Continuum Mechanics And Transport Phenomena
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Lecture – 99
Linear Momentum Balance: Fluid Mechanics vs. Momentum Transport
Part 1

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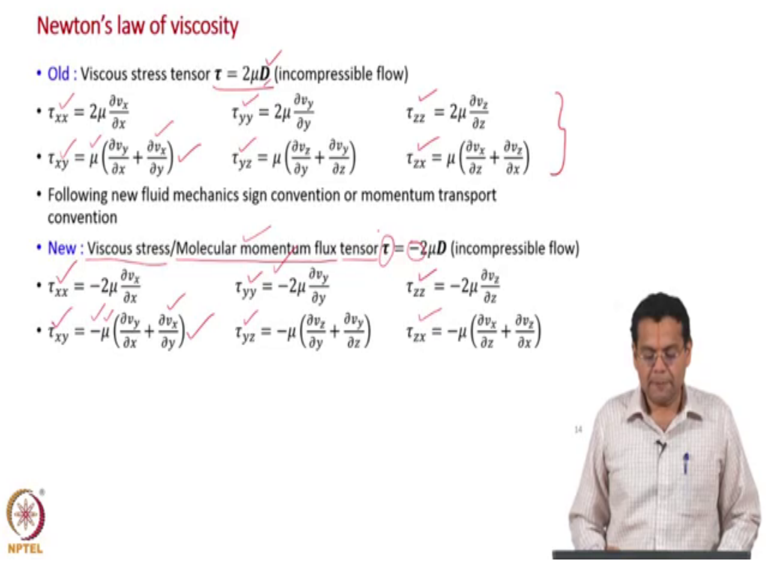

Let us write the Newton's Law of Viscosity; first on the old fluid mechanics sign convention. The viscous stress tensor is related to the strain rate tensor by the equation

$$\tau = 2\mu D$$

And in terms of components these are the equations.

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x}; \qquad \tau_{yy} = 2\mu \frac{\partial v_y}{\partial y}; \qquad \tau_{zz} = 2\mu \frac{\partial v_z}{\partial z}$$

$$\tau_{xy} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad \tau_{yz} = \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \quad \tau_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial z}\right)$$

Left hand side we have the components of the viscous stress tensor, and right hand side or in terms of the components of the strain rate tensor, which of course, we are discussed we are familiar with this.

Now, what we will do is we will rewrite the Newton's law of viscosity following the new convention. What is that? New fluid mechanic sign convention or momentum transfer convention we have already reconcile both of them. And both of them have the same sign. So, what is new? The same equation, but with the negative sign here. So, now,

$$\tau = -2\mu D$$

How do we interpret τ? If you are saying that new fluid mechanic sign convention it is viscous stress tensor no change in that at all, only sign convention has changed. But if you have same momentum transport convention then it is interpreted as molecular momentum flux tensor.

So, τ has two interpretations, one is viscous stress other is molecular momentum flux. For fluid mechanics is always viscous stress only the sign convention is different, but if you are saying momentum transport it represents the molecular momentum flux. So, now, just rewrite the same set of 6 equations including a negative sign that is all, no change at all other than including a negative sign.

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x}; \qquad \tau_{yy} = -2\mu\frac{\partial v_y}{\partial y}; \qquad \tau_{zz} = -2\mu\frac{\partial v_z}{\partial z}$$

$$\tau_{xy} = -\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad \tau_{yz} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \quad \tau_{zx} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial z}\right)$$

So, now the left and sides are the components of the viscous stress tensor, once again if it is the new fluid mechanic sign convention. If you take a momentum transport convention you interpret the left hand sides as components of the molecular momentum flux tensor that is the new viewpoint.

So, we have all our discussions based on this one-dimensional Newton of law viscosity. We are just extended that to the three-dimensional form of Newton's law viscosity. So, only conclusion from this side is whatever we have discuss for the simple form has been extended to the more general 3 D form by including negative sign. One other important point in this slide is that tau is now interpreted as the molecular momentum flux tensor.

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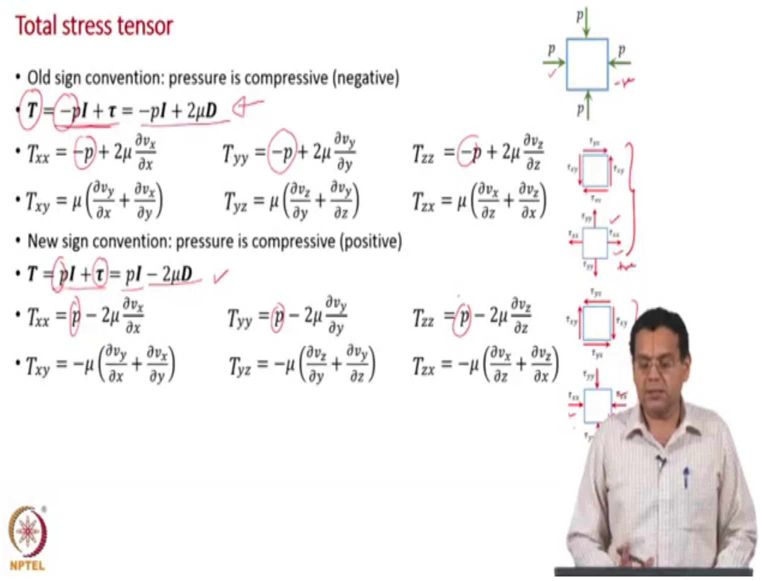

Now, how do we write the total stress tensor? So, when I write the total stress tensor we are going to compare the old fluid mechanic sign convention and the new fluid mechanics sign convention. So, let us look at it. Now, where is the difference? We have discuss already τ only difference can come from the pressure term. Now, based on the old fluid mechanic sign convention, since pressure is compressive it is negative.

$$T =- pI + \tau =- pI + 2\mu D$$

Remember pressure is always compressive that does not depend on whether old sign convention new sign convention it is all, it always acts into the control volume. So, the first two figures are for the old sign convention.

So, in this case if you take a positive phase force is along the positive axis which means that tensile is positive compression is negative. So, if you now write the components,

$$T_{xx} =- p + 2\mu \frac{\partial v_x}{\partial x}; \qquad T_{yy} =- p + 2\mu \frac{\partial v_y}{\partial y}; \qquad T_{zz} =- p + 2\mu \frac{\partial v_z}{\partial z}$$

$$T_{xy} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad T_{yz} = \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \quad T_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial z}\right)$$

This is same as what you have see in the previous slide just add −p to the normal stresses, now that change. There is no change in the shear stresses because p is just a normal stress.

Now, the new sign convention. Pressure is again compressive that cannot change please keep that in mind, but now becomes positive. Why? The last two figures are shown based on the new sign convention. Here on a positive phase forces along the negative direction. Now, look at these two arrows, they are representing compressive forces and they have, they represent a positive stress tensor component. So, the pressure which is compressive the new sign convention becomes positive.

So, now let us write down the total stress tensor.

$$T = pI + \tau = pI - 2\mu D$$

So, in terms of hierarchy first all the discussion of based on one dimension Newton's law viscosity. The previous slide we have explain that to the three-dimensional form. Now, you are further extending and writing the total stress tensor, so, very simple.

$$T_{xx} = p - 2\mu\frac{\partial v_x}{\partial x}; \qquad T_{yy} = p - 2\mu\frac{\partial v_y}{\partial y}; \qquad T_{zz} = p - 2\mu\frac{\partial v_z}{\partial z}$$

$$T_{xy} = -\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right); \quad T_{yz} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right); \quad T_{zx} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial z}\right)$$

Whatever expression we had in the previous slide based on new sign convention add a plus p to the normal stress components, because we are still fluid mechanics new sign convention, tau represent only viscous stresses. So, so to summarize this slide pressure is compressive, but it is negative in the old sign convention, positive in the new sign convention, that kind of summarizes this slide.

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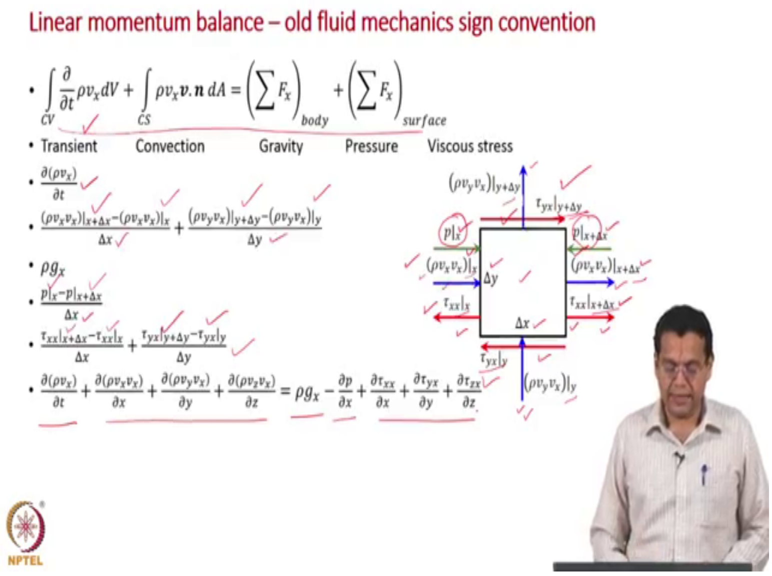

Now, what is the implication of our discussion on the liner momentum balance? Now, what we are going to do now is derive the linear momentum balance in 3 different ways. First is fluid mechanics old sign convention, then fluid mechanics new sign convention and then the momentum transport convention. Now, this just a recall whatever we are done earlier for 3D case, three-dimensional control volume here we are going to do for two-dimensional control volume, so we will just only list the important steps. So, we will start with the integral form of the linear momentum balance,

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

We took the integral form on liner momentum balance and applied to the applied that for us small control volume, three-dimensional control volume.

What we have here is a two-dimensional control volume, for sake of simplicity. So, what we have is the integral linear momentum balance for a fixed control volume, the transient term, the convection term in the left hand side then right hand side we have body forces because of the gravitational force and then the surface forces because of pressure and viscous stress. I am stressing this you will understand why I stress this here. So, let repeat transient convection of the left hand side, right hand side body and surface forces and body force because of gravity, surface force because of pressure and viscous stress.

Now, let us look at this control volume quickly all the required quantities are shown here in one diagram. Earlier when we derived we had different control volumes. Now, let us look at them. The lengths are Δx and Δy and then this blue arrow marks represent the convective momentum in and then out. Remember all of them refer to x momentum, the second velocity is v_x and this is because of mass flow in x direction and this because of mass flows in y direction. Entering at x leaving at x + Δx , entering at y leaving at y + Δy; so, all the blue colours represent the convective momentum.

Now, all the forces are shown. First, all the surface forces are shown. Let us look at pressure which is easier. Pressure is always compressive. So, it is acting into the control volume, this acting into the control volume at x and x + Δx. Now, the viscous stresses are shown and because it is the old sign convention on a positive phase the force along positive axis and similarly here on a positive phase forces along the positive axis.

So, this diagram is known to us. In terms of differences earlier we did for 3D, now it is this is for 2D. Earlier we are shown the convective momentum flux separately, pressure separately, the viscous stresses separately, now it is 2D all of them are put together. Now, if you simplify the transient term whatever terms we are now going to list all per unit volume that is after dividing by $\Delta x\Delta y\Delta z$, and so the first terms becomes

$$\frac{\partial(\rho v_x)}{\partial t}$$

What about second term? Remember, it is net rate at which momentum leaves the control volume, so

$$\frac{\left((\rho v_x v_x)|_{x+\Delta x}-(\rho v_x v_x)|_x\right)}{\Delta x}+\frac{\left((\rho v_y v_x)|_{y+\Delta y}-(\rho v_y v_x)|_y\right)}{\Delta y}$$

The body force per unit volume is

$$\rho g_x$$

Coming to the net force due to pressure, it is it is along positive x axis this along negative x axis so

$$\frac{\left(p|_x-p|_{x+\Delta x}\right)}{\Delta x}$$

Now, coming to the stresses this is positive

$$\frac{\left(\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_{x}\right)}{\Delta x} + \frac{\left(\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_{y}\right)}{\Delta y}$$

Now, if you substitute all of them, take limit, etcetera.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

This we have already discussed. Of course, written here for 3D. We have discussed for 2D, but writing for 3D. The transient term and then the convection term, the gravity term, the pressure term and the viscous stress term; so, this kind of recall only. We already discuss this for three-dimensional case.

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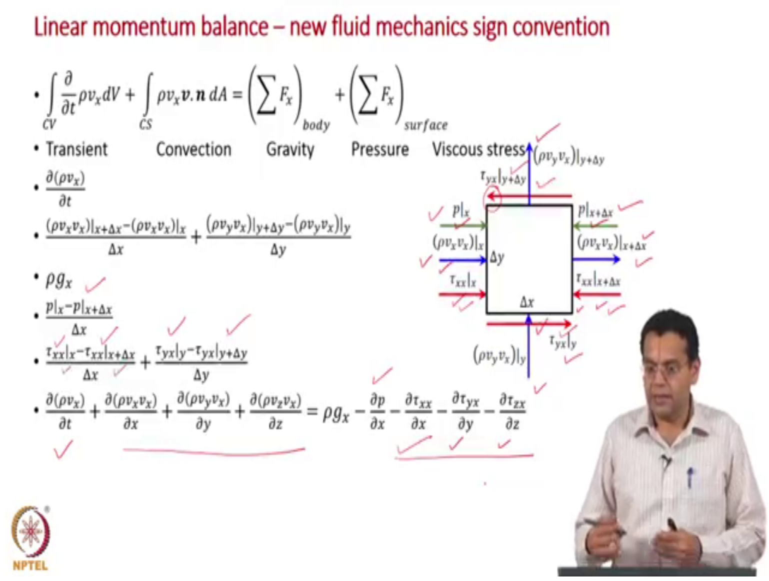

Now, what do we do? Same derivation, but using the new fluid mechanics sign convention. So, let us write the integral linear momentum balance.

$$\int_{CV} \frac{\partial}{\partial t} \rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

Left hand side we have the transient term the convection term, of course, no change because conceptually we are same, only sign convention is different. So, right hand side still we have gravity and the surface forces namely pressure and viscous stress. Once again I am stressing here, next slide this going to change. When you go to momentum transport the first equation is going to change.

So, once again we want to say transient convection in the left hand side, gravity, pressure, viscous stress on the right hand side. Now, look at the control volume. In terms of convective momentum no change, in and out, no change. Respective pressure once again, there is no change. Why there is no change in pressure? Because it is always compressive. Now, when you come to the viscous stress, the new sign convention has been adopted. What does it mean? Once again always the stresses are show in a positive sense.

So, on a positive phase as shown this force acting towards negative x axis, on negative phase I have shown this force acting towards positive x axis. Similarly, for the shear stresses on a positive phase force is along negative x axis, on a negative y phase force is along positive x axis. So, new sign convention has been adopted to show the viscous stress,, no other different at all compared to the previous representation.

So, now transient term is same.

$$\frac{\partial(\rho v_x)}{\partial t}$$

The convective momentum term is also same, remember it is out minus in. Once again I want to stress this. So, it is out minus in, net rate at which convective momentum leaves the control volume and so, we have taken what is leaving minus entering in both directions, same as previous slide

$$\frac{\left((\rho v_x v_x)|_{x+\Delta x}-(\rho v_x v_x)|_x\right)}{\Delta x}+\frac{\left((\rho v_y v_x)|_{y+\Delta y}-(\rho v_y v_x)|_y\right)}{\Delta y}$$

The gravity force is also same

$$\rho g_x$$

Pressure term is also same as previous slide, it is compressive

$$\frac{\left(p|_x-p|_{x+\Delta x}\right)}{\Delta x}$$

Now, when you come to the net viscous force terms because the sign convention has changed

$$\frac{\left(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}\right)}{\Delta x} + \frac{\left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}\right)}{\Delta y}$$

Now, take limit $\Delta x \Delta y \Delta z \rightarrow 0$. We will get

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z}$$

So, the last three terms have changed compared to the previous slide because they used a new sign convention. Earlier all the derivatives are were plus, now all the derivatives are negative, because of the changed sign convention.