Continuum Mechanics And Transport Phenomena
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Lecture - 83
Fluid in Rigid Body Motion: Pressure Distribution

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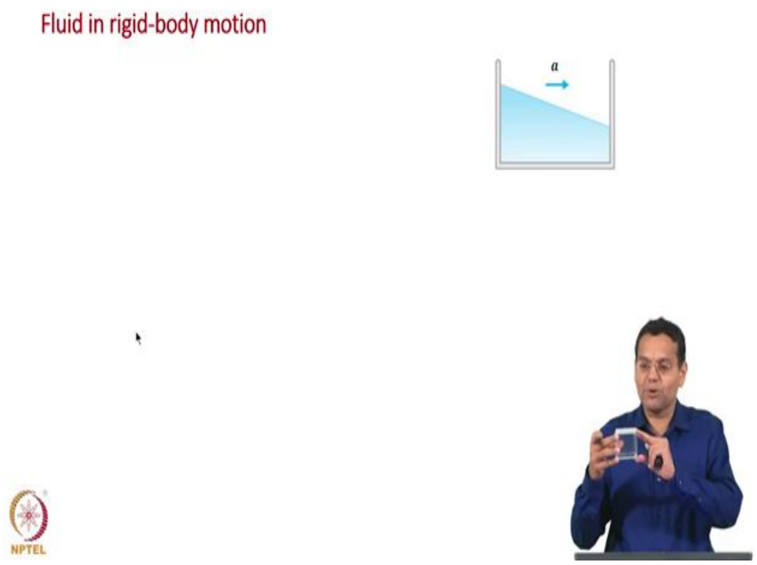

Let us move to the next level of application. What do I mean by next level of application? The previous case the previous level of application, the fluid was under rest under static condition. Now, we allowed the fluid to move, but not in a general way as a rigid body. The entire body of fluid should move, I have a container and let us say it is partly filled with liquid we are going to discuss only liquid and it is moving.

Remember this we called as translation. It also be rotate also the entire body of fluid could rotate also, there is also rigid body motion, but that will take us to cylindrical coordinates. So, to avoid that we are only considering translation of the entire body of liquid as an example for rigid body motion.

If, you look at books they had discussed both translation and rigid body rotation just to avoid cylindrical coordinates, we will discuss only translation of the entire fluid body as an example for rigid body motion, when you say fluid and rigid body motion it may look little paradoxical also. Because, we are saying fluid, but rigid body also usually the word rigid body is for solids, but here the entire body moves like a rigid body, that is a meaning of this fluid in rigid body motion.

Now, what is the best example? Of course, the best example is during summer time, when tanker lorries carry water from one place to other. Let us imagine the front of the lorry and then it travels and then it carries water. This entire body of water travels along with the tanker lorry. So, that is the best example which I initially visualize for this rigid body motion, need not be water we can see on a daily basis tanker Lorries carrying milk it could be even gasoline; so, any such liquid being transported as a best example for fluid in rigid body motion.

The fluid alone cannot be existent and move in rigid body motion, it should be in some container and that container moves as a rigid body. So, the fluid in the container also moves as a rigid body that is what is shown here and we are interested in the shape of the free surface.

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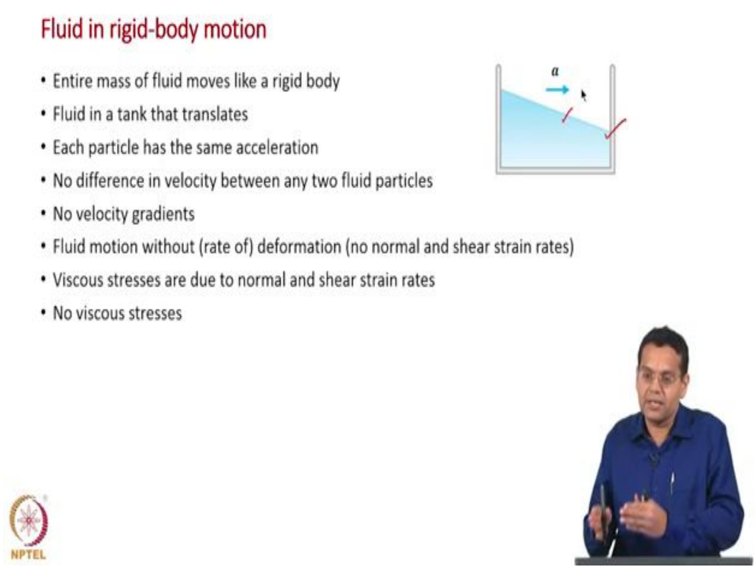

Let us proceed. The entire mass of fluid moves like a rigid body that is a definition. We have been telling rigid body several times and we said, we are discussed about rigid body when we are discussed about strain for solids. We discussed about rigid body, when we discussed about strain rate for fluids and we said rigid body motion includes translation, rotation, they do not contribute to viscous stresses. Now, time has come where we can really understand and have a practical example as well.

So, entire mass of fluid moves like a rigid body, example is fluid in a tank that translates we have seen example for that. How do you now put more physically, we have described in several ways let us get into the physics. Now, when the entire mass of liquid moves and see the case what we are discussing is the fluid accelerates. And, which means that the every particle in the liquid also has a same acceleration. Let us say the lorry accelerates, then every particle, every fluid particle also has the same acceleration, that is why the sentence is each particle has the same acceleration.

Now, what does it mean? If you consider two fluid particles there is no difference in the velocity, both have the same velocity. And, you are having a fluid body and that is translating, if you focus on let us say two fluid particles, they have the same velocity there is no difference in velocity. So, no difference in velocity between any two fluid particles, which means that there is no velocity gradient; obviously, if you take two particles which are nearby each other, no velocity difference, no velocity gradient.

Now, what did we say for normal strain rate, shear strain rate to exist that should be velocity gradient. In fact, we express the strain rates in terms of velocity gradients. Now, there are no velocity gradients here, which means that there are no normal strain rates, there are no shear strain rates, there is no defamation or more precisely there is no rate of defamation, because of discussing fluids we should say there is no rate of defamation in this case, which means that there is no viscous stresses.

Look at the hierarchy nicely we have discussed from rigid body motion to now viscous stresses and this is what we had discussed earlier also rigid body motion does not contribute to viscous stresses, only when there is normal strain rates, shear strain rate, it contribute to viscous stresses.

So, let us quickly repeat this, I have a moving body of fluid, moving as a rigid body motion it just translates. And, if you take any two fluid particles there is no difference in velocity, there is no velocity gradient, there is no strain rate, either normal strain rate or shear strain rate, because there is no velocity gradient.

And, because there are no strain rates there are no viscous stresses, normal viscous stresses and shear viscous stresses also, no viscous stresses at all. Like to mention in the previous case also there were no viscous stresses. This case also there is no viscous stresses, but there is a difference.

The previous case there are no viscous stresses, because the fluid was under rest condition. In this case there is no viscous stress not because of fluid is stationary, it is moving in a rigid body motion. Because, it is moving in a rigid body motion, there are no velocity gradients no strain rates, hence there is no viscous stress, that is why we put both these under the same category.

Remember we said the title for the present section of discussion is fluid under rest and under rigid body motion, what connects these two is no viscous stresses. So, if you ought to be more formal title the first level of applications are where there are no viscous stresses. And one in which, one in which the fluid is not at all moving other in which the fluid is moving like a rigid body.

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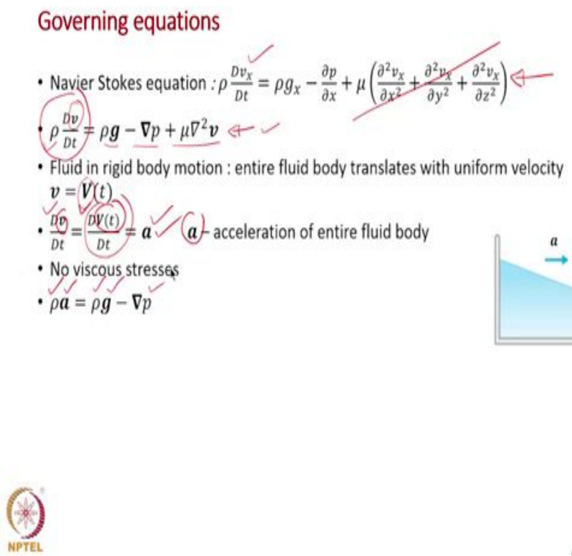

What are the governing equations? We are now discuss the physics let us look at the governing equations. So, let us write the Navier Stokes equation

$$\rho \frac{Dv_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

Now, when we derive the Navier Stokes equation, we expressed this equation in a compact form, in a vectorial form as shown here. Please recall the discussion earlier under Navier Stokes equation.

$$\rho \frac{Dv_x}{Dt} = \rho g - \nabla p + \mu \nabla^2 v$$

First, we derive the Navier Stokes equation, then we expressed from a material particular point of view and then express that vectorially as shown here.

Now, when the fluid is in rigid body motion, the entire body translates with uniform velocity, that is what we said. If, you take two fluid particles there is no difference in velocity. What does it mean? There is one velocity for the entire body of fluid, every point have the same velocity and, that velocity we will take as

$$v = V(t)$$

Why is a new nomenclature introduced. v is the velocity of the fluid, we said every fluid particle here as the same velocity that velocity we take as V(t). Remember this V vectors imposed by us remember v is also a function of time. What is the best example? Let us say the lorry starts and then it accelerates, that is V as a function of time.

$$\frac{Dv}{Dt} = \frac{DV(t)}{Dt} = a$$

In this case, v = V(t), best example for V vectors a function of time is just imagine the lorry starting from rest and then accelerating. And, now so, $\frac{DV(t)}{Dt}$ is the acceleration; acceleration of the entire fluid body, which is equal to acceleration of the lorry. Please keep that in mind and that acceleration is a known quantity or given quantity or imposed by us. Unlike the acceleration due to gravity, which is the body force this acceleration is imposed by us, that depends on how fast the velocity changes, how fast the velocity of the lorry changes, that determines this acceleration.

So, the acceleration is a value imposed by us, the fluid body also accelerates at the same acceleration as the lorry. So, having understood that, the acceleration of the fluid is same as that of the lorry and that is denoted by a this is a given quantity, let us proceed further, we have discussed that there are viscous stresses. So, let us see how does the equations simplify

$$\rho\frac{Dv_x}{Dt} = \rho g - \nabla p + \mu\nabla^2 v$$

$$\rho a = \rho g - \nabla p$$

Usually the given data is in terms of acceleration either directly or indirectly. So, left hand side ρa, right hand side we have ρg and then of course, minus gradient of p. The third term does not appear because there are no viscous stresses,

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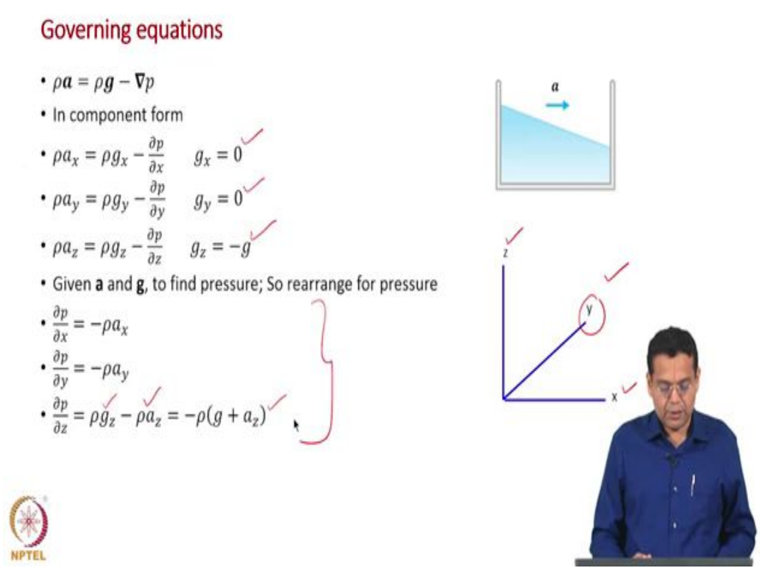

$$\rho a = \rho g - \nabla p$$

Now, let us put this equation in component form the x component

$$\rho a_x = \rho g_x - \frac{\partial p}{\partial x}; \qquad g_x = 0$$

And the y component of that vectorial equation,

$$\rho a_y = \rho g_y - \frac{\partial p}{\partial y}; \qquad g_y = 0$$

and the z component of that vectorial equation

$$\rho a_z = \rho g_z - \frac{\partial p}{\partial z}; \qquad g_z = -g$$

Now, to proceed further as we have done in the case of fluid in rest, we should decide the coordinate axis. So, we will choose the same coordinate axis, which I have chosen earlier x is horizontal axis, z is a vertical axis, and y is perpendicular to the slide. Now, once I have chosen this coordinate axis $g_x = 0$, $g_y = 0$, and $g_z = -g$ same as what we have discussed earlier.

That is where it is important to choose the coordinate axis; so, that we can assign values for g_x, g_y and g_z. So, let us rearrange this equation the way in which we rearrange this equation is that the unknown here is pressure. So, we take all the gradients of pressure on the left hand side and the known quantities on the right hand side. So, let us rearrange

$$\frac{\partial p}{\partial x} = -\rho a_x$$

And, similarly the y direction

$$\frac{\partial p}{\partial y} = -\rho a_y$$

And in the z direction,

$$\frac{\partial p}{\partial z} = \rho g_z - \rho a_z = -\rho(g + a_z)$$

So, these are the three equations that described the pressure distribution in a fluid in rigid body motion.

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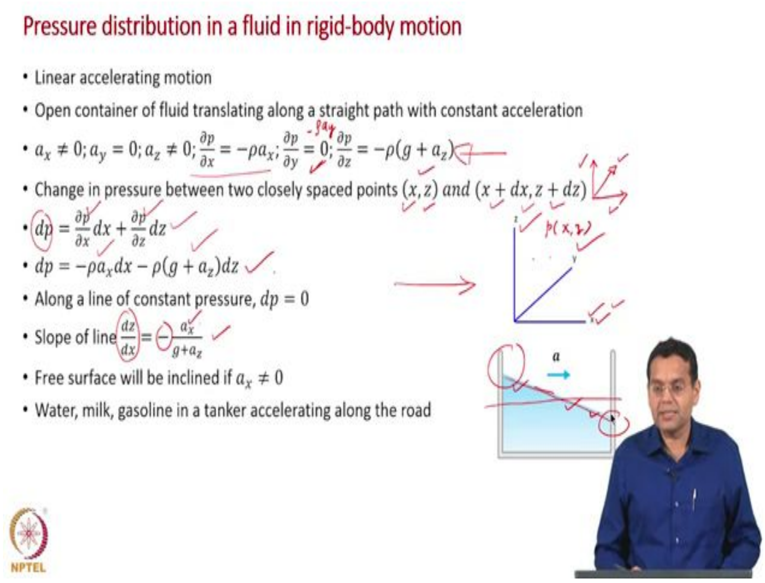

So, now let us proceed further let us see how do we use this equation to analyse further. Now, we are going to consider the linear accelerating motion, what does it mean? The example is open container of fluid translating along a straight path and constant acceleration. What does it mean? let us take the lorry and it accelerates along x direction. Other example what we can take is let us say you have a say body of fluid in a lift and the lift accelerates in the z direction.

So, we are going to consider acceleration in the x direction, other possibility is acceleration in the z direction. The example is container of let us say bucket of water in a lift which is either accelerating up or accelerating downwards. We are not going to consider acceleration in the y direction, just for simplicity, but we consider acceleration in the x direction and z direction as well, both acceleration can also be there.

So,

$$a_x \neq 0; \quad a_y = 0; \quad a_z \neq 0$$

Now, the equation are,

$$\frac{\partial p}{\partial x} = -\rho a_x$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho(g + a_z)$$

Now, we will consider in this x z plane two points and look at the difference in pressure between these two points. So, let us say one point is at (x, z) and the other point is at $(x + dx, z + dz)$. Now, in this case pressure is a function of (x, z) depends on two variables. So, the total differential dp can be written in terms of the partial derivatives

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$$

So, we know that the total differential can be expressed in terms of partial differential according to this equation.

Now, we will substitute for

$$\frac{\partial p}{\partial x} = -\rho a_x; \qquad \frac{\partial p}{\partial z} = -\rho(g + a_z)$$

So,

$$dp = -\rho a_x dx - \rho\,(g + a_z)\,dz$$

Now, as we said we are interested in the shape of the free surface. What is free surface? Free surface is one which is exposed to the atmosphere and along the free surface the pressure is a constant.

So, along we will consider a line of constant pressure which means $dp = 0$. So, when you substitute $dp = 0$, you can find the slope of the surface; a slope of the line as

$$\frac{dz}{dx} = -\frac{a_x}{g+a_z}$$

So, what is the physical interpretation of this equation, let us say a lorry is accelerating in this direction, a_x is positive this equation tells you that the slope will be negative. And, that is why in all the if you cause right from the beginning we have been seeing surface sloping in this direction, which has a negative slope, a_x is positive $\frac{dz}{dx}$ which represent the slope of the surface or the line is negative.

That is why we have drawn the surface in this orientation, which also tells you that to begin with the liquid level will be a horizontal surface and the lorry is accelerating in this direction, the on the backside the level will increase, on the front side the level will decrease. Of course, this we have seen water milk gasoline in a tanker accelerating along the road that is the best example which you can think of.

Now, also like to mention that suppose if the lorry is not accelerating. Then of course, $a_x = 0$ the surface will be horizontal. Of course, we are in the more interesting case is where the lorry is accelerating.

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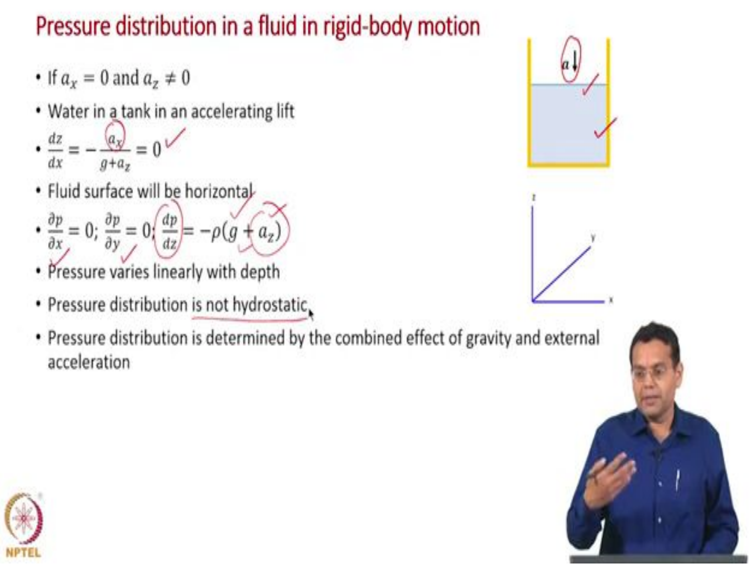


Now, let us take the other case, where you have a container in a lift, let us see what happens? If, $a_x = 0$, because the lift travels only vertically up and down. So,

$$a_x = 0; \qquad a_z \neq 0$$

Example is of course, water in a tank in an accelerating lift and that is what is shown here ok. The acceleration is shown in the downward direction could be in the upward direction also. So, now what happens to this slope

$$\frac{dz}{dx} = -\frac{a_x}{g+a_z} = 0$$

We get the slope to be 0, which means the surface will be horizontal and that is why a horizontal surface has been shown here. So, if you take a bucket of water put in an elevator the surface will not have a slope, it will just be flat; fluid surface will be horizontal.

Now, the equations are

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho(g + a_z)$$

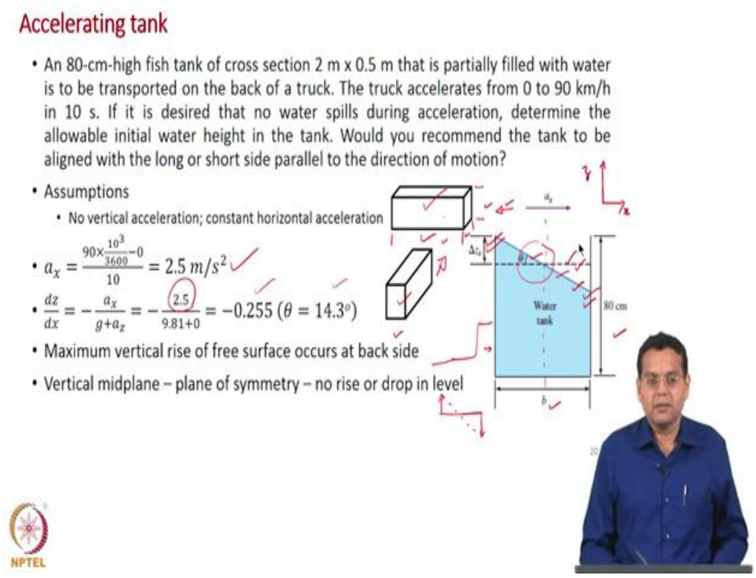
Now, pressure varies only along the z direction. Pressure varies linearly with depth same conclusion which I have discussed earlier, but now what is the difference, we have this extra a_z term, pressure varies linearly with the depth.

Now, it has two contributions; one is gravity acceleration due to gravity and then other is a externally imposed acceleration. Both of them together determine the variation of pressure with the distance. So, pressure varies linearly with depth, but two contributions are there. Acceleration due to gravity of course, that is natural and one externally imposed acceleration, acceleration of the lift.

Now, what are the terminology? We do not call this as hydrostatic pressure distribution, We call this as pressure distribution only, we do not use the word hydrostatic, we say pressure distribution and it is not hydrostatic why?, only if it is due to gravity alone we call that as hydrostatic, but now the pressure variation is due to acceleration due to gravity and external acceleration. And, hence we do not call that as hydrostatic just pressure distribution.

This, what we discussed pressure distribution is determined by the combined effect of gravity and external acceleration. I think you should understand the word external acceleration meaning, that is imposed by external agency like the acceleration of the lorry.

Example: (Refer Slide Time: 23:30)



Let us apply the equations for an accelerating tank, let us read the example we have a 80 centimetre high fish tank different user shown here of cross section 2 meter by 0.5 meter. That is partially filled with water is to be transported on the back of a truck, we are given the height as well 80 centimetre.

So, that is why in general this dimension is shown as b. So, we have that this tank partially filled with water transferred to the back of a truck, the truck accelerates from 0 to 90 km/h in 10 seconds, that is why I have been telling you, the acceleration is externally imposed. Either given directly or indirectly, in this case we are given indirectly in terms of change in velocity over a time period.

If, it is desired that no water spills during acceleration, determine the allowable initial water height in the tank to what level you can fill the water. And, that depends on the orientation, would you recommend the tank to be aligned with the long or short side parallel to the direction of motion, that is why both the configurations are shown. What does it mean, let us say you have a truck, which is of course accelerating the x direction would you place this fish tank in first configuration or in the second configuration.

The first configuration where the longer side is parallel to the direction of motion. And, second case is where the shorter side is parallel to the direction of motion, that is why all these configurations are shown.

So, once again theoretically equation lot of practical application, may not have commercial importance, but still we use this to do a simple decision whether should my fish tank be kept in first configuration or should be kept in second configuration and, then what is the level to which I can fill the water so, that there is no spillage. Interesting question to answer based on a theoretical equations, which we have discussed.

So, assumptions; no vertical acceleration constant horizontal acceleration, we have discussed x axis and then z axis, we discussed acceleration along x and acceleration along z, we are considering only x acceleration. So, no vertical acceleration constant horizontal acceleration, that acceleration can depend on time that will make it more complicated; so, just constant acceleration the constant value.

So, let us calculate the acceleration,

$$a_x = \frac{change\ in\ velocity}{time\ interval} = \frac{90\ X \frac{10^3}{3600} - 0}{10} = 2.5 \frac{m}{s^2}$$

We are given velocity changes from 0 to 90 km/h. So, let us convert SI units time interval is 10 second. So, you will find the acceleration as 2.5 m/s^2.

Now, let us find out the slope of the free surface

$$\frac{dz}{dx} = -\frac{a_x}{g + a_z} = -\frac{2.5}{9.8 + 2.5} = -0.255 \quad (\theta = 14.3^0)$$

What is that we have done? We have used the expression which I derived found out the slope of this free surface and expressed that in terms of angle you get 14.3 degrees. Now, as we have discussed earlier maximum vertical rise of free surface occurs at the backside at the rear side, front side there is a decrease in level. The dotted line represent the level of water to begin with, when the truck starts accelerating, the liquid level increases in the rear side decreases in the front side.

And, now it so, happens that this vertical plane the middle plane is the access of symmetry. So, you have a liquid surface and it goes this way something like tilts about the mid plane.

So, of course, that is what is shown here the liquid level to begin with and how it changes. So, at the midpoint there is no change in water level. The vertical mid plane of symmetry no rise or drop in level.

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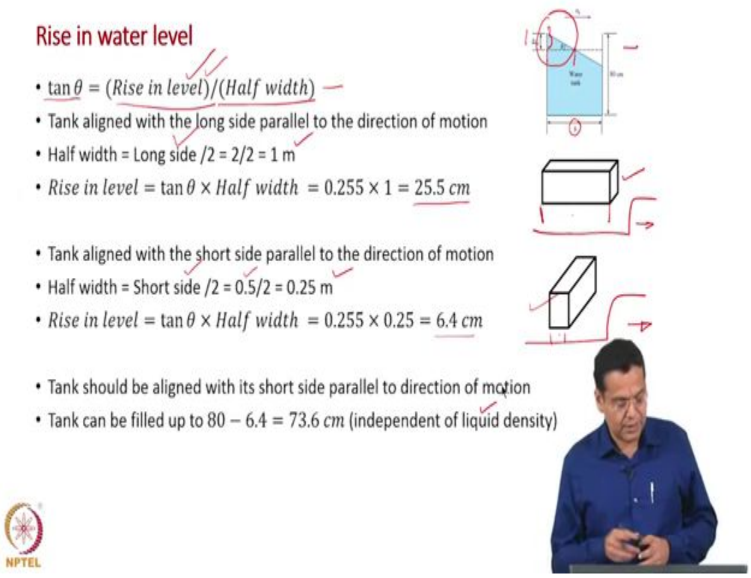

What is the rise in water let us see what is the rise in water level and how do we decide the orientation. Now, based on this triangle,

$$\tan tan\ (\theta) = \frac{Rise\ in\ level}{Half\ width}$$

Now, let us consider both the cases; first tank aligned along with the long side parallel to the direction of motion this case, you have the truck and then the fish tank is kept this way. The long side parallel to the direction of motion of course, the truck is moving in this direction.

Now, what is half width, because we have kept in this direction half width is

$$Half\ width = \frac{Long\ side}{2} = \frac{2}{2} = 1\ m$$

So, the rise in level from the previous equation is

$$Rise\ in\ level = \tan tan\ (\theta)\ \ X\ Half\ width = 0.255\ X\ 1 = 25.5\ cm$$

Now, let us take the other case tank aligned with the short side parallel to the direction that is this one. The short side is now parallel to the direction of motion what is the half width now

$$Half\ width = \frac{Short\ side}{2} = \frac{0.5}{2} = 0.25\ m$$

Now, the rise in level is,

$$Rise\ in\ level = \tan tan\ (\theta)\ \ X\ Half\ width = 0.255\ X\ 0.25 = 6.4\ cm$$

Now, becomes very clear, that we should choose this orientation, because the level in increase in water level is only 6.4 centimetre.

So, tank should be aligned with it is short side parallel to the direction of motion. And, what is the level up to which you can fill the water you can fill the water up to $80 - 6.4 = 73.6\ cm$.

So, the initial water level is 73.6 and when the truck accelerates this will go up by 6.4 centimetre. So, it will be just up to the brim 80 centimetre, there would not be any spillage here.

And, of course, remember this quantity is independent of the liquid density nowhere we considered liquid density. Nowhere liquid density played a role. So, to conclude this example a very simple example nice example, you would not imagine that this equation plays a role in deciding whether to transport a fish tank this way or this way. Also tells you what is the level up to which you can fill the water so, that there is no spillage. So, that is a very good example.

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So, let us summarize this part of the applications of Navier-Stokes and as a Navier-Stokes one simple few terms in Navier-Stokes. And, what I discussed earlier, the rigid body in motion is also discussed in the beginning chapters of a fluid mechanics book. Almost after fluids under rest this discuss, we also discussed that way, but the path we have reached there is different.

So, let us summarize this part of the application of Navier-Stokes equation, where we I considered very few terms the simplest and very simple applications. And, we considered first fluids at rest we considered we discussed the hydrostatic pressure distribution both for liquids and gases. Then, we considered fluids in rigid body motion and where the whole fluid body is subjected to motion and we discussed the pressure distribution.

Now, we can understand why hydrostatic pressure distribution here and pressure distribution here. Of course, to begin with an outline do not have been clear now we can clear distinguish why it is hydrostatic pressure distribution and pressure distribution here.