Continuum Mechanics And Transport Phenomena
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Lecture - 93
Planar Poiseuille Flow: Governing Equations

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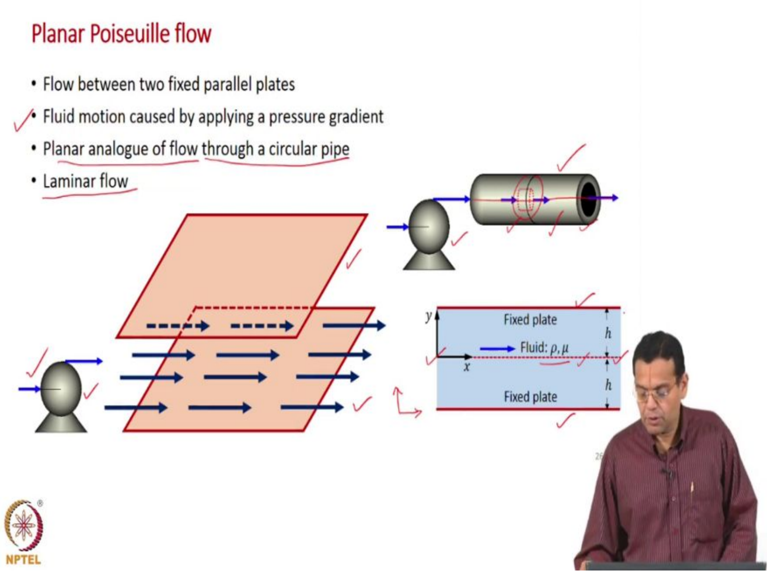

We are going to the next application of Navier Stokes equation to viscous flows where the flow takes place between two parallel plates and both of them are fixed we call this as a Planar Poiseuille flow as we have seen in the introduction.

Flow between two fixed parallel plates we have come across this geometry several times though of course, not telling the word Poiseuille flow. So, we have two plate both of them are fixed. So, now, when both of them are fixed what drives the flow, in the earlier case the top plate was moving?

So, that said the fluid in motion between the two plates when both the planes are fixed you need an external agency to make the fluid flow between the two plates. That is why second bullet says fluid motion caused by applying pressure gradient that is shown little more practically here.

We have the two plates which are fixed and then you have a external let us say pump which drives the liquid between the two plates. So, that is why we call this as fluid motion caused by applying a pressure gradient. This applied pressure gradient or this pump is what which causes the flow between two plates. Now, and that create a velocity profile between the two plates.

Now what is a practical significance of these example like we saw in the earlier case the practical applications were viscometer and then shaft bearing etcetera. Now what is the application of this I would say extremely practical example if the geometry were cylindrical.

So, what we are discussing is a Planar analogue of flow through a circular pipe of course, throughout the industry even household you have lot of pipes they are all cylindrical and what we are saying is these two plates fixed which is analogous to the fixed wall of the pipe. So, what are discussing is in Cartesian coordinates we are not discussing this flow through pipe because that requires cylindrical co ordinates. I was also shown a cylindrical shell here. So, that requires cylindrical coordinates.

What here what is shown here is also a pressure driven flow which means a pump pushes the liquid through the pipe so, that the actual practical scenario pump sending a liquid through a circular pipe. Instead of that we are discussing taking two parallel plates which are fixed because the pipe wall also does not move. Similarly here also the two plates does not move and so, that is that is why we say it is a planner analogue of flow through a circular pipe. And of course, both are pressure driven when we say pressure driven what you should understand is you have a pump which sense the liquid between the plates.

And of course, we are going to consider laminar flow we have discussed two flow regimes laminar and turbulent when we discuss about in viscid flows which was our second level of application. So, here we are going to do and we saw in viscid flow happens at high Reynolds number and now we are discussing the other extreme which is laminar flow so, this happens at low Reynolds number.

Now in terms of their geometry that two fixed plates are shown here and axis is show here and I like to mention the reason why we are chosen axis at the center $y = 0$ at the centre of the plates. Reason is that in the planner couette flow example our axis x axis was here and then y equal to 0 was at the bottom plate.

Now, the reason for choosing this axis at the centre line is that remember we are discussing this as a Planar analogue or Cartesian analogue of the cylindrical pipe, for the cylindrical pipe; obviously, let us say r = 0 will be the axis of the pipe. So, you also like to choose axis which is similar to the axis which you would choose for flow through pipe or flow through a circular pipe.

That is why axis meaning y = 0 is chosen at the centre of the plates. And so in fact, in terms of expression also it looked almost similar whatever expression we are now going derive for the velocity profile for this flow between the two parallel plates will be similar to the velocity profile which we would derive for a flow through a circular pipe.

And of course the fluid as property density ρ and then viscosity μ and now what about the distance between the plates? The distance between the plates is 2h, but in terms of y coordinates, y = h at top and y = −h at the bottom that should be kept in mind, because y = 0 is the axis.

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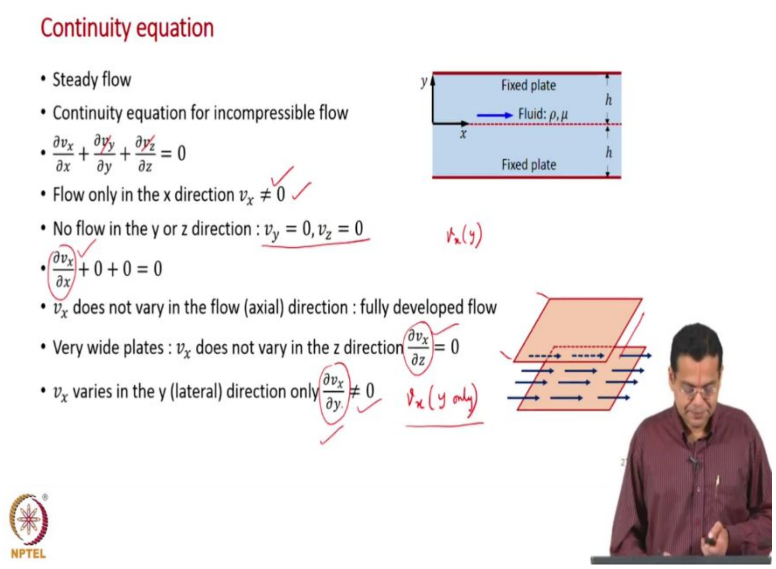

So, as in the last example and in the case of Planar couette flow will start the continuity equation and we will assume the flow is steady we will take the flow to be incompressible. Let us write the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

And like in the earlier case we will consider flow on1y in the x direction and no flow in the y direction or z direction so,

$$v_x \neq 0; \qquad v_y = 0; \qquad v_z = 0$$

Let us substitute in the continuity equation

$$\frac{\partial v_x}{\partial x} = 0$$

The same conclusion which we arrived for the Planar couette flow as well, v_x does not vary in the flow or axial direction which we says fully developed flow of course, v_x can have a profile as a function of y, but the whole profile will not change in the flow direction.

Now we consider very wide plates what do you mean by that this is a width that is very wide which means that there is no variation of the x velocity, v_x does not vary in this direction. So,

$$\frac{\partial v_x}{\partial z} = 0$$

Which means that v_x varies in the y or lateral direction only

$$\frac{\partial v_x}{\partial y} \neq 0$$

So, we have v_x as a function of y only. Now, we said $v_x \neq 0$ and by taking no flow in the y and z direction based on the continuity equation we concluded there is no variation of v_x along x and we made assumption that v_x does not vary along z. And so, leaves us with the condition that v_x varies only along y.

Now, if you look at this discussion this exactly same as what we did for the Planar couette flow there is no change. In fact, the same bullet us are taken in terms of text it is same only the figures are changed. In both the cases the conditions $v_x \neq 0$, $v_y = 0$, $v_z = 0$ and $\frac{\partial v_x}{\partial x} = 0$, $\frac{\partial v_x}{\partial z} = 0$, $\frac{\partial v_x}{\partial y} \neq 0$ they are all exactly same.

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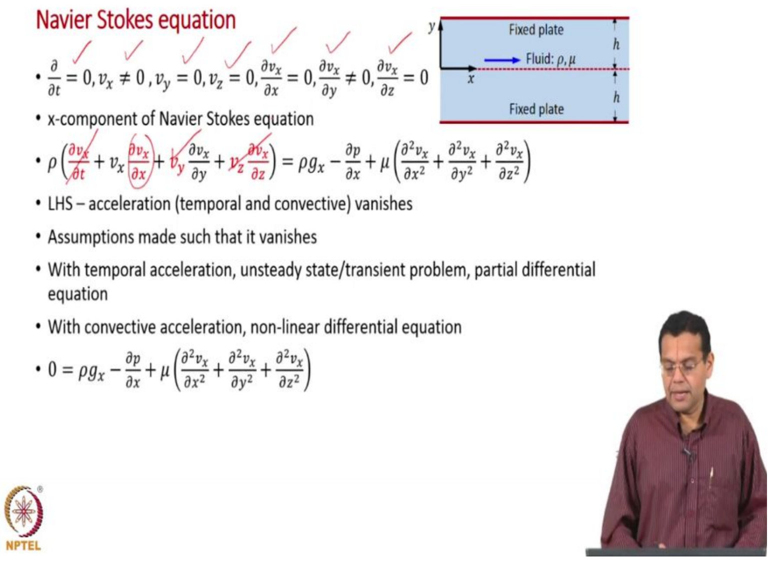

Now let us proceed with the Navier Stokes equation, let us state all the conditions either assumptions or what we arrived from the continuity equation. So,

$$\frac{\partial}{\partial t} = 0; \quad v_x \neq 0\,, \ v_y = 0\,, \ v_z = 0\,, \ \frac{\partial v_x}{\partial x} = 0\,, \ \frac{\partial v_x}{\partial z} = 0\,, \ \frac{\partial v_x}{\partial y} \neq 0$$

Let us write the x component of Navier Stokes equation

$$\rho\left(\frac{\partial(v_x)}{\partial t} + v_x\frac{\partial(v_x)}{\partial x} + v_y\frac{\partial(v_x)}{\partial y} + v_z\frac{\partial(v_x)}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, if we apply the assumptions then the entire left hand side vanishes. So, LHS like in the previous case of course, assumptions all are same assumptions all are same.

So, assumptions are made such a way that the acceleration term in the left hand side vanishes both the temporal acceleration and the convective acceleration vanishes. As I said assumptions are made such that the left hand side vanishes, we also discussed what is the difficulty if we are not made such assumption.

If the transient term was there it would have resulted in partial differential equation, the convective acceleration terms are non-linear that is more difficult to solve.

$$0 = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, the left hand side becomes 0 exactly same as in the last case no difference at all once again this slide is also in terms of bullets, contents, it is same as last example except for the figure.

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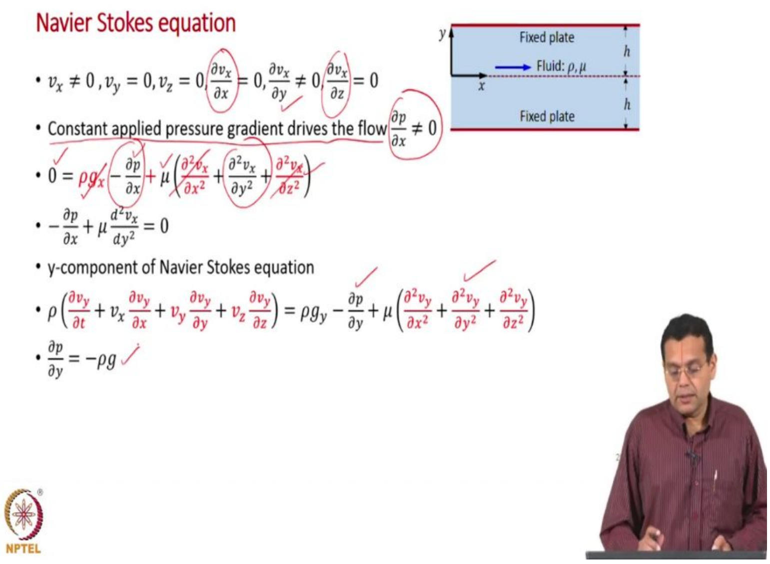

Now, let us write the conditions

$$v_x \neq 0\,,\ v_y = 0\,,\ v_z = 0\,,\ \frac{\partial v_x}{\partial x} = 0\,,\ \frac{\partial v_x}{\partial z} = 0\,,\ \frac{\partial v_x}{\partial y} \neq 0$$

Now we are going to look at the right hand side of the x component of Navier Stokes equation this where the difference comes and the first case which is the Planar couette flow case there was no applied pressure gradient.

So, there was no pressure drop along the length of the flow, in this case this is where in fact, we started the whole description that is the constant applied pressure gradient which drives the flow and as I told you have a pump which pushes the liquid between plates

$$\frac{\partial p}{\partial x} \neq 0$$

So, the $\frac{\partial p}{\partial x}$ was equal to 0 in the case of couette flow. In the case of Poiseuille flow $\frac{\partial p}{\partial x}$ is not equal to 0, pressure decreases along the direction of flow. So in fact, $\frac{\partial p}{\partial x}$ will be negative, because pressure will have to decrease along the flow because it causes the fluid to flow between the plates and hence there is a nonzero pressure gradient along the direction of flow.

$$0 = 0 - \frac{\partial p}{\partial x} + \mu\left(0 + \frac{\partial^2 v_x}{\partial y^2} + 0\right)$$

So, left hand side is 0 as we have seen in the last slide, right hand side of course, gravity acts along y direction. So, $\rho g_x = 0$ because $g_x = 0$. Now, coming to the viscous stress terms on the right hand side $\frac{\partial v_x}{\partial x} = 0$ so, second derivative is certainly 0, v_x does not vary with z so second derivative is once again 0. So,

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2} = 0$$

So, that is the simplified form of the Navier Stokes equation in the x direction, the entire left hand side became 0, right hand side we have the pressure gradient term and the viscous stress terms. Let us look at the y component of Navier Stokes equation

$$\rho\left(\frac{\partial(v_y)}{\partial t} + v_x\frac{\partial(v_y)}{\partial x} + v_y\frac{\partial(v_y)}{\partial y} + v_z\frac{\partial(v_y)}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

$$0 = \rho g_y - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0)$$

v_y is 0 so, the left hand side is 0 and in the right hand side we do have the gravity term of course, we have the pressure gradient term and of course, the viscous stress terms are 0 because v_y is 0. So, y component simplifies to

$$\frac{\partial p}{\partial y} =- \rho g$$

Once again this particular part are the y component of the Navier Stokes equation is same as what you have seen for the Planar couette flow case.

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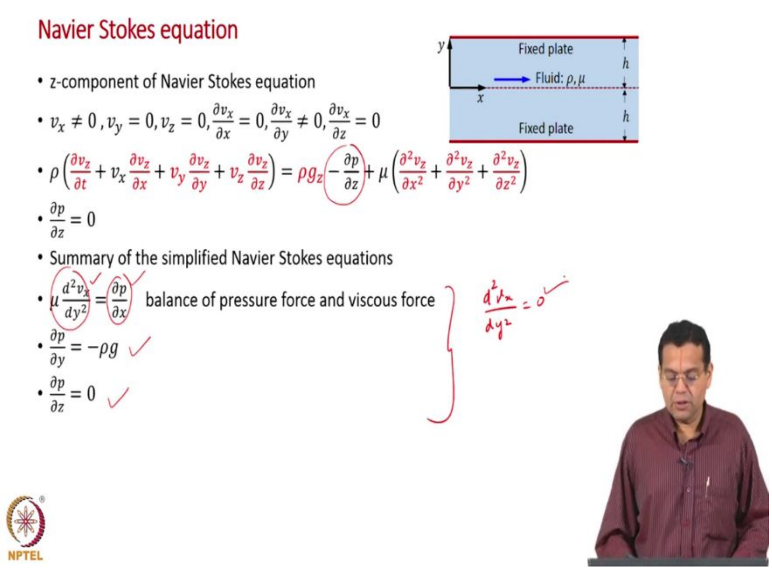

Now, coming to the z component of Navier Stokes equation let us write down the conditions

$$v_x \neq 0\,,\ v_y = 0\,,\ v_z = 0\,,\ \frac{\partial v_x}{\partial x} = 0\,,\ \frac{\partial v_x}{\partial z} = 0\,,\ \frac{\partial v_x}{\partial y} \neq 0$$

So, the z component of Navier Stokes equation is

$$\rho\left(\frac{\partial(v_z)}{\partial t} + v_x\frac{\partial(v_z)}{\partial x} + v_y\frac{\partial(v_z)}{\partial y} + v_z\frac{\partial(v_z)}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

$$0 = 0 - \frac{\partial p}{\partial z} + \mu\,(0 + 0 + 0)$$

This is simplified to

$$\frac{\partial p}{\partial z} = 0$$

Once again this the z component of Navier Stokes equation is same as what we have seen in the previous case of Planar couette flow.

Let us summarize all the three Navier Stokes equations x component we have

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2} = 0$$

What does it tell you? It is the balance of pressure force and then viscous force. We have pressure force and then the viscous force and the Navier Stokes equation has got simplified to the balance of just pressure forces and viscous forces. Of course, both of them are net we should say net pressure force per unit volume net viscous force per unit volume. And y direction it is just a hydrostatic equation which is

$$\frac{\partial p}{\partial y} = -\rho g$$

And, z direction,

$$\frac{\partial p}{\partial z} = 0$$

So, these are the simplified Navier Stokes equation compare the Planar couette flow case the last two equation are same.

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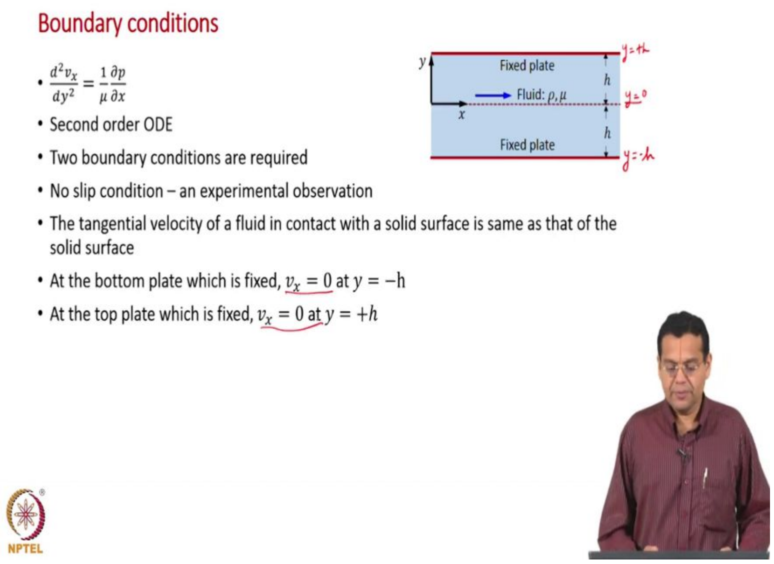

Now, we will have to state the boundary conditions and that is the equation we have

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

It is a second order ordinary differential equation. We need two boundary conditions to solve this second order ODE and as in the case of the Planar couette flow we will use a no slip condition and as I mention no slip condition is an experimental observation which tells you that the tangential velocity of a fluid in contact to the solid surface is same as that of the solid surface.

So, we will come across no slip condition very frequently any almost any fluid mechanics problem you take you will come across no slip conditions. So, let me repeat in this case your both the plate are fixed, what this condition says is the tangential velocity the tangential velocity of a fluid contact the solid surface is same as that of the solid surface. And in this case the both the plates are fixed. So, velocity of fluid layer just in contact with the bottom plate is 0 and velocity of the fluid layer just in contact with the top plate that is also 0.

So, at the bottom plate which is fixed $v_x = 0$ at $y =- h$, and so, at the top plate $v_x = 0$ at $y =+ h$. So, both at the bottom plate and the top plate $v_x = 0$.