Continuum Mechanics And Transport Phenomena
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Lecture – 88
Bernoulli Equation : Irrotational Flow

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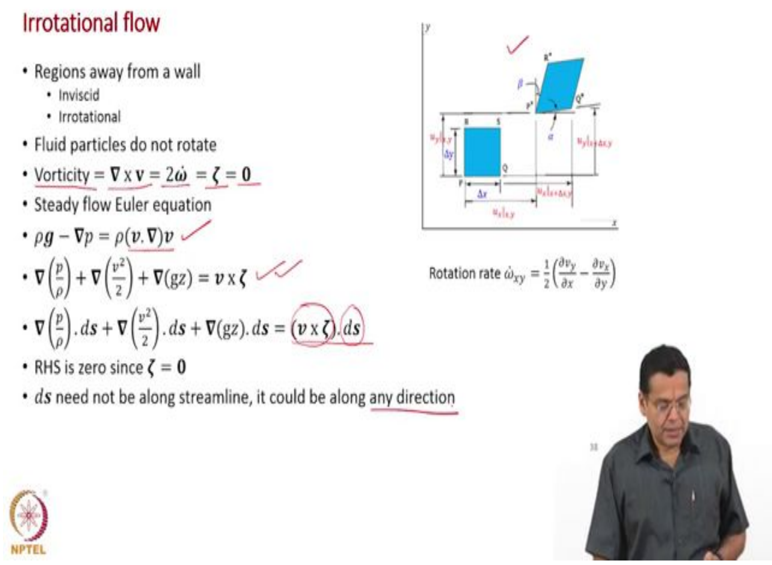

Ok, what we will now do is derive another form of Bernoulli's Equation which is for Irrotational Flow. So, we have already derived Bernoulli's equation for inviscid flow which are which happens at high Reynolds number and they are they are current regions away from the wall and which we called as inviscid flow. Now, we say in addition to that the flow is irrotational. What do you mean by that? This diagram is familiar to us, it shows a fluid element at time t and the same element at time t + Δt. It can undergo normal strain rate, shear strain rate, also undergo rotation rate.

Now, the condition what we are saying is that, if there is no rotation of the fluid element; when we say rotation rigid body rotation of the fluid element. There can be change in length, there can be change in angle which means there can be non-zero normal strain, rate non-zero shear strain rate, but rotation is 0 that is what we mean by irrotational ok. Fluid particles do not rotate that is what it means. How do you quantify it? We have seen the vorticity is given by

$$\nabla \, x \, v = 2\dot{\omega} = \xi = 0$$

So, the vorticity vector is equal to 0 vector ok, that is how we quantify irrotational flow.

So, we said the $\nabla \, x \, v$ represents rotation and now since we are considering flows which are inviscid and irrotational. So, the vorticity is 0 vector, we started with the steady flow Euler equation; so, we will do here that as well.

$$\rho g - \nabla p = \rho \, (v.\nabla) \, v$$

In fact, this derivation most of the part almost till one particular stage this derivation is same as what we derived for the other form of Bernoulli equation ok. So, this is the steady state Euler equation.

$$\nabla \left(\frac{p}{\rho}\right) + \nabla \left(\frac{v^2}{2}\right) + \nabla \, (gz) = v \, x \, \xi$$

And then we used the vector identity and obtained this form of the equation. We also divided by ρ which means the assumption of incompressible carries over here as well. So, till this stage whatever we have derived earlier applies here as well.

Now, what we did next was we took a small differential length along the stream line and took a dot product of the right hand side and the left hand side with that vector differential vector ds vector.

$$\nabla \left(\frac{p}{\rho}\right) .ds + \nabla \left(\frac{v^2}{2}\right) .ds + \nabla \, (gz) \, .ds = (v \, x \, \xi) \, .ds$$

Now, why did we do that? Our objective was to our idea was to make the right hand side 0 and because, this $v \, x \, \xi$ and the ds vector are perpendicular to each other; we took a small differential length along the stream line then took a dot product so, that right hand side is 0.

Now, for the present case the right hand side is 0 automatically, because we have taken vorticity to be 0 ($\xi = 0$). In the earlier case we made the right hand side 0 after taking a dot product with a differential length along this streamline. But, now right hand side is 0 anyway because we are considering irrotational flow which means that the ds vector need not be along streamline.

Even if you take along any line the right hand side is going to be 0, our idea of taking differential length along streamline was that to make the right hand side 0. Now, that necessary does not arise because, the vorticity is 0 so, right hand side is anyway 0. So, d s need not be along streamline, it could be along any direction and this is what you are going to see in the next slide.

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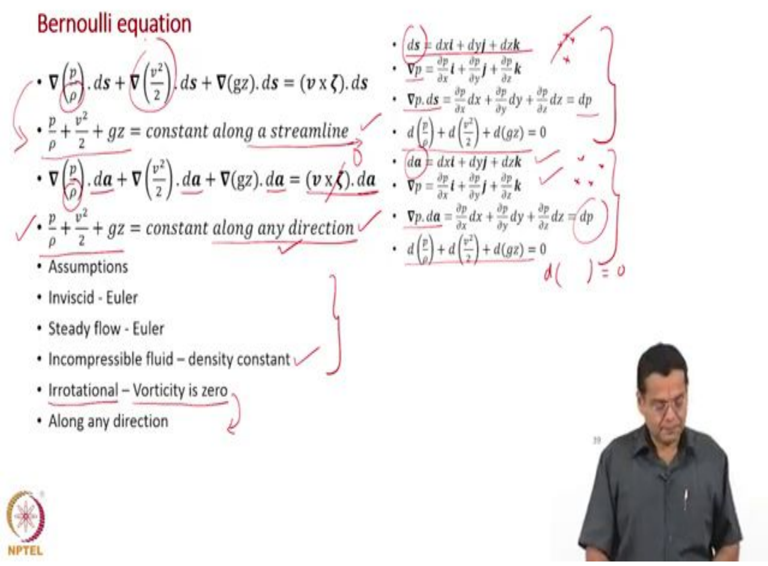

$$\nabla\left(\frac{p}{\rho}\right).ds + \nabla\left(\frac{v^2}{2}\right).ds + \nabla\,(gz)\,.ds = (v\ x\ \xi)\,.ds$$

So, let us recall what we had done for this Bernoulli's equation the last two steps, we took the dot product with the ds vector and then the right hand side was 0.

$$\nabla\left(\frac{p}{\rho}\right).ds + \nabla\left(\frac{v^2}{2}\right).ds + \nabla\,(gz)\,.ds = 0$$

And, then we had these steps where

$$ds = dxi + dyj + dzk$$

$$\nabla p.ds = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

So, we could express this equation first in terms of differential and then we said the sum of these three terms namely pressure, kinetic and potential energy per unit mass is equal to constant along the stream line. And, I also told you that we take two points which are very near that is what the differential means, but they should be along the stream length, that is where this sum is a constant along the streamline.

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = constant$$

So, that is the case if we take only inviscid flow, now because we are taking irrotational flow also; now as we discuss the previous slide I need not take ds vector which is along the stream line, I can just take any vector. So, let us say this is our streamline, I can take any two points along any direction.

$$\nabla\left(\frac{p}{\rho}\right).da + \nabla\left(\frac{v^2}{2}\right).da + \nabla\,(gz)\,.da = (v\;x\;\xi)\,.da$$

So, this da vector is such a small differential length once again, but along any direction. Why do we do that? Because, vorticity is 0 vector. So, right hand side is anyway 0 so, I can and take two points along any direction. Now, what happens? If you repeat the same steps as we have done earlier, but instead of ds vector I have a vector some da vector along any direction, once again expressed as $dxi + dyj + dzk$.

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = constant\ along\ any\ direction$$

The sum of the three energies pressure, kinetic and potential energy per unit mass is constant. But now along any direction the a vector can be along any direction and this sum is constant along any direction. So, even so purposefully I have not shown the derivation detail because, we should be able to arrive at this conclusion just based on physics itself. Based on the condition that if it is irrotational the right hand side is 0, statement number 1.

Statement number 2 because, right hand side is 0 I can take dot product with a vector along any direction, differential vector along any direction. So, immediately based on these two statements, we should be able to conclude that that some of these three energy terms is a constant along any direction.

Now, what are the assumptions?

- Inviscid that is anyway we started with Euler,
- Steady state flow because we took the steady state form of Euler equation.
- Incompressible fluid remember moment we bring in density inside the gradient in both the derivations, the density is taken as constant which is incompressible fluid.
- Irrotational – Vorticity is zero

We have now seen two forms of the Bernoulli's equation. Now, what is the extra condition here, extra assumption? Irrotational flow which means vorticity is 0, that is extra condition for the second form of Bernoulli's equation and which results the statement that the sum of the energies three energy terms, is the constant along any direction.

That is in this vorticity being zero leads to the conclusion that the sum is a constant along any direction. Vorticity is not zero, if it is not irrotational, if it is rotational flow then the sum of the energy is a constant along a streamline only, streamline only. It cannot be constant along any direction, though we said incompressible fluid it is for incompressible flow also.

Just like we discussed for a continuity equation, what do I mean by compressible flow? Incompressible fluid is something like water, flow of water, incompressible flow is like flow of air at low velocities. So, it is applicable for that condition also.

Now, like to mention that if you look at a typical fluid mechanics book Bernoulli's equation is discussed let us say the third chapter. First chapter in fundamentals, second chapter considers fluids at rest and rigid body motion, third chapter itself Bernoulli's equation is discussed. How do they derive then is that they take a streamline, consider a moment of a particle along a streamline.

They consider acceleration which is our this term this term and then for the forces they consider only pressure force and then gravity force, that is all they consider. They do not consider viscous forces, that is what we have also done; though we started with the Navier-Stokes, we wrote down the Euler equation neglecting the viscous stresses both converts to the same physical significance. But, the derivation is done in the beginning. Now, the derivation is once again done in the later part of the book as well after deriving Navier-Stokes and Euler equation and that is the derivation we are discussing.

We are not discussing the of course, in principle they are same. So, the methodology of derivation what we have adopted or what we have followed is available usually in the later part of the book. The first part of the book it is taking a particle, looking at the forces that is what we have done we have also done indirectly. So, if you want in terms of reference this derivation will be available in later part of the book. The derivation done in the first part of the book, in principle it is same physically it is same may look slightly different.

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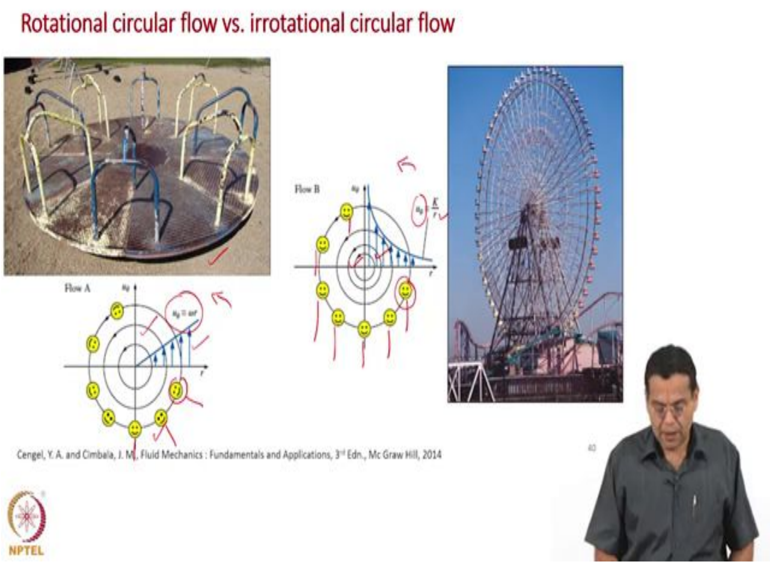

Since we are discussing about rotational, irrotational flow, I thought I will discuss this difference as well nicely brought out in this book Cengel and Cimbala. What we are going to differentiate is between two types of circular flows: one is a rotational circular flow, other is a irrotational circular flow. So, let us look at the two figures here, it says flow A and flow B. What are what is shown here are streamlines and the velocity vectors. Because, we are discussing circular rotational from we cannot avoid a polar coordinates, though we avoid it as much as possible.

So, here what is shown here is u_θ, what is u_θ? Velocity in the tangential direction, velocity in the θ direction, and that varies linearly with r, with the radial direction. And, that is what is shown here, velocity varying linearly with r and this is circular flow showing streamlines and the velocity vectors. Now, let us show other look at the flow B, where once again it is a circular flow because the stream lines are circular no doubt about it. But, in this case the velocity once again the tangential direction, in that θ direction varies inversely with r inversely with r.

And, that is what is shown here as you go away from the center the tangential velocity decreases or u_θ and r is a hyperbolic form ok, that is what is shown here. So, flow A circular, flow B circular and flow A, the tangential velocity increases with r, flow B tangential velocity is inversely proportional to or decreases with r. Now, if you identify a fluid element here and if you look at the top view, what happens? It rotates along with the flow that that is what you see here; but this rigid body rotation the whole body just rotates along with the flow. And, give a nice example that is what this book does, what is shown here is a roundabout which is ah shown here which of course, children play.

And, if you look from the top view you can see them rotating along with the roundabout. Of course, they undergo a rigid body rotation of course, just that just like this smile here also a smile is here which undergo rigid body rotation. And so, this is a rotational circular flow.

Now, let us look at the other example. In this case, if you add into a fluid element for moment let us take the front view that is a physical example also, then what do you see? They always stay up right. They do not undergo any rotation, all of them are staying upright of course, smiling face.

What is the physical example? Fairy wheel or giant wheel, where if you look at the front view of course, whomever is there and let us say every position they remain always upright. Of course, they have to remain upright based on the configuration of this fairy wheel or giant wheel. So, in this case once again it is circular flow, but the fluid elements remain always upright. So, this example where circular flow, but it is irrotational.

So, the first case circular, but undergoes rigid body rotation. The second case, if you look at the orientation of fluid element it always remains upright. Or, if you look at this giant wheel or fairy wheel, the position of the people sitting in let us say in every basket or whatever they remain upright. So, it is a good example to distinguish between circular, rotational circular flow and irrotational circular flow.