Continuum Mechanics And Transport Phenomena
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Lecture – 82
Hydrostatic Pressure Distribution in Gas

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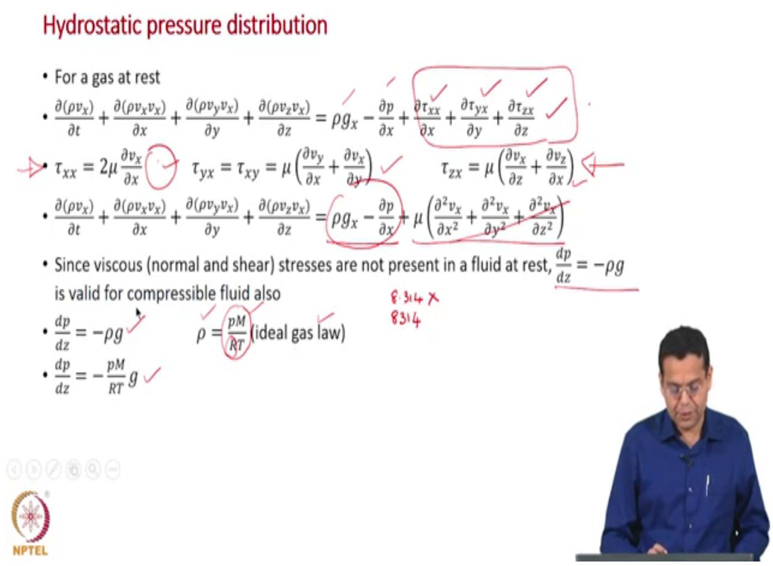

Now, we will look at Hydrostatic Pressure Distribution in a Gas. What does it mean? You have a container filled with air and how does pressure vary? Now, of course, if you take a small container like this and look at pressure distribution hardly any pressure distribution. So, unless it is a very long column or the height is large there would not be any difference in pressure.

So, the example which I going to take is at ground level the atmospheric pressure and then let us say you are going up the hill what is the variation in pressure how does it pressure and change and what is the practical significance of that; a very nice example of practical example as well.

Let us start. We will derive the equation and see numerical application. For a gas at rest first question what we like the answer is we said we are going to apply this for a gas at rest and we said the gas is compressible. First question arises whether the equation which we derived the balance of pressure and gravity force is it applicable for a compressible gas? First question is that we should make sure that the equation applicable for gas for a compressible fluid.

Remember we said the Navier–Stokes equation what we derived is for incompressible fluid only, that is not for a compressible fluid, but now we are going to apply of course, for a under static condition and so, first we should check whether the hydrostatic equation balance of pressure and gravity is it applicable for a gas at rest. So, for that let us recall back our derivation of Navier-Stokes.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

How did we derive? We took the linear momentum balance and then for the viscous stress terms on the right hand side we substituted the Newton's law of viscosity.

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x}; \quad \tau_{xy} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad \tau_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

When we substituted I have not taken the term which involves the $\nabla . v$ which applicable for a compressible fluid. So, these terms of course, would not change whether it is compressible or incompressible. We had one $\lambda\nabla . v$ term by not writing that this Newton's law of viscosity is for incompressible fluid.

So, the way in which we derive is we took the viscous stresses on the right hand side and then substitute the Newton's law of viscosity. When we substituted we have not taken the term here which was $\lambda\nabla . v$ which is for compressible fluids. So, this form of the Newton's law of viscosity is for incompressible fluids and of course, when we substituted we got the Navier–Stokes equation.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, which means that the assumption of incompressible fluid is restricted to the three viscous stresses terms only that does not influence the left hand side nor the gravity force nor the pressure force. The assumption of incompressible fluid plays a role only in this term; only this term will change whether we are going to apply for an incompressible fluid or a compressible fluid.

We have already seen for a fluid at rest there are no viscous stresses and hence the equation which I have written relating gravity and pressure is valid for compressible fluid also.

$$\frac{dp}{dz} =- \rho g$$

So, let me just summarize we start with a linear momentum balance substitute the compressible form of Newton's law viscosity and obtain the Navier–Stokes equation for incompressible fluid. But, we are now going to apply that for a fluid under rest condition for with there are no viscous stresses at all. So, that assumption has no role to play here. We are considering only the body force and the surface force due to pressure.

So, with respect to those two terms there is no question of incompressible assumption. So, the hydrostatic equation which we derived is applicable and I say hydrostatic equation I mean $\frac{dp}{dz} =- \rho g$ is valid for compressible fluid also that is a starting point. Having understood that it is applicable for compressible fluid let us proceed further.

This is our starting equation,

$$\frac{dp}{dz} =- \rho g$$

Now, because the fluid is compressible the density depends on pressure and that comes from a equation of state from our thermodynamics class. Now, the most simplest equation of state is the ideal gas law.

$$\rho = \frac{pM}{RT}$$

So, we will assume the ideal gas law to be valid and that is also because as I said we are going to apply this for a case where there is around atmospheric pressure. The pressures are not going to be very high. So, ideal gas law can be very well be used.

Also like to mention at R as the value of 8314, the unit of ρ is kg/m^3; R in SI unit is 8.314 Joule per gram mole Kelvin, but because ρ is in kg/m^3 , R should be substituted as 8314 Joule per kg mole Kelvin.

So, it can be cause of confusion to like to stress that; even the problem you will see R as R is substituted as 8314 and not as 8.314; if it is SI unit we should do substitute 8.314. Now, we will substitute the expression for rho in terms of pressure in the hydrostatic equation.

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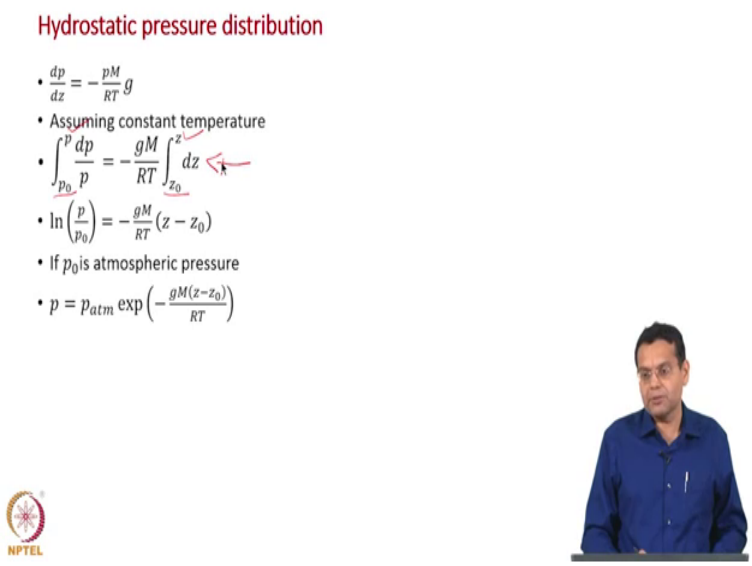

$$\frac{dp}{dz} =- \frac{pM}{RT}g$$

Now, we will have to integrate as we are done in the earlier case, but earlier for the case of water rho was just a constant now rho depends on pressure. Now, to proceed further we need to assume how temperature varies, whether temperature is a constant or temperature varies.

Now, to begin with for simpler case we will assume temperature is constant. Let us integrate,

$$\int_{p_0}^{p}\frac{dp}{p} =- \frac{gM}{RT}\int_{z_0}^{z} dz$$

Now, very simple integration,

$$-\ \ln ln\left(\frac{p}{p_0}\right) =- \frac{gM}{RT}\left(z - z_0\right)$$

Let us write this. If p_0 is a atmospheric pressure. So,

$$p = p_{atm} \exp exp\left(- \frac{gM}{RT}\left(z - z_0\right)\right)$$

So, what is use of this equation? If you know the atmospheric pressure at z_0 you can find out what is the variation of pressure with respect to z. Now, our interest as I told you is to study the variation of pressure let us say across as we go up the hill and we know that in the atmosphere the temperature varies with z. So, the previous derivation where we assumed the temperature to be constant is not exact. To have a better equation you will also take into account the variation of temperature with z and that is what we are going to do now.

Where are we now? We are deriving equation for variation of pressure with height in a column of gas. The present derivation what we have done is for the case where temperature is constant. Now, we are going to account where temperatures varies with z and that is what we are going to do now.

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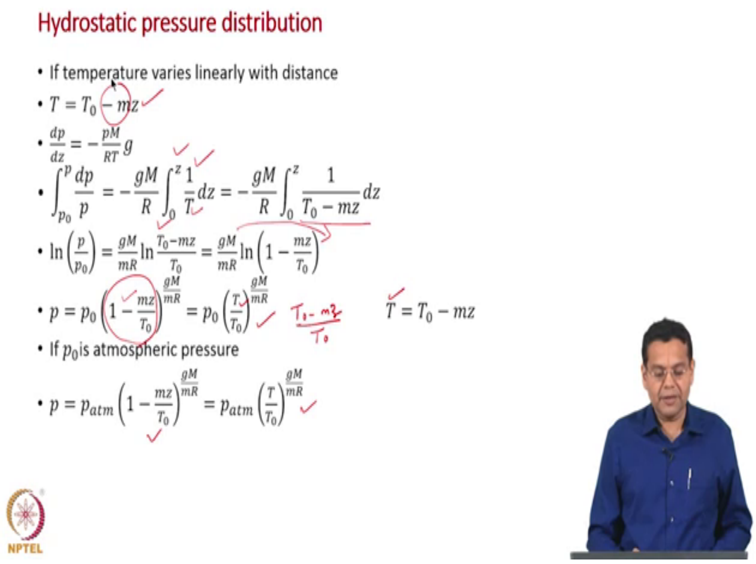


If temperature varies so, how does a temperature vary, we should know that. We will assume the temperature varies linearly with distance and that information is given to us. If temperature varies linearly with the distance

$$T = T_0 - mz$$

That is also in line with our application; I think this derivations can easily be understood if we keep the application in mind. At the surface of the earth some the datum level the temperatures T_0 and as you go up temperature decreases that is why we have taken this relationship.

So, now let us go back to our equation

$$\frac{dp}{dz} =- \frac{pM}{RT} g$$

Let us integrate

$$\int_{p_0}^{p} \frac{dp}{p} =- \frac{gM}{R} \int_{0}^{z} \frac{1}{T} dz =- \frac{gM}{R} \int_{0}^{z} \frac{1}{T_0 - mz} dz$$

Now, I cannot take T out of the integral sign, T is a function of z. Now, integrate from instead of z_0 to make it simple taken 0 to z. So, now simple integration

$$\ln ln \left(\frac{p}{p_0}\right) = \frac{gM}{mR} \ln ln \left(\frac{T_0 - mz}{T_0}\right) = \frac{gM}{mR} \ln ln \left(1 - \frac{mz}{T_0}\right)$$

We can also simplify this further as

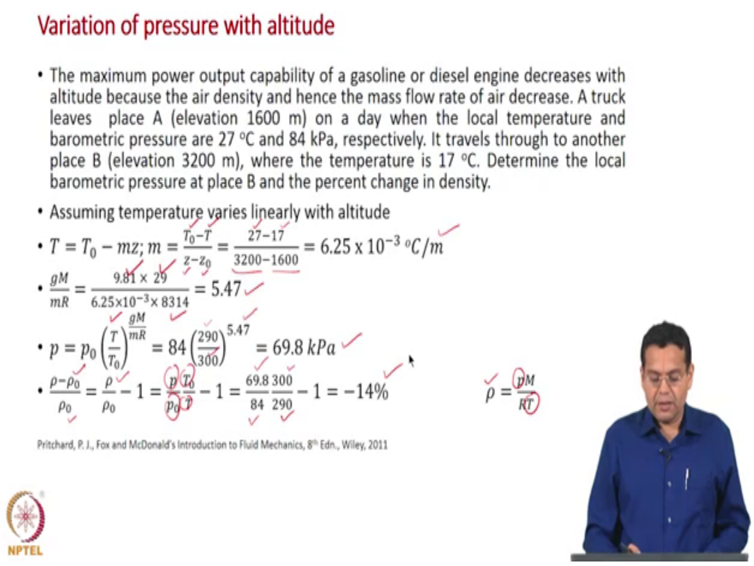
$$p = p_0 \left(1 - \frac{mz}{T_0}\right)^{\frac{gM}{mR}} = p_0 \left(\frac{T}{T_0}\right)^{\frac{gM}{mR}}$$

So, now if p_0 is atmospheric, then

$$p = p_{atm} \left(1 - \frac{mz}{T_0}\right)^{\frac{gM}{mR}} = p_{atm} \left(\frac{T}{T_0}\right)^{\frac{gM}{mR}}$$

Remember capital M is molecular weight, small m is slope of the relationship between T and z. Though this tells you how pressure varies with z if you look at this relationship there is no z explicitly, but T depends on z. If you look at this relationship z explicitly appears in the equation the tells you how p varies with z.

Example: (Refer Slide Time: 13:41)



Let us look at an example I read the first line that will show the real practical significance of this application. The maximum power output capability of a gasoline or diesel engine decreases with altitude because the air density and hence the mass flow rate of air decrease that is a practical relevance of application which I derived. What does it mean? that is example going to see, let us say you have a truck and of course, which works on a gasoline or diesel engine, mostly diesel engine and as it travels up a hill there is change in pressure.

Of course, there is a decrease in temperature there is change in pressure because of change in pressure there is a change in density and because density changes mass flow rate changes and you will see decrease in density as you go up the hill which means there is a decrease in mass flow rate which means the maximum power output decreases.

So, that is a practical relevance, though it may look like a very theoretical example of studying pressure as a function of height density depends on pressure. So, we are starting density as a function of height, but density determines the mass flow rate that determines the power output. So, that is a practical relevance.

Let us proceed further. The truck leaves place A elevation 1600 meters not really ground level, but some other higher level on a day when the local temperature and barometric pressure are 27 degree centigrade and 84 kilo Pascal. That is why the pressure is 84 kilo Pascal, it is not 101 kilo Pascal which would have been the pressure at the ground level already this truck is starting at a higher elevation and that is why the pressure is 84 kilo Pascal.

It travels through another place B whose elevation is 3200 meters, both the elevations are from ground level usually elevations are given from ground level and the temperature is lower there which is 17 degree centigrade. Determine the local barometric pressure at place B and the percentage change in density. So, we are we like to know what is the pressure there and since density depends on pressure what is the percentage change in density.

So, simple application, but very good practical relevance; once again from Fox and McDonald that is why we derive the expression for pressure distribution a gas in under two different conditions. First we took temperature is constant, but a more practical cases where temperature is vary; that is why we did this derivation once again taking the variation of temperature with height.

Assuming temperature varies linearly with altitude we will take this equation

$$T = T_0 - mz$$

The m is a slope. How do you find out the slope? We are given two locations, two temperatures. So,

$$m = \frac{T_0 - T}{z - z_0} = \frac{27-17}{3200-1600} = 6.25\ X\ 10^{-3}\ degree\ C/m$$

Now, let us calculate the combination of parameters it is dimensionless

$$\frac{gM}{mR} = \frac{9.81\ X\ 29}{6.25\ X\ 10^{-3} X\ 8314} = 5.47$$

Now, let us write the equation and substitute the values

$$p = p_0 \left(\frac{T}{T_0}\right)^{\frac{gM}{mR}} = 84\left(\frac{290}{300}\right)^{5.47} = 69.8\ kPa$$

That is the pressure at the higher elevation. There is a decrease in pressure from 84 kilo Pascal to 70 kilo Pascal roughly. Now, what other objective one question is what is the local barometric pressure at place B, that is what we have determined. More importantly to relate to the effect on power output we will have to find out the change in density. So,

$$\frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1$$

Now, ρ depends on $\frac{p}{T}$ both of them change. So,

$$\frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = \frac{p}{p_0}\frac{T_0}{T} - 1 = \frac{69.8}{84}X\frac{300}{290} =- 14\%$$

So, you see the 14 percent decrease in density and that will have a corresponding influence on the maximum power output. So, theoretical equation and we have seen the practical relevance of that and as we kept one of our objective is to look at engineering applications we try to solve examples which are mostly engineering relevance.