Mathematics for Chemistry

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Module - 05
Lecture - 24
System of 1st order ODES, Matrix Method

We have seen First Order Differential Equations, and we have looked at various methods to solve them.

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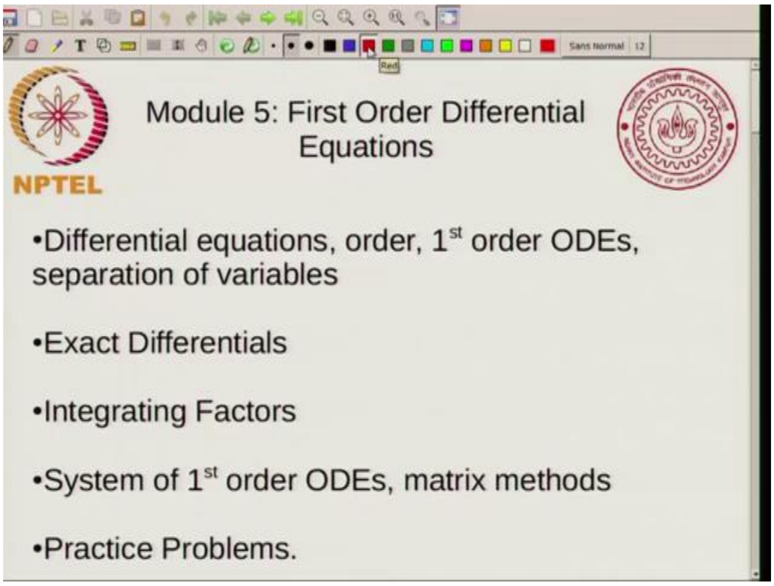

So, what the first method that we looked at was what was called separation of variables, then we looked at using the ideas of exact differentials to solve first order differential equations. After that we looked at how you can find integrating factors, even when your first order differential equation cannot be separated or it is not exact you can use integrating factors to solve first order differential equations.

Today I will talk about the system of first order differential equations and then I will talk about linear equation and how you can use matrix methods to solve linear first order differential equations, or even a system of linear first order differential equations. So, let us get to system of first order differential equations.

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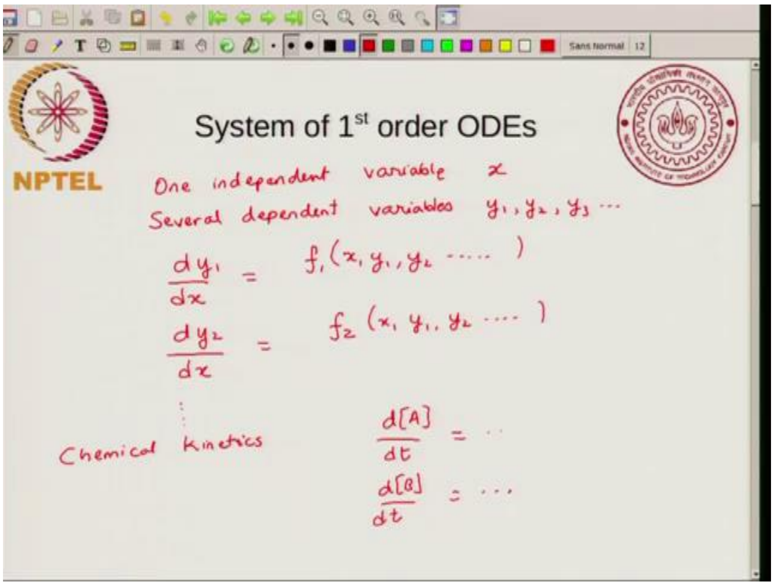

So, the idea is the following. Suppose you have an one independent variable, let me call it x and you have several dependent variables y 1, y 2, y 3 etcetera. Then your system of first order differential equation looks something like this. So, it might look like d y 1 by d x is equal to f 1 of in general x y 1 y 2 some function. So, it is the derivative of the first dependent variable. There is only one independent variable, so that is why I just have d y by d x. So, this is one equation. Then you have d y 2 by d x will be some other function f 2 of x y 1 y 2 so on and so on. All the way up to how many other variables you have. So, this is a system of first order differential equation.

So, what you mean is it is not just one first order differential equation its several first order differential equations and that is because you have several dependent variables, but you have only one independent variable. So, this is a system of first order differential equation. And this is something that we encounter very often. So, you might remember your chemical kinetics; in chemical kinetics we often have something like d A by dt is equal to something, then you have d B by dt equal to something and so on.

So, we often encounter such system of first order differential equations especially in chemical kinetics, but also in several other areas you might encounter them. Now we look at a particular class of equations. So, look at a particular class of equations called linear first order differential equations. Before that I will just mention one thing that you can do something; you can convert a higher order differential equation to first to a system of first order differential equations.

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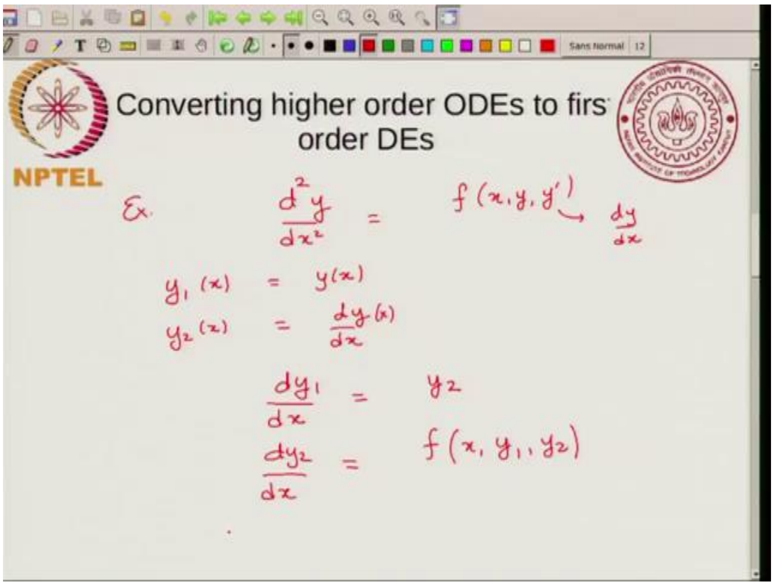

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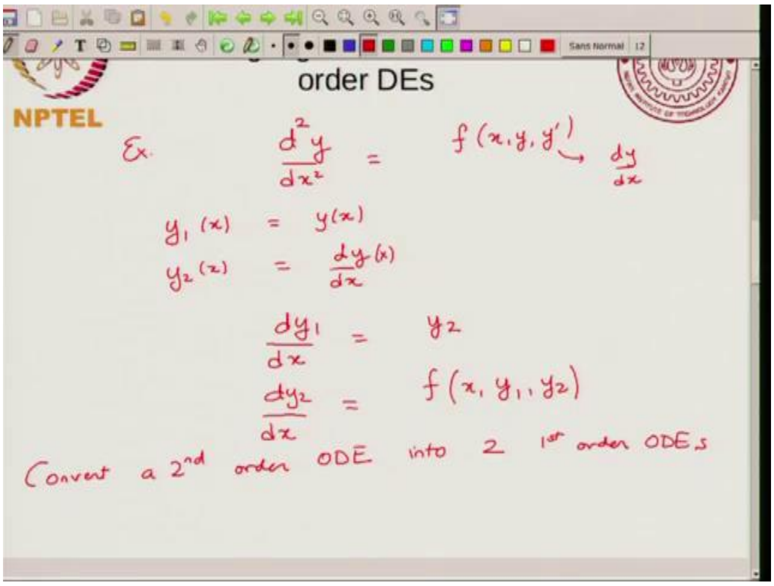

So example; suppose you have a differential equation d square y by d x square is equal to let us say f of x y and I will just say y prime. So, at some function of x y and y prime this is the second derivative, it is some function of x y and the first derivative of y. So, y prime is d y by d x. Suppose you have a second order differential equation. Now you can say let y 1 of x equal to y of x and let y 2 of x equal to d y by d x; it is a function of x.

Suppose you use this substitution, then your system of equations becomes. So, since y 2 is d y by d x I can write this equation in the following form. So, d y 1 by d x is equal to y 2, and I can write d y 2 by d x; d y 2 by d x is d square y by d x square. So, this is my f of x y 1 this is y 1 this is y 2. So, what I did is I converted my second order differential equation to two first order differential equations. I converted it to two first order differential equation it is as though you can imagine that x is the only dependent variable and y 1 and y 2 are two independent variables or x is the independent variable; y 1 and y 2 are two dependent variables. And this has exactly the form that we had earlier; this has exactly this form, so d y 1 by d x is some function of all these variables d by 2 by d x is some function of all these variables.

So, same way you have d y 1 by d x. Now in this case it depends only on y 2 it does not depend on y 1 or x, but that is also fine. And then similarly you have d y 2 by d x is some function of x y 1 and y 2. So, what this means is that you can convert second order ODE into 2 first order ODE's. And you can extend this; you can convert the third order ODE into three first order ODE's and so on. You can do this for third order differential equations two. And you can extend it to any order.

So, basically you can convert a higher order differential equation into first order differential into a system of first order differential equations. Now, let us look at linear first order differential equations.

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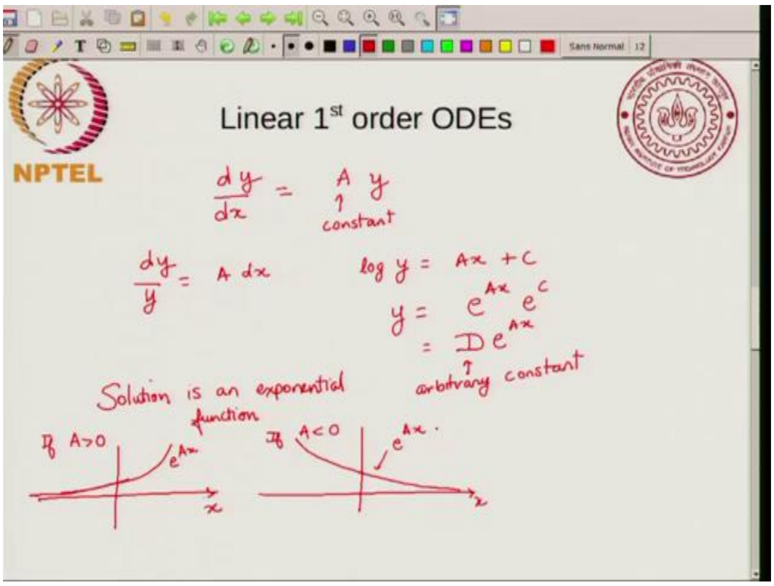

So, a linear first order differential equation means it looks it has a form d y by d x is equal to some constant times y. So, each term has y up to power 1. Now if you have this equation then you can solve this. So, A is a constant. So, you can solve this linear first order differential equation by writing d y by y is equal to a d x or you can write log of y is equal to Ax plus a constant of integration or I can take exponential on both sides; I can write y is equal to e raised to Ax into e raised to c. And I can just call e raised to c as some constant d e raised to x. So, d is an arbitrary constant.

In other words, a linear first order differential equation the solution is exponential. So, solution of linear first order differential equation is an exponential function. I will just write it here solution is an exponential function. Now, if A is greater than 0 then the solution; is to greater than 0 then it is exponentially increasing. So, for x greater than 0 it looks like this. So, along the positive x axis it looks like this. Now when x is less than 0 then it will just come all the way to 0. So, when x goes to minus infinity it will go to 0. So, you will have some function that looks like this. If A is greater than 0 or if A is less than 0 then you will have something which looks exactly the opposite way; so it goes to infinity and then here it goes to 0.

That is what your function looks like, that is what the solution e raised to x. So, this is the graph of e raise to x, this is also e raised to x. So, for a linear first order differential equation with a constant coefficient that has e raised to x as the solution.

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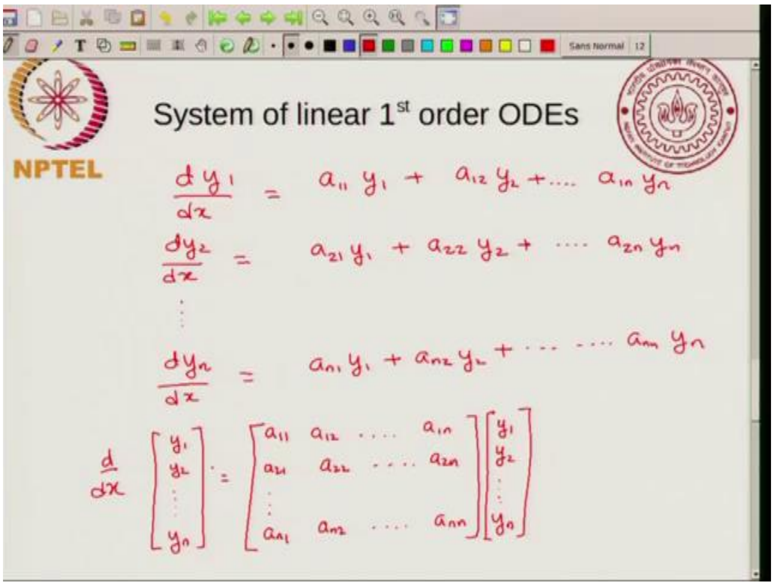

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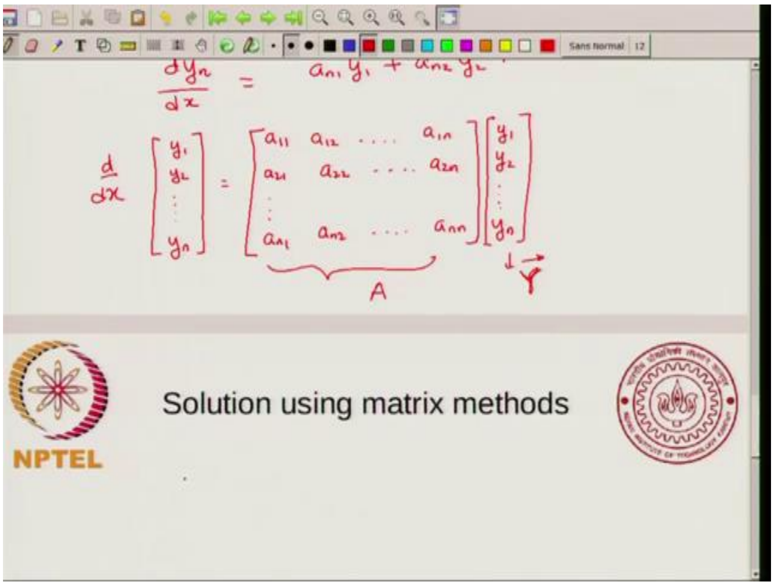

What happens if you have a system of linear first order differential equations? Suppose I had d y 1 by d x is equal. So, this is a 11 times y 1 plus a 12 times y 2 plus let us say you go all the way up to a 1n times y n. So, y 1 y 2 etcetera up to y n are the dependent variables. And what I did is I allowed the differential equation for y 1 as some constant times y 1 plus some constant times y 2 plus all the way. So, this is a linear first order differential equation, it is linear all these dependent variables. Similarly if I have d y 2 by d x is equal to a 21 y 1 plus a 22 y 2 plus all the way up to a 2n y n. And you can go all the way to d y n by d x is equal to a n1 times y 1 plus a n2 times y 2 all the way up to a nn times y n.

Now, this system of differential equations, I can write it in a very nice form. I will write this in the following form. So, I will make a column vector that looks like y 1 y 2 up to y n, and I will take a d by dt of this d by d x of this. Basically, I take this is same as taking d by 1 by d x d y 2 by d x and so on. And I can write this using our matrix methods as a 11, a 12 a 1n, a 21 a 22 a 2n, a n1 a n2 a nn this times y 1 y 2 up to y n. So, what I have on the left hand side is a derivative with respect to x. So, I do not have just the vector I have the derivative with respect to x, so each term you have to take the derivative with respect to x. And what I have on the right hand side is just this matrix multiply by y 1 y 2 up to y n.

Notice: what I did is I converted this system of linear first order differential equations to something that looks like a matrix equation. Now, I can write this in a slightly different form. Maybe call this as y I will just say capital y which is a vector. And let me call this matrix A.

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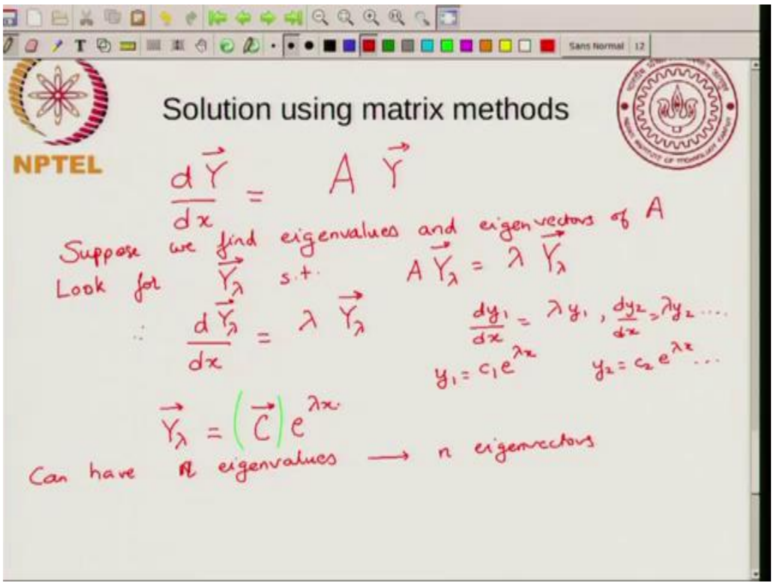

So, then I can write d of Y vector divided by d x which is the left hand side that I can write as A times y. Now, this looks just like a matrix equation; this is A times Y which is equal to d Y by d x. So, how do you solve this differential equation? Now, one of the things I had said when we discussed matrixes is that one of the most powerful techniques in matrixes is that of finding eigenvalues and eigenvectors.

So, suppose we find eigenvalues and eigenvectors of A. What do you mean by finding eigenvalues and eigenvectors? We look for of Y lambda these eigenvectors such that A times Y lambda is equal to lambda times Y lambda; so looking for vector such that this is satisfied. Now for these vectors, so therefore we can immediately write d Y lambda divided by d x is nothing but lambda times Y lambda. So, why did I write this, because you look at this equation; this is just suppose Y was not a vector then you would say d Y by d x equal to lambda Y the solution it is a linear equation the solution is exponential. But now we notice that this is an eigenvalue lambda. So, for each of the components of Y you have a linear equation.

So the solution: that means, each of the components satisfies an exponential equation. That means, I can say that d y 1 by d x equal to lambda y 1 d y 2 by d x equal to lambda y 2 and so on. And the solution of this I can write as y 1 is equal to I will just call it c 1 e to the lambda x y 2 equal to c 2 e to the lambda x and so on. So, what that means, is that I can write my Y lambda as some coefficient vector I will just call it C times e to the lambda x, where c is basically a vector which has c 1 c 2 up to c n.

So, what we said is that this general equation has; if you are able to find the eigenvalues then this is one form y lambda equal to c e to the lambda x is one form that actually satisfies the differential equation. If y has n columns and A is an n by n matrix then you can have n eigenvalues. So, you can have N or n eigenvalues. General, yes it is correct that you can have n eigenvalues each of the coefficients of these eigenvalues will be seen here. So, you can have n eigenvalues and you can have n eigenvectors and corresponding n eigenvectors. So, what is it that I want to get across here? If you know the eigenvalues and eigenvectors, then I should write this slightly differently. So, I will just put this in brackets.

What I want to say is the following: that if you know the eigenvalues and eigenvectors then you can write the solution of this differential equation. I just emphasize to this once again. Remember y 1 is nothing but a one of the coefficients of the eigenvectors. And since you know y 1 you can actually write the general solution.

So, what is the idea of this whole thing? So, you start with d y by d x you write it as A times y, you solve for the eigenvalues and eigenvectors. Once you are solved for the eigenvalues and eigenvectors then you know that any eigenvalue and eigenvector will have this as a solution. So, the solution can be written as some constant times e to the lambda x. That would be a solution of this differential equation. And so if you know the eigenvalues you can solve the differential equation using these matrix methods.

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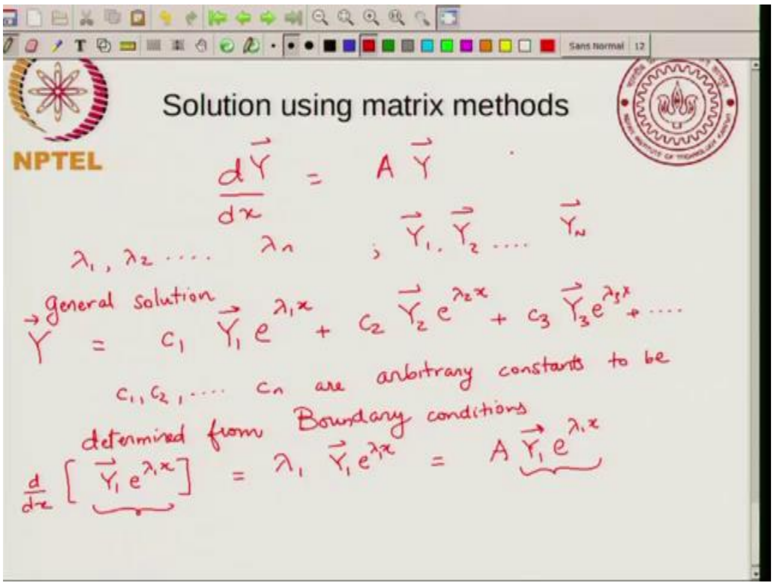

So, what is the strategy? You have d Y by d x equal to A times Y, so this was your equation. And you get eigenvalues let us say lambda 1 lambda 2 up to lambda n, and you get the corresponding eigenvectors y 1 y 2 up to y n. And then you can write your general solution: now what is the general solution? A general solution will have undetermined constants and it is not hard to show that you can write the general solution as a linear combination of all these eigenvalues multiplied by the eigenvector. So, each term will look like y 1 e to the lambda 1 x times c 1 plus c 2 times y 2 e to the lambda 2 x plus c three times y 3 e to the lambda three x and so on.

So, general solution I will just write y is equal to this; where c 1 c 2 up to c n are arbitrary constant, to be determined from boundary conditions. So what did I do? I took this y and I wrote this let us just consider one of these terms. Suppose I just consider y 1 e to the lambda 1 x. Suppose, I just look at y 1 e to the lambda 1 x. Now if I take the derivative of this with respect to x. Suppose I just take d by d x of this you can clearly see that I will just get lambda times y 1 e to the lambda x. And so this is just lambda times y. And what we said this is equal to lambda times y 1 is just; so each of these actually solves satisfies this differential equation, because you know that lambda 1 times y 1 is just A times y 1.

And so what you are doing is, you are using the method of eigenvalues and eigenvectors to actually get independent solutions. And then you are using A times y 1 e to the lambda 1 x. Therefore, what you are saying is that y 1 e to the lambda 1 x is the solution of this differential equation; the original differential equation that you have. Similarly y 2 e to the lambda 2 x is also a solution, y 3 e to the lambda 3 x is also a solution and so on. And so you can take a linear combination of all these solutions and get a general solution of this differential equation. That has lot of un-arbitrary constant.

Now each of these arbitrary constants you can determine from the boundary conditions. And so this is a way to solve a system of linear first order differential equations, and using this idea of eigenvalues and eigenvectors. So, let us just go back once again and just remind ourselves that where we got the system of differential equations. We got it from saying that each of these differential equations is linear and so I can write this in matrix form. So, d by d x of y 1 up to y n is this and we identified each of these as y d y by d x and this with matrix A. And so we could write our differential equation in this form. This system of differential equation looks like a differential equation for a vector.

And once you find the eigenvalues and eigenvectors then you can use these eigenvalues and eigenvectors in order to write the general solution. So, what we have to do is to find this I find these eigenvectors and these eigenvectors satisfy this differential equation, because d y by d x is a y which is A times y lambda which is lambda times y lambda. And what that gives us is that d y lambda by d x is lambda times y lambda. And this immediately tells us that y lambda should have this exponential factor, we can go ahead and write it as some constant times an exponential.

And what this tells us we have n eigenvalues and we have n eigenvectors. So, we can write the general solution in this form and you can verify that each of these terms will satisfy the differential equation. So, each of these terms in the general solution, each of these terms will actually satisfy the differential equations.

This concludes discussion on first order differential equations. Now in the last lecture of this module I will be doing some practice problems to show you how you can use all these methods that we have learnt.