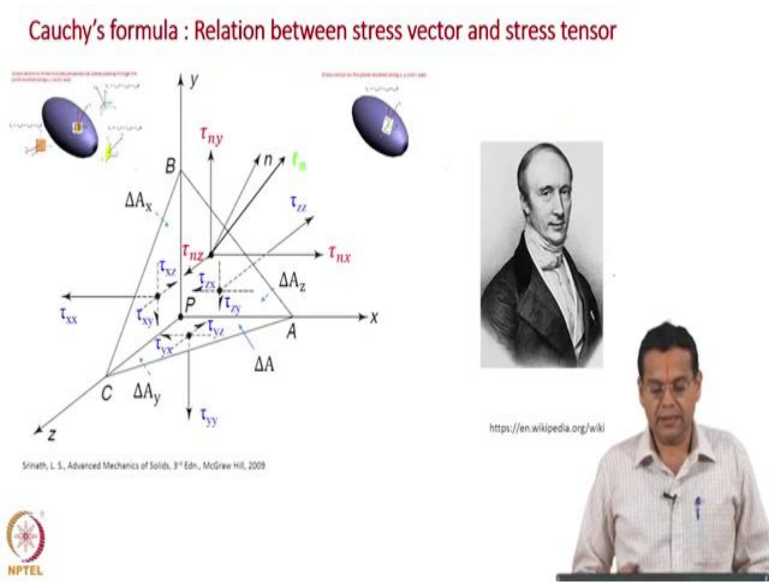

Then, we related these two the stress vector to the components of stress tensor using the Cauchy's formula.

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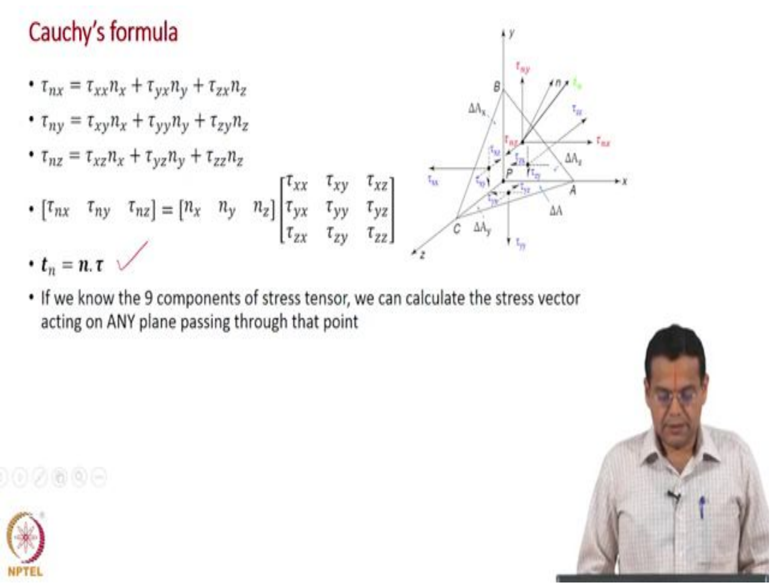

So, we completed this block we extended this knowledge to fluids then we discussed these two blocks, total stress, fluid under rest and pressure and viscous stress tensor.

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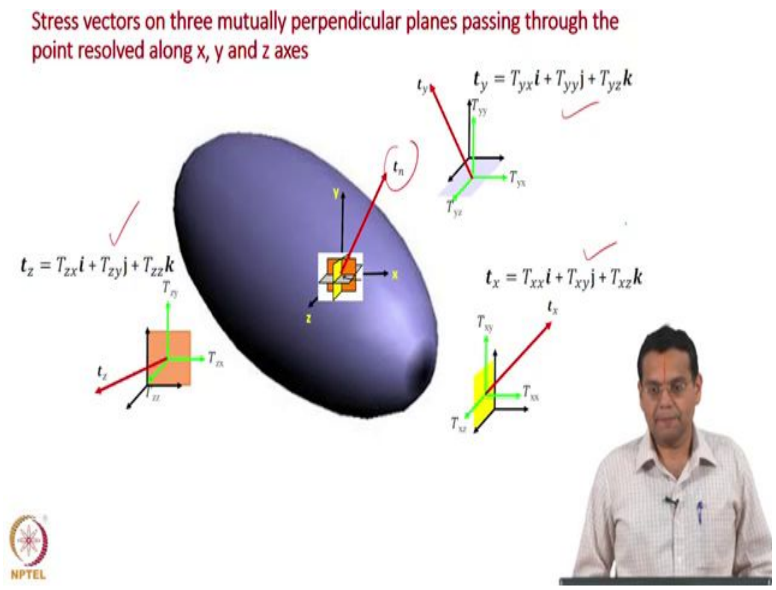

Some slides from here. So, once again stress vector, but now components of total stress tensor in terms of T.

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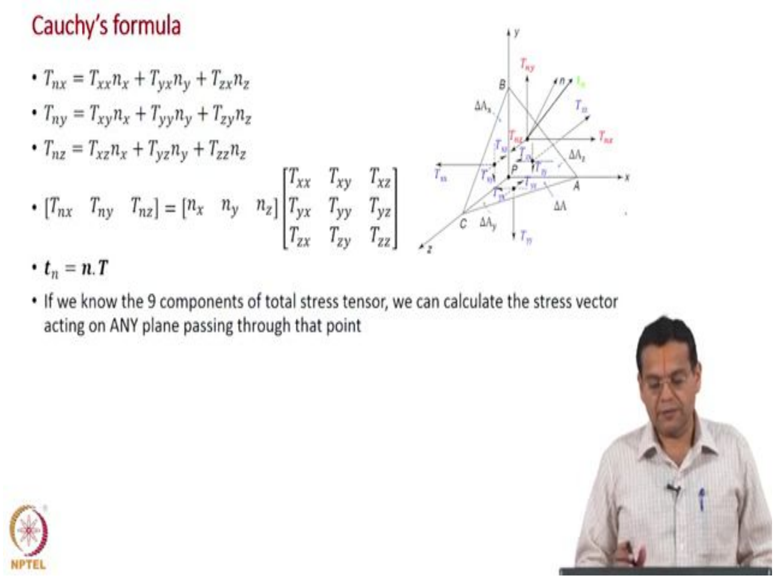

Once again we related stress vector two components of total stress tensor by the Cauchy's formula.

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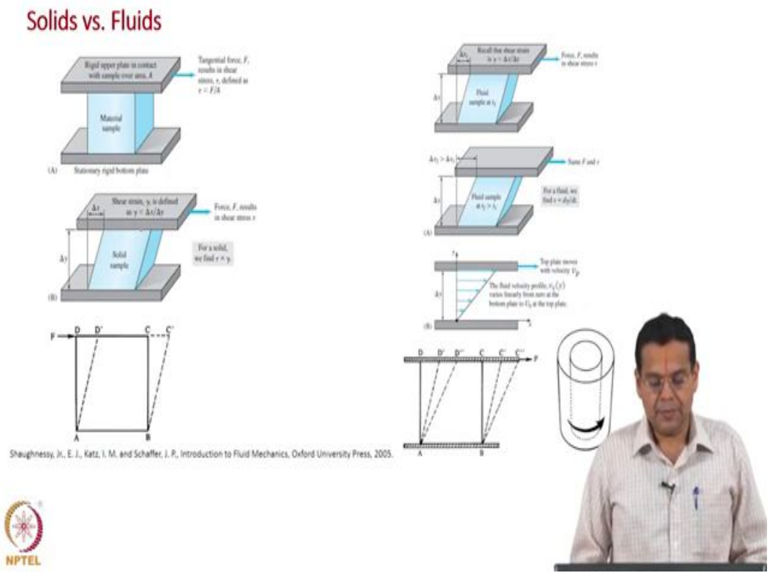

Then to proceed further we discuss the difference between solids and fluids we said fluid under rest cannot sustain shear stress.

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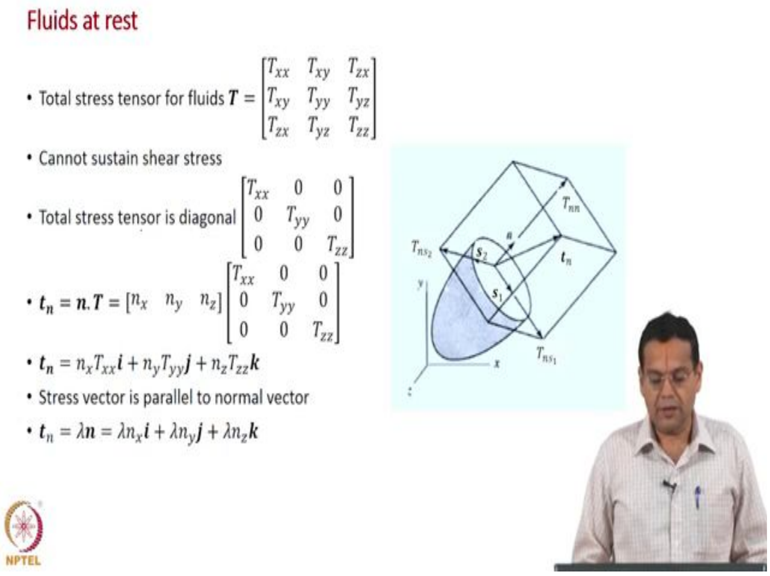

Based on that, we discussed the stress in a fluid under static condition, which is normal stress, which is pressure the thermodynamic pressure and is compressive hence $-p$.

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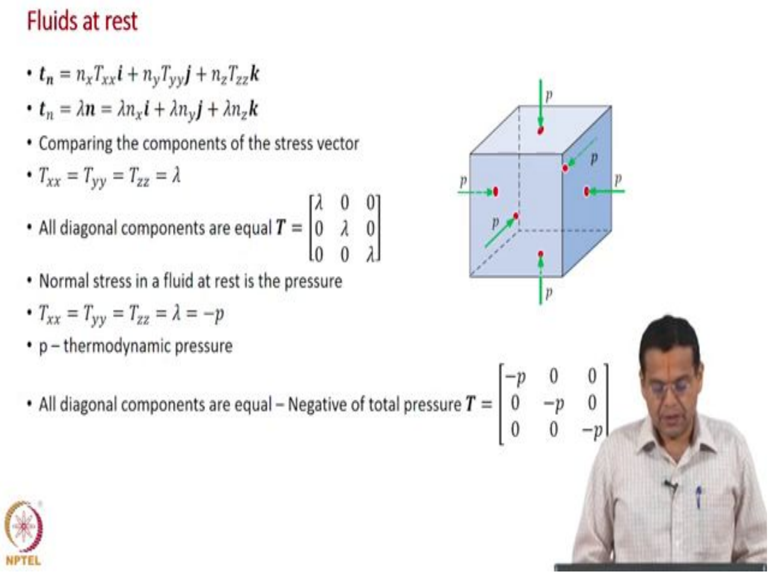

So, we got the component of total stress tensor under hydrostatic condition.

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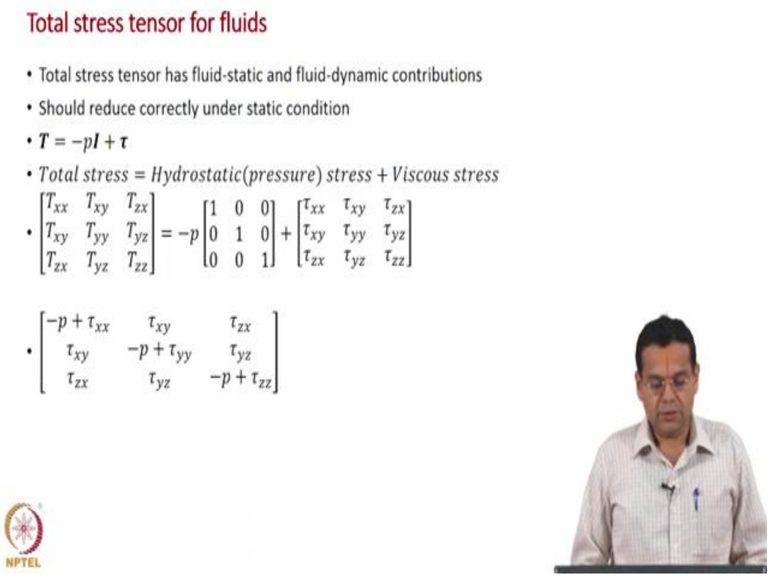

So, we expressed total stress, in terms of hydrostatic stress and a viscous stress. So, and that is what we discussed here, fluid at rest total stress tensor in terms of pressure and viscous stress tensor.

So, having understood the surface forces, we could we substituted here and then completed the differential form of linear momentum balance, gravity was already discussed we included pressure and viscous stress on the right hand side and few slides from there.

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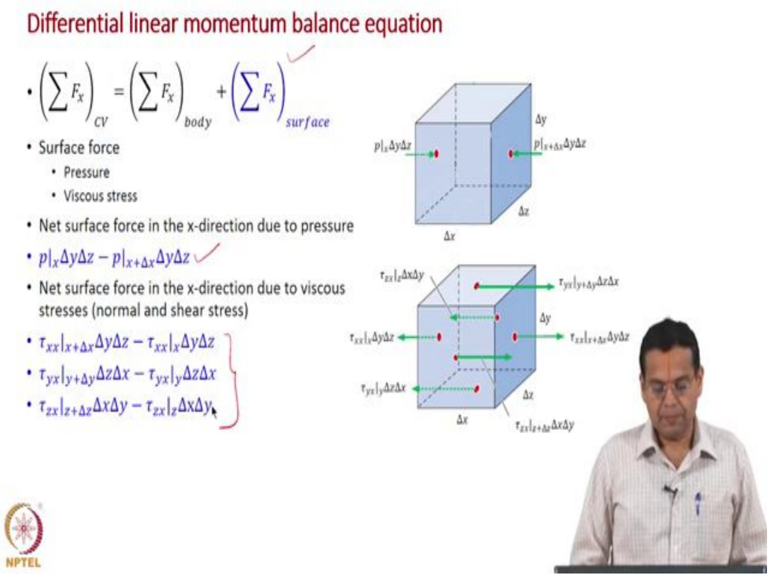

So, now our attention was on the surface forces, specific attention was on the surface forces, we express the surface force for pressure and for the viscous stress.

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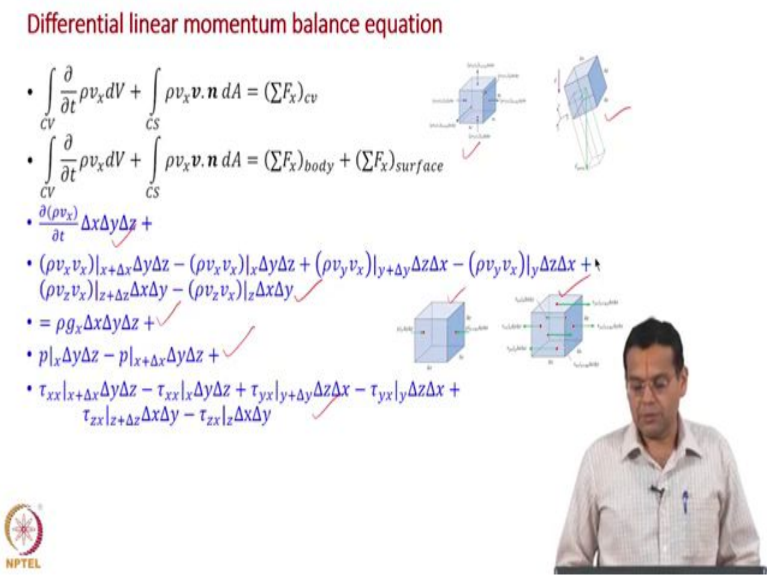

And, then we combined with our earlier transient term convection term and the gravity on the right hand side, along with that we included the surface forces due to pressure and viscous stresses represented by all this 4 I would say control volumes.

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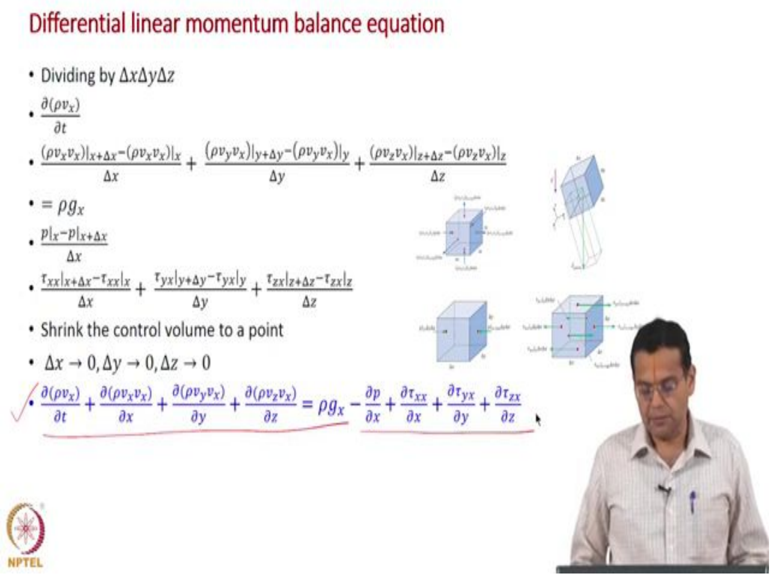

And, then we derived the complete form of the linear momentum balance. I would say we completed the linear momentum balance, till this point we already derived, these two terms or these two set of terms where newly added. At this stage we were here differential form of the linear momentum balance with gravity pressure viscous stress on the right hand side.

Look liked we are almost there, but then we will have to know more to proceed further. It is like reaching a big station and one station before you are put for a long time you do not get a signal to proceed further. Something like that we are there, what was objective we did a degree of freedom analysis and found that the viscous stresses are not known they are unknowns. We will have to express in terms of the velocities of course, now we not the velocity gradients.

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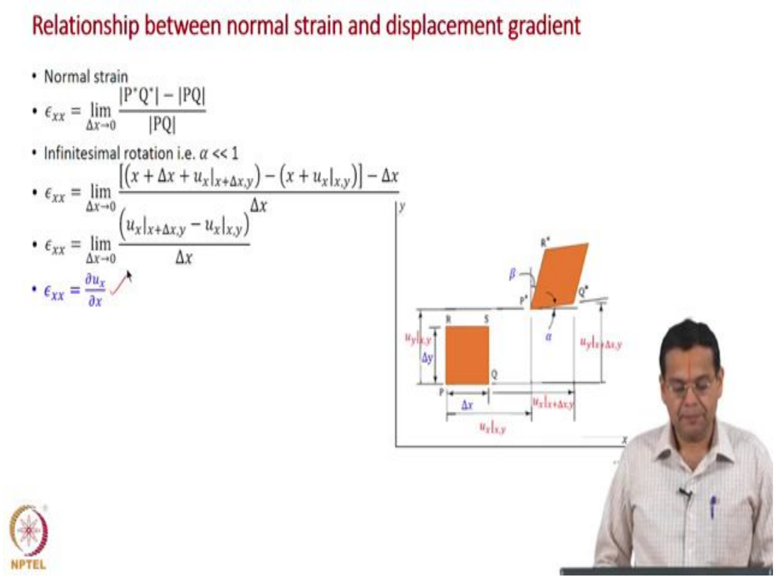

Then, we took a the second diversion to solid mechanics, discussed about normal strain, shear strain, express them in terms of the displacement gradients.

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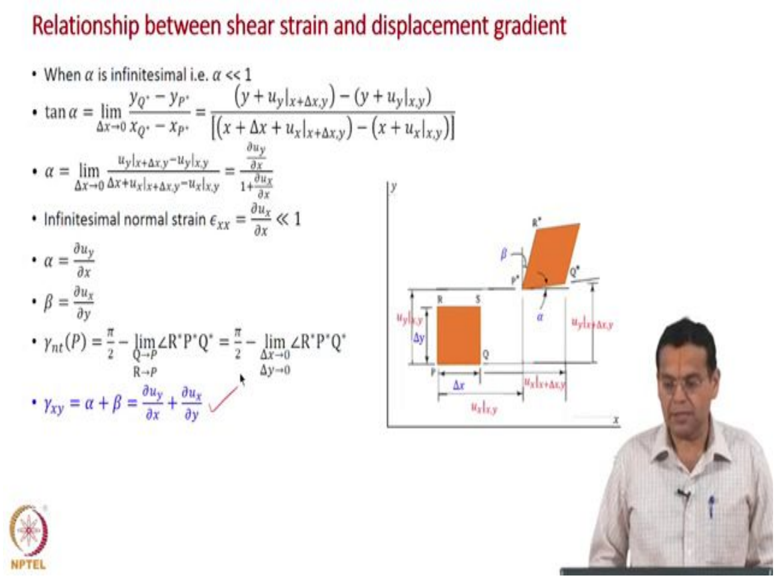

The normal strain in terms of displacement gradient, the shear strain in terms of displacement gradients.

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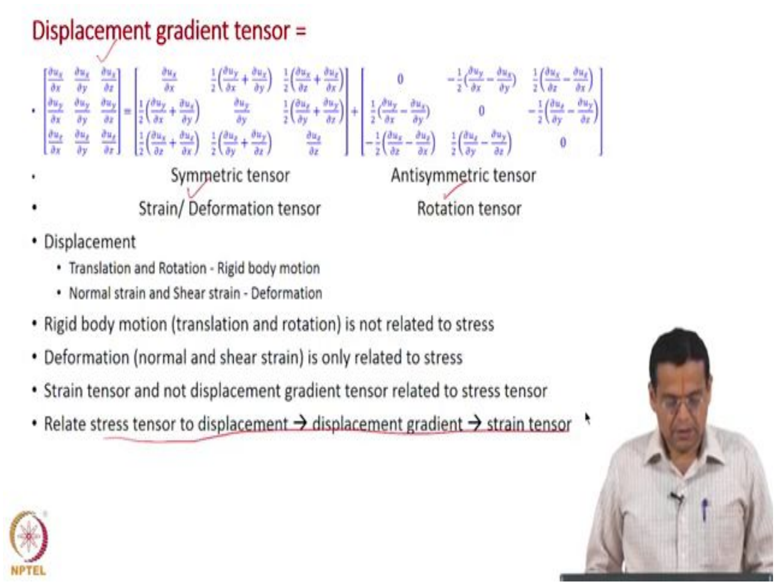

And, more importantly we discussed that the displacement gradient tensor a sum of strain tensor and rotation tensor. And, we also said that the stress tensor has to be related to the strain tensor. Now, at that point we have completed these two blocks. We will have to extend that knowledge analogues to fluids, where we discuss the strain rate and strain rate tensor.

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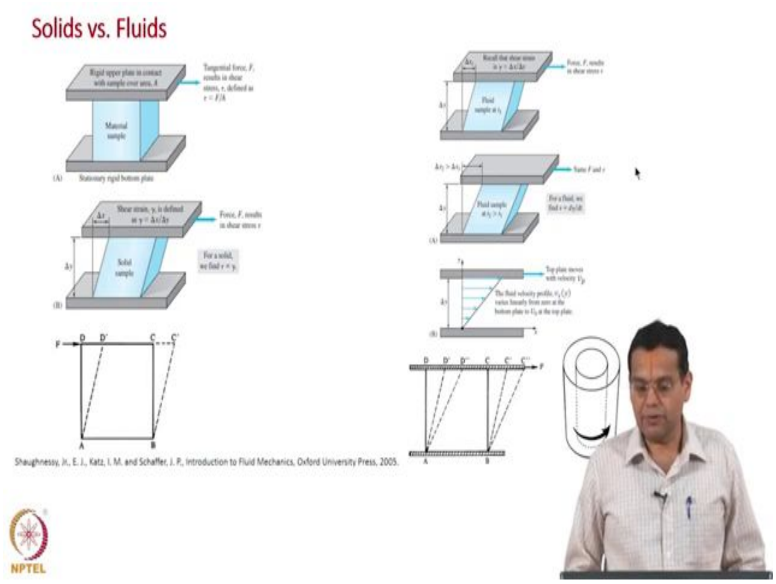

Few slides from there, we have once again discuss the difference between solids and fluids to understand that it is strain for solids and strain rate for fluids. Force depends on stain for solids and force depends on strain rate for fluids.

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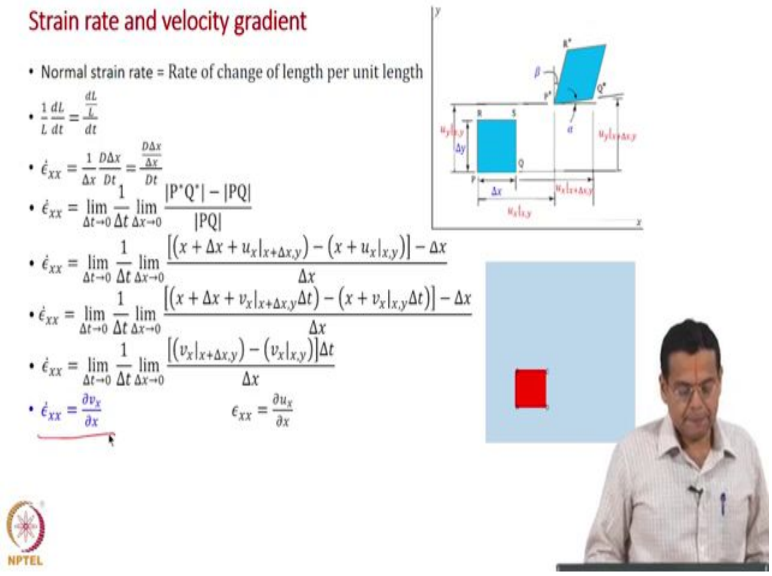

And, we discussed normal strain rate and shear strain rate, we said they are material derivatives of normal strain and shear strain, and then we related the normal strain rate to velocity gradient, shear strain rate to velocity gradient

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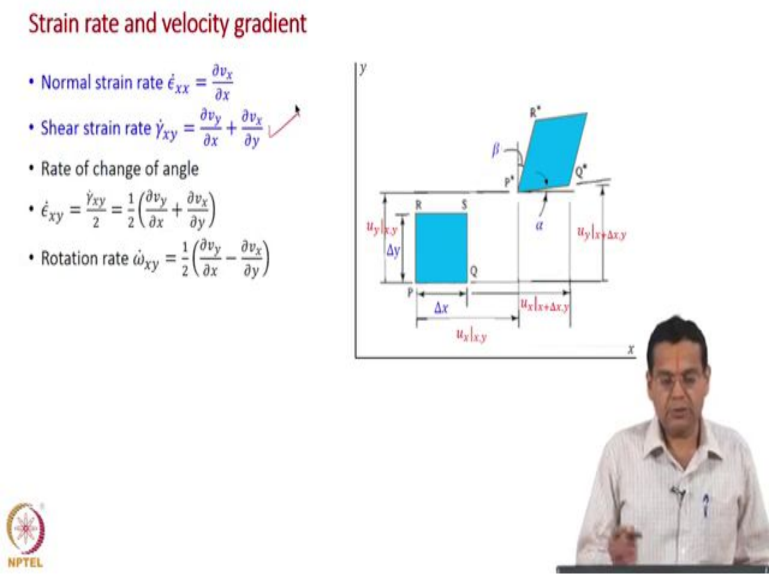

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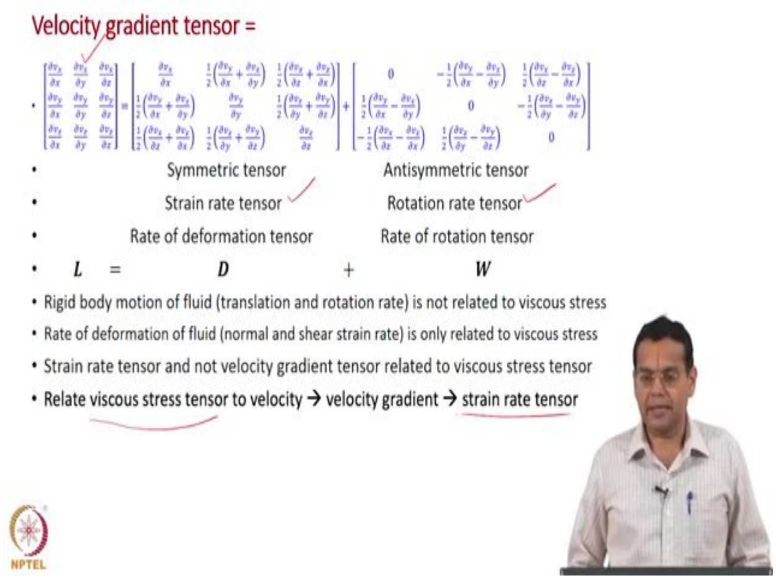

And, once again more importantly we discussed, the velocity gradient tensor a sum of a strain rate tensor and rotation rate tensor. Once again we said the objective is to relate viscous stress tensor to the strain rate tensor.

So, at that point we had discussed these two blocks. Now, having understood stress and strain and viscous stress and strain rate, next step was to relate these two for solids through Hooke's law and for fluids through Newton's law of viscosity. So, once again we took a diversion to solid mechanics and discuss the assumptions now, because they are material dependent.

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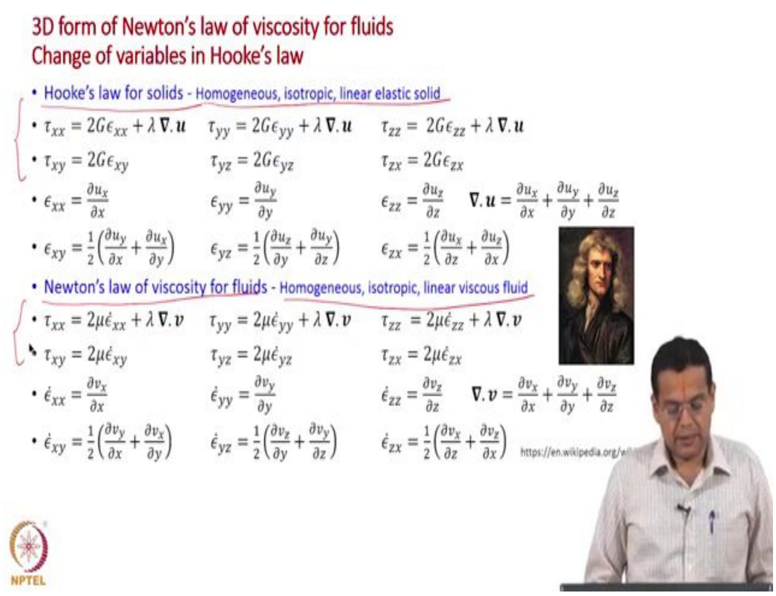

And, came back to fluid mechanics just wrote down the Newton's law. And, all those are summarized in one slide here Hooke's law of solids assumptions and for Newton's law of viscosity and the assumptions here. And, these six equations represent a Hooke's law, by change of variables we obtain 6 equations representing Newton's law of viscosity and that point we had completed these two blocks. So, now it is a matter of substitution of this here and obtaining here and that is what we have discussed just now.

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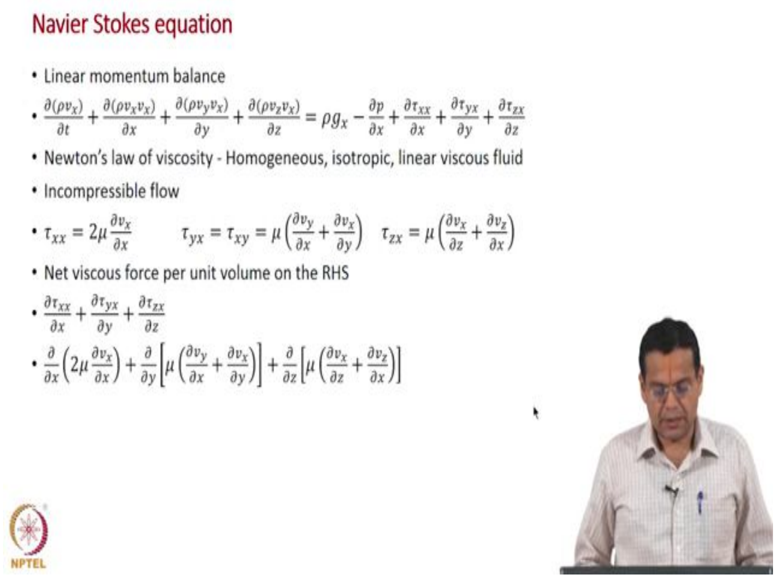

Took the linear momentum balance equation, then substituted the Newton's law of viscosity, few more assumptions we made and obtained the Navier Stokes equation.

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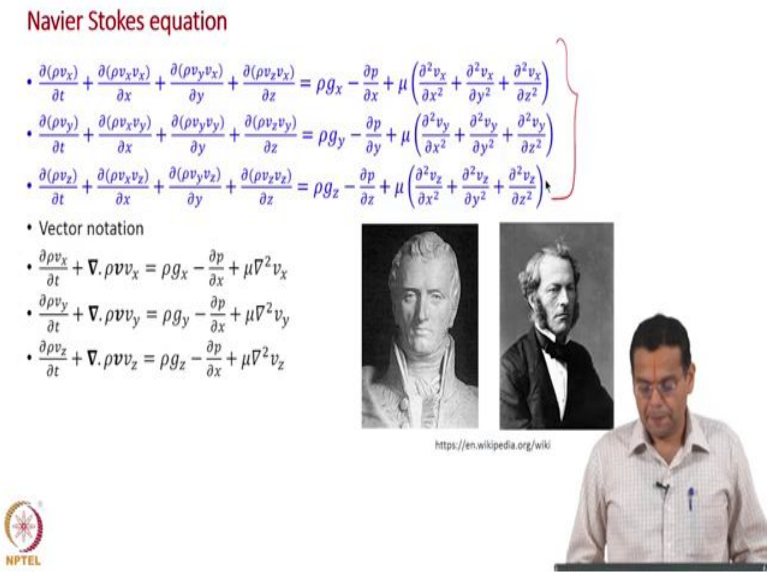

And, that we reach the destination ok. It has been a long journey and that is why it has been called as journey to the Navier Stokes. Also like to mention that usually derivations are considered boring.

Now, we are seen a derivation we can say that we have been deriving from Newton's law to Navier Stokes over several hours, several lectures, derivation can be as challenging as interesting as knowledge as knowledge gaining as we have seen ok. We have learned so, many concepts so, many assumptions wherever required and so, derivation helps to know the scope of application of a particular equation.

And, once we derive you also know the physical significance of each of the terms in an equation. Otherwise, they are there is some term the $\frac{\partial v_x}{\partial x}$. But, now if you look at $\frac{\partial v_x}{\partial x}$ you know the physical significance of that particular term. Similarly, all other terms in the equation.

Those advantages of doing a detailed derivation, fundamentals becomes very strong. So, now, we will have to look at applications of the Navier Stokes equation, which we will discuss in the next classes.