Dynamics of Structures
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Lecture - 14
Non-periodic Excitations

Welcome back every one, we are going to continue our discussion from the last class in which we learn about Duhamel integral and we also saw that how to obtain solution of a single degree of freedom system subject to any arbitrary excitation.

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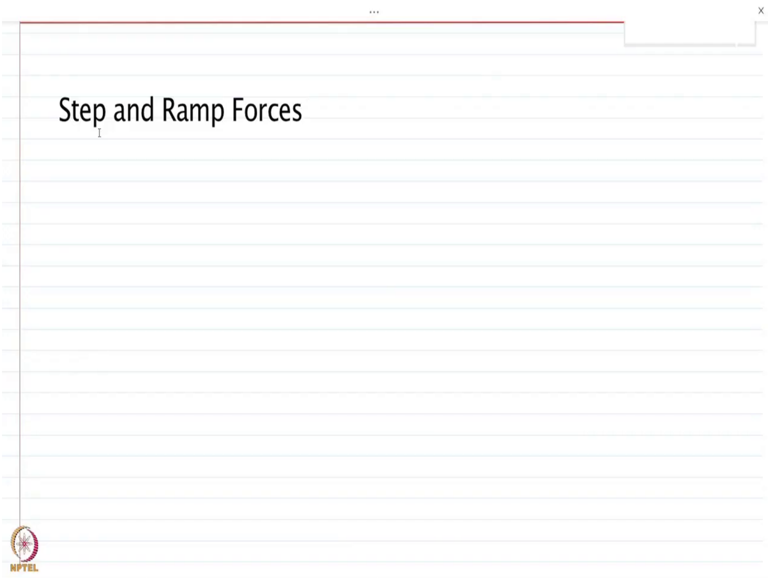

Now we are going to discuss some of the specific cases of pulse excitation as well as a step forces and then see whether how we can obtain the analytical response of a single degree of freedom system. Last class we saw that if we have some arbitrary excitation.

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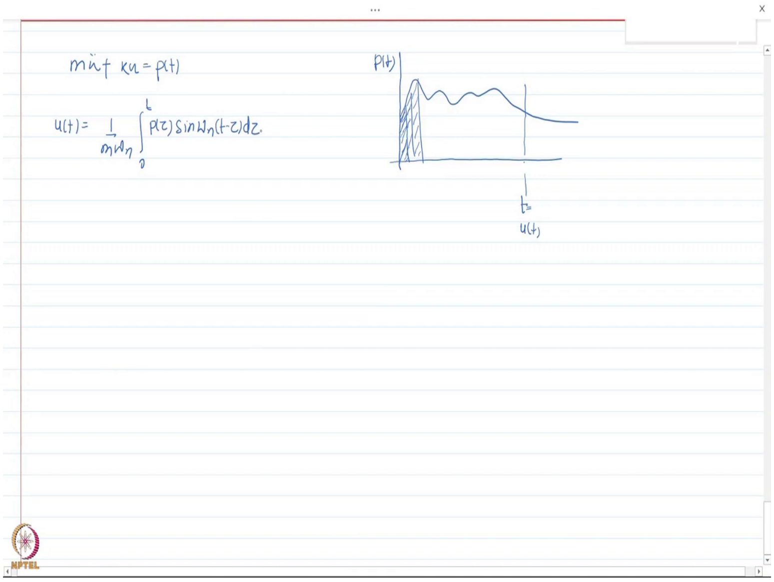

So, if we have the equation for a single degree of freedom system which is-

$$m\ddot{u} + ku = P(t)$$

$P(t)$ is some arbitrary excitation. So, we derived an integral which is called a convolution integral to find out the response to any arbitrary excitation $P(t)$ and the idea behind that, was that if I have any random arbitrary varying load or excitation $P(t)$.

Then the response at any time *t* to get this we divided this force into very small duration forces and we said that this small duration forces would basically represent an impulse. So, they would represent each of them an impulse and the response at time *t* would be sum of responses due to each of impulses. We know the response due to an impulse is basically free vibration with initial velocity and we sum up all those responses up to time *t* to get the response $u(t)$ at time *t*.

So, we derived the expression, first for Duhamel convolution integral and we saw that we can substitute the value of unit impulse response function to get the Duhamel integral. So, I am just going to write the final expression. So, for an undamped system we got this expression.

$$u(t) = \frac{1}{m\omega_n} \int_0^t P(\tau) \sin \omega_n (t - \tau) d\tau$$

So, we can utilize this expression to get the response and similarly for damped system as well, we can write down the Duhamel integral.

$$u(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\xi\omega_n(t-\tau)} \sin \omega_D (t - \tau) d\tau$$

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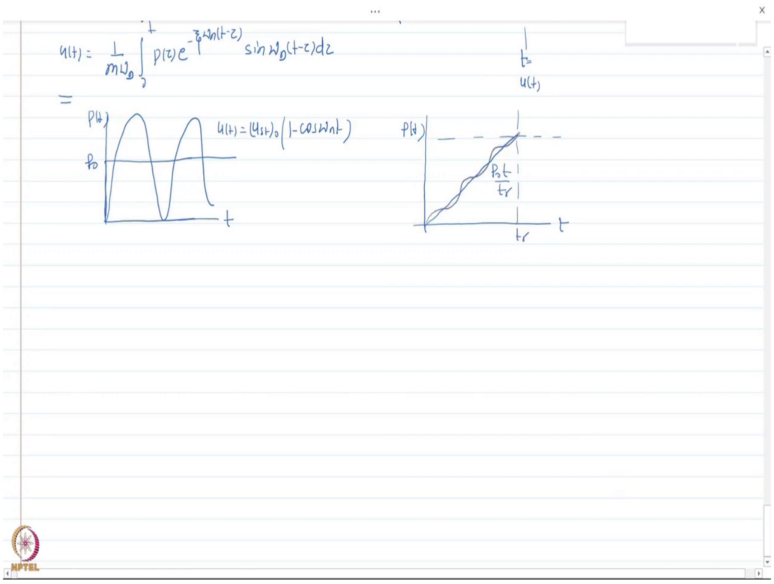

We discussed that in some cases Duhamel integral is very useful to get the analytical response of function $u(t)$ something like this and in some cases, you can use the conventional method of solving the differential equation. It will purely depend on the problem. So, we will have to inspect the problem and look at the integrant that you have inside that integration to see which would be easier.

So, utilizing this basically we obtain response to a step force which is represented something like this. So, this is a step force with suddenly applied load of magnitude P_0 and then it is maintained over time is P_0. We found out the response to this as $u(t) = (u_{st})_0 (1 - \cos(\omega_n t))$ where $(u_{st})_0 = P_0/k$.

Then we found out the response due to a linearly increasing force which let's say this is $P(t)$ here and this is t and let us say this force is represented by $\frac{P_0}{t_r} t$ where P_0 is basically this amplitude which is achieved over time duration of t_r. So, during the linearly increasing phase we try to obtain the solution $u(t)$ as-

$$u(t) = (u_{st})_0 \left[\frac{t}{t_r} - \frac{\sin(\omega_n t)}{\omega_n t_r} \right]$$

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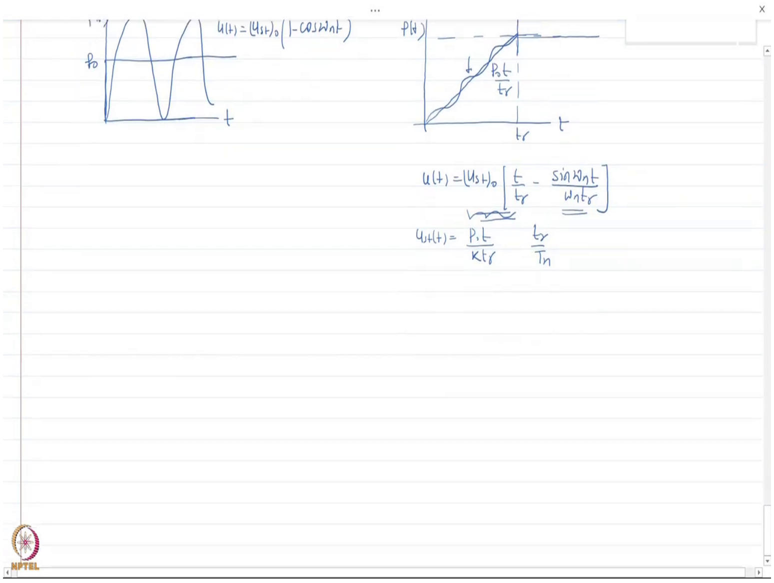

So, as you can see it is vibrating at frequency ω_n around its static solution which is represented by this here. So, $u_{st}(t) = \frac{P_0 t}{k t_r}$ which is basically this function here. So, we saw that $\frac{t_r}{T_n}$ is a very important parameter here and we are further going to discuss that today. So, we stopped at this point and let us now discuss what happens to the response during the constant forced phase.

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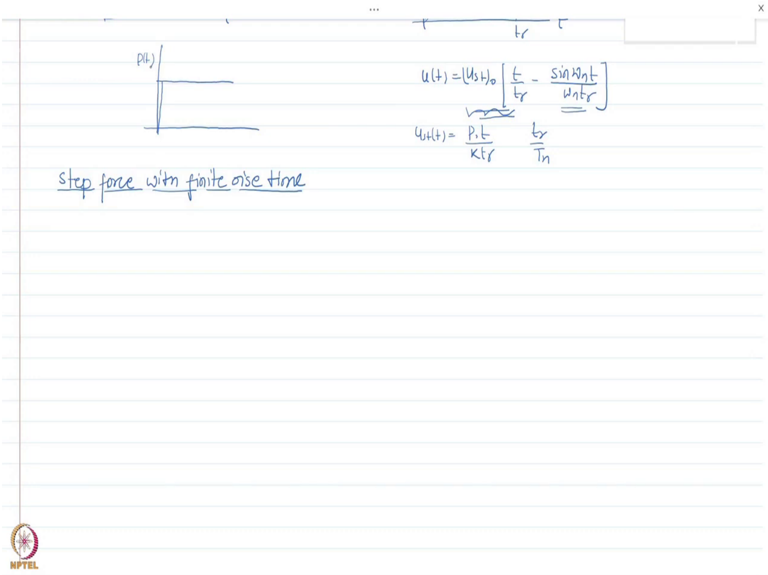

So, what we are going to do today is basically a step force with finite rise time and this is a better representation of force that is applied in real life. So, if you think about it, it is almost impossible to apply a certain force. There is always some duration however, a small. It (duration) may be over which the amplitude of the force P_0 is achieved. So, in general I can represent any force that reaches to its maximum value P_0 through a step force with a finite rise time, we can also call it a ramp force.

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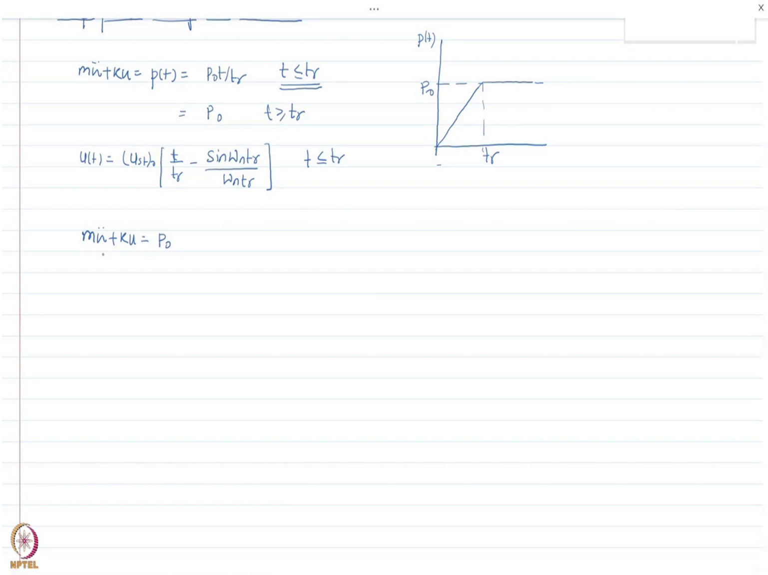

So, basically the problem statement that I am applying a force which goes to its amplitude P_0 over a time duration t_r and then it is maintained at that value P_0. So, if I want to find out the response to this step force, let us see how to do that. So, our differential equation would be $m\ddot{u} + ku = P(t)$.

$P(t)$ is represented as
$$P(t) = \begin{cases} \dfrac{P_0}{t_r} t & t \le t_r \\ P_0 & t \ge t_r \end{cases}$$

For rising force phase, $t \le t_r$ and for constant force phase of course $t \ge t_r$. So, for this (rising phase) we have already obtained the solution and let me write that again.

$$u(t) = (u_{st})_0 \left[\frac{t}{t_r} - \frac{\sin(\omega_n t)}{\omega_n t_r} \right] \quad t \le t_r$$

Now let us obtained solution for the second part.

If you look at the second part, we have something like this and we know this is nothing, but the differential equation for a step force that we had done previous to this step of force with a finite rise time.

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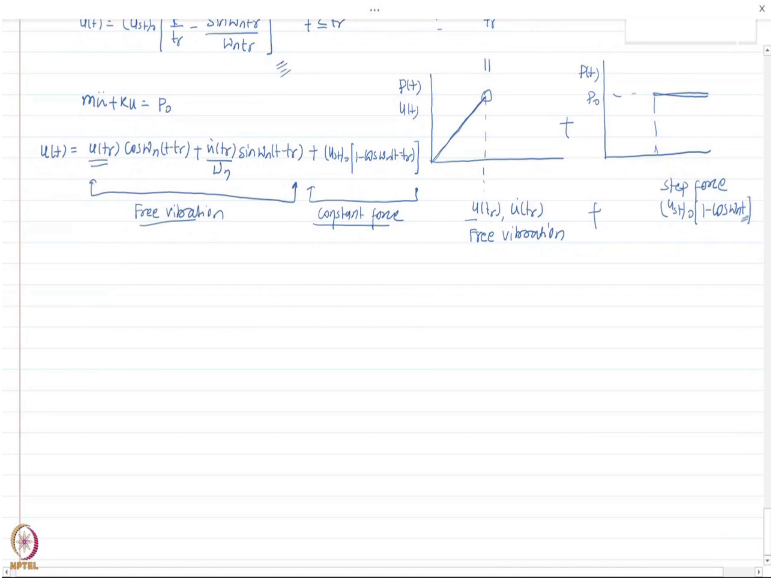

So, I am going to divide this force into two parts. So, this (rising phase) and this (step force) here. So, let us say and this is P_0. So, up to this point (at the end of rising phase) my single degree of freedom system would have achieved some displacement and velocity. So, let us say it would have achieved some $u(t_r)$ and $\dot{u}(t_r)$ for a linear system.

If I am representing the total solution to this step function as a sum of solution to this linearly increasing phase up to time t_r and then a step force from point t_r onwards. It would be basically with this an initial condition ($u(t_r)$ and $\dot{u}(t_r)$), it would undergo free vibration.

However, note that the motion starts here at t_r instead of $t = 0$ and in this case we already know what is the solution for this one is it? We know that for this step force the solution is basically. So, this is step force and the solution I can write it as

$$u(t) = (u_{st})_0 \left[1 - \cos(\omega_n t)\right]$$

Remember this is starting at $t = t_r$. So, this t will have to shift to t_r.

So, the total response can be represented as free vibration response due to linearly increasing phase and because it is starting at $t = t_r$ and not t equal to 0. I am basically shifting it by t_r so, that my time variable is represented by $t - t_r$. And this I will write it like-

$$u(t) = u(t_r)\cos\omega_n(t - t_r) + \frac{\dot{u}(t_r)}{\omega_n}\sin\omega_n(t - t_r) + (u_{st})_0\left[1 - \cos\omega_n(t - t_r)\right]$$

Now, this is the free vibration phase response (1st part) and this (2nd part) is the constant force phase. Now in this I can substitute the value of $u(t_r)$ and $\dot{u}(t_r)$, by substituting $t = t_r$ in this expression (rising phase for $t \le t_r$) first and then differentiating it (rising phase) and then again substituting $t = t_r$.

So, I will do that what I am going to do here just write the final expression-

$$u(t) = (u_{st})_0\left[1 + \frac{1}{\omega_n t_r}\left[\left(1 - \cos(\omega_n t_r)\right)\sin\omega_n(t - t_r) - \sin(\omega_n t_r)\cos\omega_n(t - t_r)\right]\right]$$

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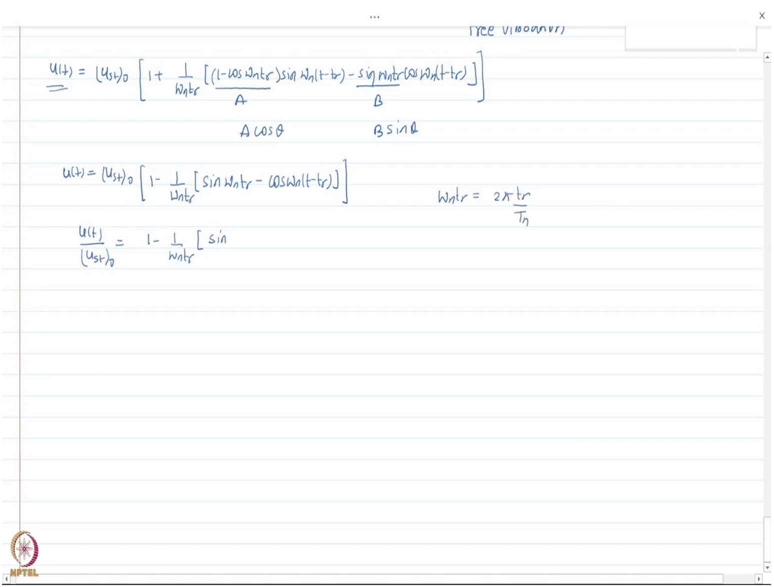

So, this is the expression if you look at it this is nothing this is a constant here, you can call it A and this is another constant is B these are coefficients are- $A=\left(1-\cos\left(\omega_n t_r\right)\right)$ and $B=\sin\left(\omega_n t_r\right)$. So, this is basically something of form $A\cos\theta$ and then $A\sin\theta$ and we will utilize this property later when we have to find out the maximum value of $u(t)$.

But right now, let us further simplify this-

$$u(t)=\left(u_{st}\right)_0\left[1-\frac{1}{\omega_n t_r}\left[\sin\left(\omega_n t_r\right)-\cos\omega_n\left(t-t_r\right)\right]\right]$$

So, one more thing remember, I can write it as $\omega_n t_r=2\pi\frac{t_r}{T_n}$.

$$\frac{u(t)}{\left(u_{st}\right)_0}=1-\frac{T_n}{2\pi t_r}\left[\sin\left(2\pi\frac{t_r}{T_n}\right)-\cos\left(2\pi\frac{t}{T_n}-2\pi\frac{t_r}{T_n}\right)\right]$$

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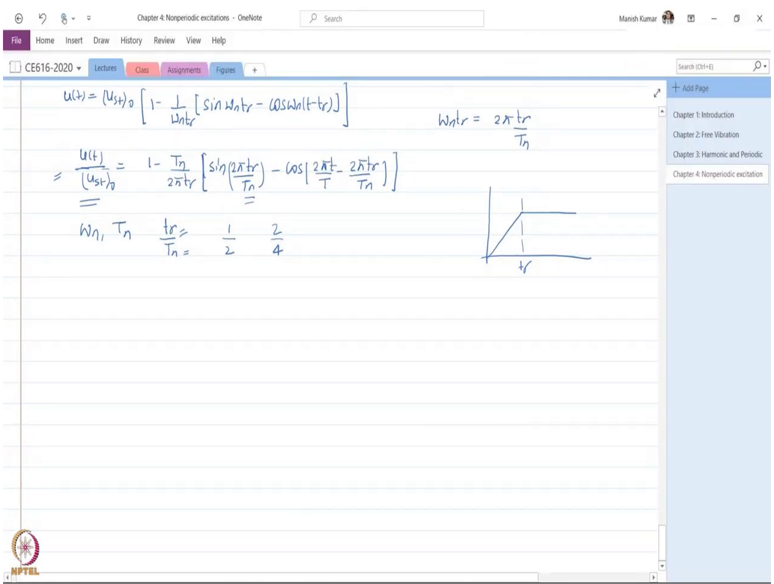

So, if you look at this carefully my system is vibrating at frequency ω_n or the time period T_n.

So, that is vibrates as its natural frequency ω_n and my response actually depends on the value of ${t_r}/{T_n}$. Note that it does not individually depends on t_r or T_n. If you look at it is always in the ratio form anywhere in this expression here for ${u(t)}/{(u_{st})_0}$. So, it depends on the ratio ${t_r}/{T_n}$.

So, I might have different ratio of ${t_r}/{T_n}$ for example, ${1}/{2}$ or ${2}/{4}$ as long as the ratio is still the same, I would get similar kind of dynamic response for my system. So, that is an important point that the response actually depends on the value of ${t_r}/{T_n}$ which is the rise time divided by the time period of the system. So, that need to be kept in mind. So, if we want to plot this for different value of ${t_r}/{T_n}$, I will have the response of the system.

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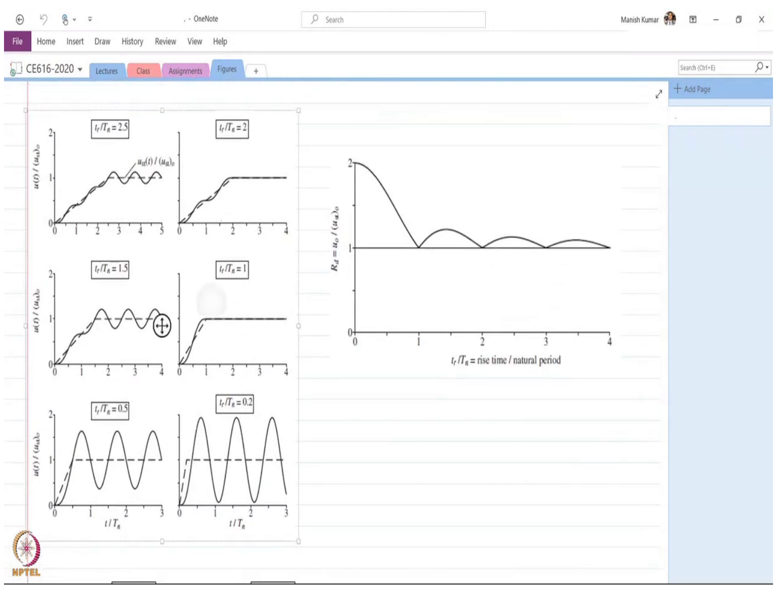

So, let me just copy that and then show you how does it look like.

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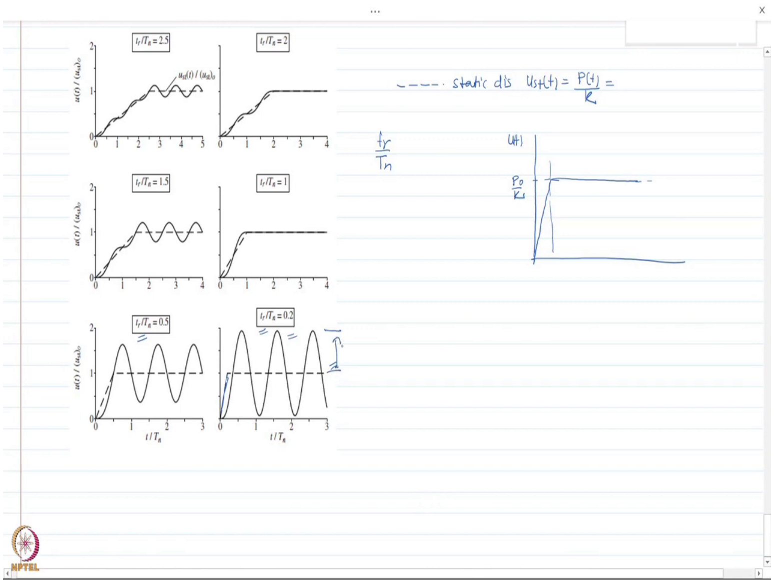

So, if you look at carefully here. So, what these plots basically represent the ratio $\left(u(t) / (u_{st})_0 \right)$ of the dynamic displacement versus peak static displacement and these have been plotted for different values of t_r/T_n and the horizontal axis is basically the normalized time.

So, if you look at it when my t_r is very small like this $\left(t_r/T_n = 0.2 \right)$ case. Note that there are two lines here, the dotted line is for static displacement which is nothing, but $u_{st}(t) = \frac{P(t)}{k}$. So, this is the static displacement assuming that system does not have any mass. I can write it as this and for this case this dotted line that has been shown here.

The dynamic displacement is basically shown here using the solid line. Now let us start with the smallest value of t_r/T_n. So, let me draw representation of this one when t_r/T_n is something like this it is starting from 0, but the rise time is very small. The response that I see here, it is similar to a step force and that should be obvious considering for very small value of t_r/T_n the force actually resembles as step force.

So, this is P_0/k displacement and then my system basically vibrates around a static equilibrium position which is around the point P_0/k. So, I can see here that the system actually vibrates around a static equilibrium position and the difference between the dynamic and the static is quite large (for $t_r/T_n = 0.2$).

Now as I go from $t_r/T_n = 0.5$, I see that there is a still lot of vibration. So, there is still significant difference between static and dynamic. However, in this case the difference has actually been reduced compared to $t_r/T_n = 0.2$.

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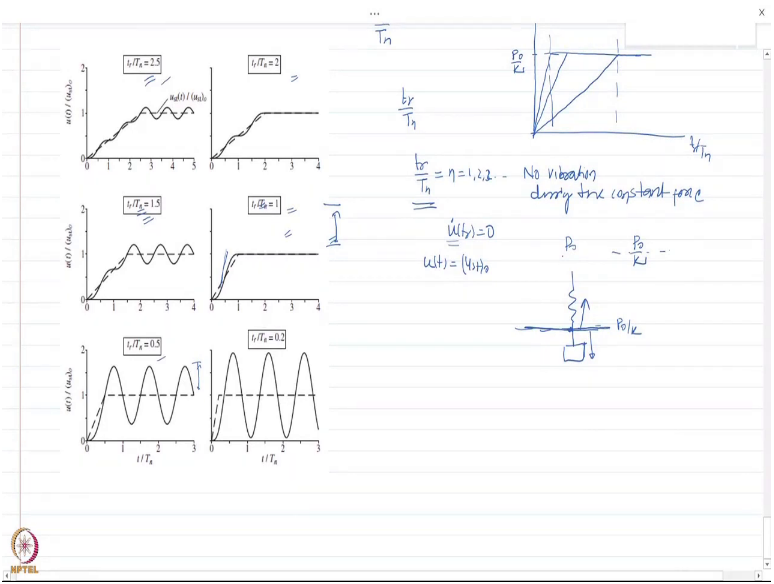

So, as I increase the rise time the difference between the static displacement and the dynamic peak displacement actually reduces and that is further enforced by the fact when you look to response for $t_r/T_n = 1.5$ and then further response for $t_r/T_n = 2.5$.

For the value very large like $t_r/T_n = 2.5$, you can see the dynamic solution almost trailing the static solution. So, it is almost like very close to the static solution that means, if I have rise time which is very large then my dynamic solution is almost close to the static solution and I do not see much difference or amplification of the response.

So, in this case depending upon the t_r/T_n dynamic characteristic of the response is determined. So, t_r/T_n is a very important parameter to characterize the response to a step force with finite rise time. There are two additional cases when $t_r/T_n = 1$ and when $t_r/T_n = 2$ or for a matter of case whenever you will have $t_r/T_n = n$ where *n* is equal to 1, 2, 3 so on integer value. So, what you will see that at the end of the linearly increasing phase you do not get any oscillation.

So, no vibration during the constant force phase and why is that happens? It is actually when you have this condition that velocity $\dot{u}(t_r) = 0$ at time $t = t_r$. So, for that case the $\dot{u}(t_r) = 0$ basically if you substitute in the expression for $u(t)$, you will get $u(t) = (u_{st})_0$ which would be constant and non vibrating with time.

And if you want to imagine this physically, think it like the equilibrium position under the action of load P_0, is at a displacement P_0/k. So, if you apply a static load of P_0 the equilibrium position is P_0/k. Let us see I have a spring which is vibrating around this equilibrium position $\left(P_0/k \right)$. Now when it comes to this displacement P_0/k and if your velocity is 0 then what will happen?

Remember in previous instances of free vibration, when you had system undergoing through the equilibrium position it had the maximum velocity and zero displacement measure from the equilibrium position. So, if I measure the displacement from the equilibrium position when it crosses here, it will have zero displacement and it would be in equilibrium because that is the equilibrium under the load P_0. So, if it's an equilibrium and there is no velocity then there is nothing, that would take the system beyond this (equilibrium position) and then bring it back here.

So, and that happens for very specific case when t_r/T_n is equal to any integer. So, for those cases when the system passes through equilibrium, it has zero velocity and then there is nothing that would take the system further or continue the vibration. So, that is why for these cases you get this kind of response. No vibration during the constant force phase.

If this is clear now let us look at if you remember for a harmonic excitation. We had defined a function R_d which we said that, it is the ratio of maximum peak dynamic displacement (u_0) divided by the peak static displacement $(u_{st})_0$.

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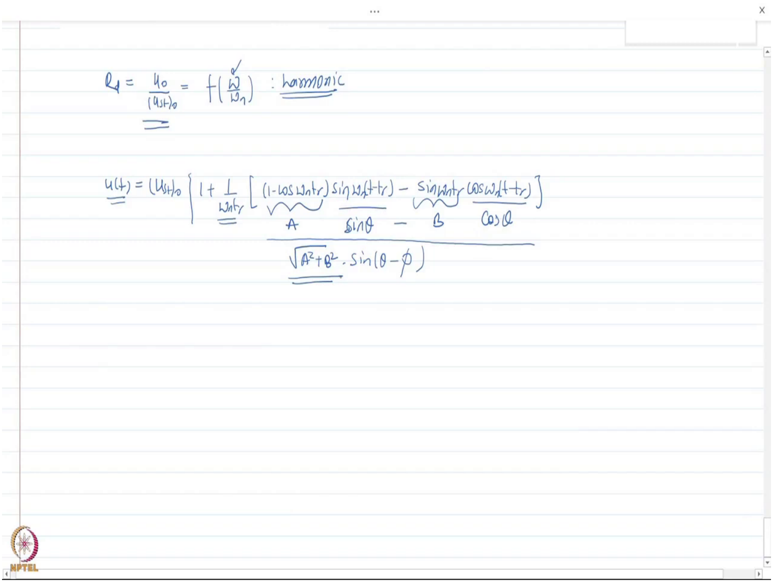

And that was a function of ω/ω_n and this was for the harmonic case. Now I want to do something similar for a step force with finite rice time over the ramp load. However, there is no excitation frequency ω in this case. So, let us see what do we get as the value of $u_0/(u_{st})_0$ and then we will try to go more into the expression.

So, if you remember $u(t)$

$$u(t) = (u_{st})_0 \left[1 + \frac{1}{\omega_n t_r} \left[\left(1 - \cos(\omega_n t_r)\right) \sin \omega_n (t - t_r) - \sin(\omega_n t_r) \cos \omega_n (t - t_r) \right] \right]$$

Now if you look at it as I previously stated that-

$$A = \left(1 - \cos(\omega_n t_r)\right) \text{ and } B = \sin(\omega_n t_r).$$

Now the expression becomes-

$$u(t) = (u_{st})_0 \left[1 + \frac{1}{\omega_n t_r} \left[A \sin \omega_n (t - t_r) - B \cos \omega_n (t - t_r) \right] \right]$$

Further this expression can be as-

$$u(t) = (u_{st})_0 \left[1 + \frac{1}{\omega_n t_r} \left[A\sin\theta - B\cos\theta \right] \right]$$

So, maximum value of the dynamic displacement basically depends on the maximum value of this expression $(A\sin\theta - B\cos\theta)$ here and as we know if I have a function of the $A\sin\theta - B\cos\theta$, maximum value is always $\sqrt{A^2 + B^2}$ because I can write this $(A\sin\theta - B\cos\theta)$ expression as $\sqrt{A^2 + B^2}\sin(\theta - \varphi)$ whatever I can write it and the maximum value is $\sqrt{A^2 + B^2}$.

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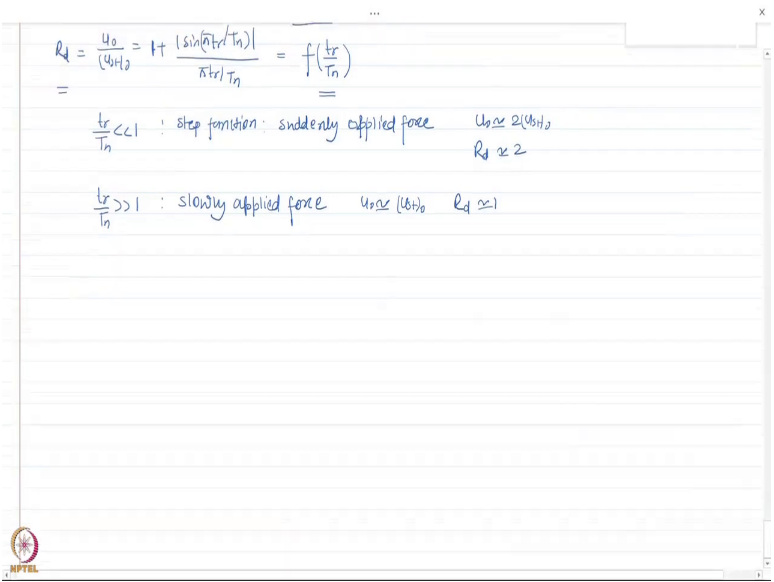

So, I am substitute it here and I simplify this I will find out that-

$$\frac{u_0}{(u_{st})_0} = 1 + \frac{\left|\sin\left(\pi t_r / T_n\right)\right|}{\pi t_r / T_n} = R_d$$

So, again if you look at carefully and let us say this is my R_d now and this depends on the value of t_r/T_n.

So, like for a harmonic excitation of a single degree of freedom system, the parameter on which the response modification factor R_d depends on is ω/ω_n, for a ramp loading that parameter is t_r/T_n. I can go ahead and plot this function R_d as a function of t_r/T_n. So, for different values of t_r/T_n I am going to basically plot this function.

Now, I can do that mathematically or before doing that mathematically, let us see for extreme cases what happens. I have already told you when $t_r/T_n << 1$, for those cases this is almost like a step function or suddenly applied force. For suddenly applied force we saw that my u_0 was approximately $2(u_{st})_0$. That means, R_d was actually equal to 2 and that we saw that from this graph $\left(t_r/T_n = 0.2\right)$ if you look at here, R_d was almost close to 2 here.

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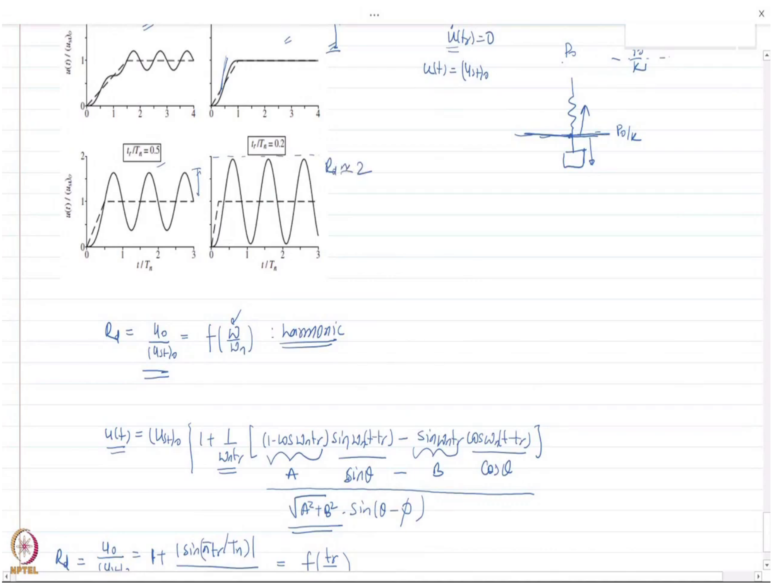

So, other extreme case is when $t_r/T_n >> 1$. So, this would be like a slowly applied force. So, for a slowly applied force we saw that u_0 was almost same as the static displacement $(u_{st})_0$.

So, $u_0 \simeq (u_{st})_0$, because my dynamic displacement was actually very much comparable to the static displacement throughout the time. So, $R_d \simeq 1$. So, for these two extreme cases we know the value of R_d for very small value of t_r/T_n and for very large value of t_r/T_n.

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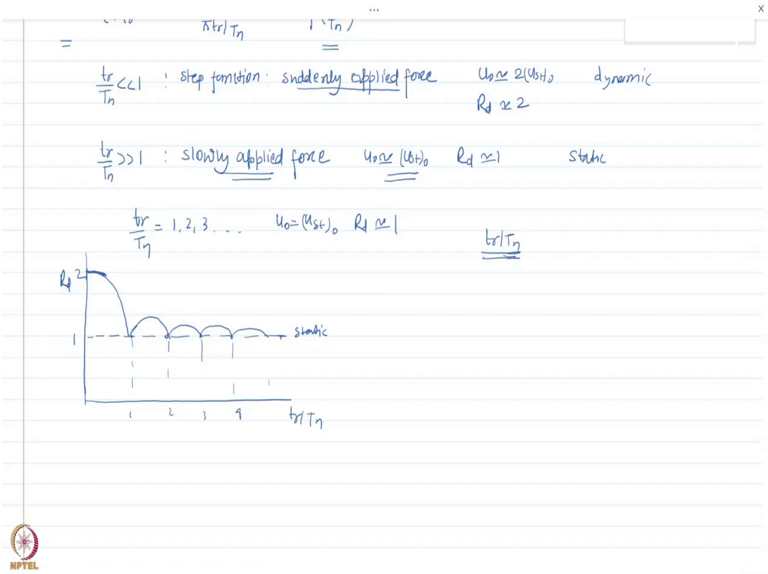

Now, we also said that if value of t_r/T_n equal to some integer says 1,2,3….so on and then for those cases there is no vibration during the constant force phase and if there is no vibration it means that static and dynamic are basically same. So, again for those cases as well the peak dynamic displacement (u_0) is actually equal to peak static displacement $(u_{st})_0$. So, again $R_d \simeq 1$.

So, we utilizing these three cases and let us see if we are able to plot R_d versus ${t_r}/{T_n}$. So, we have said that when ${t_r}/{T_n}$ is very small the value is actually 2. So, it is somewhere around here when it $\left({t_r}/{T_n}\right)$ is very large then it is close to 1 and we have also said that it (R_d) is equal to 1 at integer value of ${t_r}/{T_n}$. So, let us say this is 1, this is 2, this is 3, this is 4 and so on, on horizontal axis.

So, basically the actual response looks like something like this with this decreasing with time. So, depending upon the value of ${t_r}/{T_n}$ we can calculate R_d and the same thing can also be obtained using the expression for R_d, but we have just tried to obtain this using the analytical or discussion that we just presented. Now a point here to note that how much different it (R_d) would be from the static displacement.

It depends on the value of ${t_r}/{T_n}$ and remember these ${t_r}/{T_n}$ decides whether it is going to be a suddenly applied force or whether it is going to be a slowly applied force. So, whether it would have dynamic effect or whether it would be almost same as static and it would behave like as static. Now this is a very important parameter because most of the load that is being applied is, let us say 10 kN.

Before this course you did not care about knowing that how the load was applied. So, what was the time variation of the load? And if somebody says to you that something is applied slowly it does not simply mean, that it takes long time to actually apply that load. You also need to know that how long is the time with respect to the time period of the system. So, that is an important parameter.

So, I like to view an example. So, whether something is slowly applied or whether it is a static force and whether load is suddenly applied or whether it is a dynamic, it depends on this overall ratio of ${t_r}/{T_n}$.

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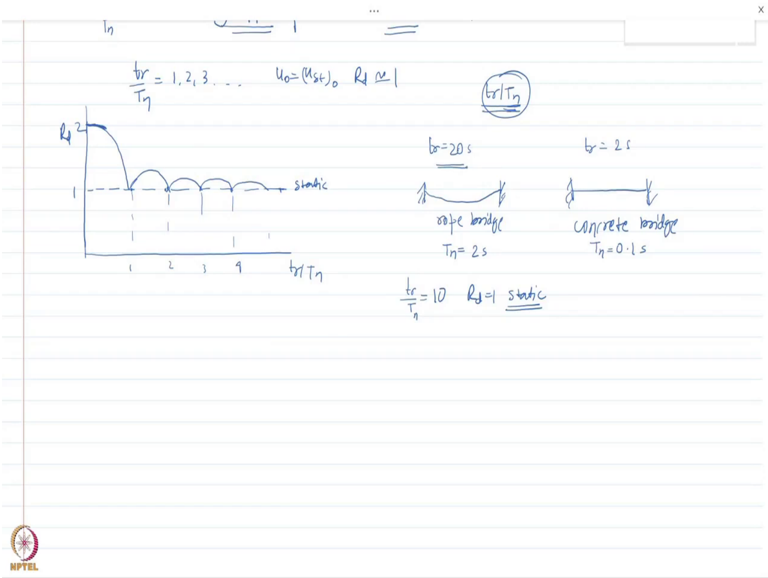

So, let us say I have two bridges, one bridge is basically a rope bridge which is very flexible and the second one is a concrete bridge. Now let us say for the rope bridge my time period is around 2 second and for concrete bridge is around 0.1 second. Which is typical you know because it is very flexible the rope bridge it would have very large time period compared to concrete bridge which has a very small time period of 0.1 second.

Now whether something is slowly applied or whether it is suddenly applied depends on this ratio ${t_r}/{T_n}$. So, let us say I want to apply a load so, that the dynamic effect is very less. So, let us say ${t_r}/{T_n}$ ratio is very high (says 10), then I can say that there is no dynamic amplification or my R_d would be equal to 1. So, it would behave like a static. Now to achieve the static type condition in this case (rope bridge) the rise time has to be $2\times 10=20$ second and in this (concrete bridge) case the rise time could be just $0.2\times 10=2$ second.

So, for this rope because it is very flexible, you need to be applying the load at a much slower pace or over a very large duration of time to achieve similar kind of response compared to concrete bridge in which you can apply the load much quickly and still be able to obtain similar kind of response. You can observe this realistic as (Refer Time: 34:52) as well. If you jump on a rope bridge it will start oscillating. So, you will see much more dynamic effect.

So, remember you are applying similar kind of load over a similar duration. So, if you jump on a rope bridge or you jump on a concrete bridge, you are applying similar load or a similar duration. However, because the time period of the two bridges are different, you would see different response. So, keep that in mind that whether something is static or dynamic, it depends on the ratio t_r/T_n and whether something is applied slowly or suddenly it is not only a function of time t_r.

So, time is relative, it is actually depends on in a structural dynamic. So, whether something is slow or fast it depends on the ratio of the duration with respect to the time period of the system. So, this is something that you need to keep in mind that t_r/T_n is a very important parameter. So, with this we saw that how to get the response of a system subject to a ramp loading or a step force with the finite rise time.

Now, what we are going to discuss a different type of excitation which is called pulse excitations. So, we are going to get into pulse excitation.

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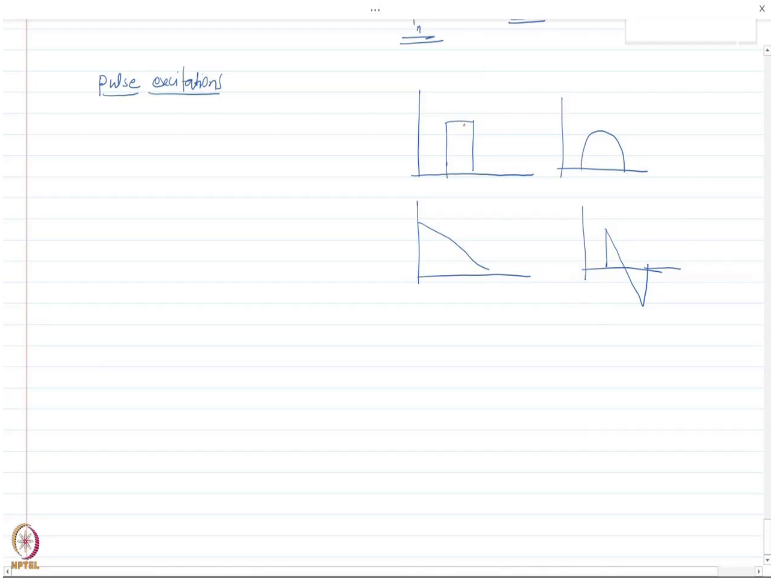

Pulse excitation is basically any type of excitation that has a finite duration of application and then it is 0. So, you have a load that is applied over a finite duration and then it is 0 before and after that. So, some of the examples (Refer Slide Time: 36.50) would be a load that is applied like this or a load let us say which is applied like this or load that is applied like this ok or you could have a different type of load.

So, any type of load that is applied only over a finite duration of time is called a pulse load and as you can imagine, if you apply a pulse load then the system does not achieve any steady state response. It would always be vibrating and depending upon whether the system has damping or not it might come to rest for a damped system or might always keep on vibrating with the initial conditions provided by this pulse type motion.

So, you know there are different type of pulse excitation. What we are going to study first is basically a very common type of pulse excitation which is rectangular pulse excitation.

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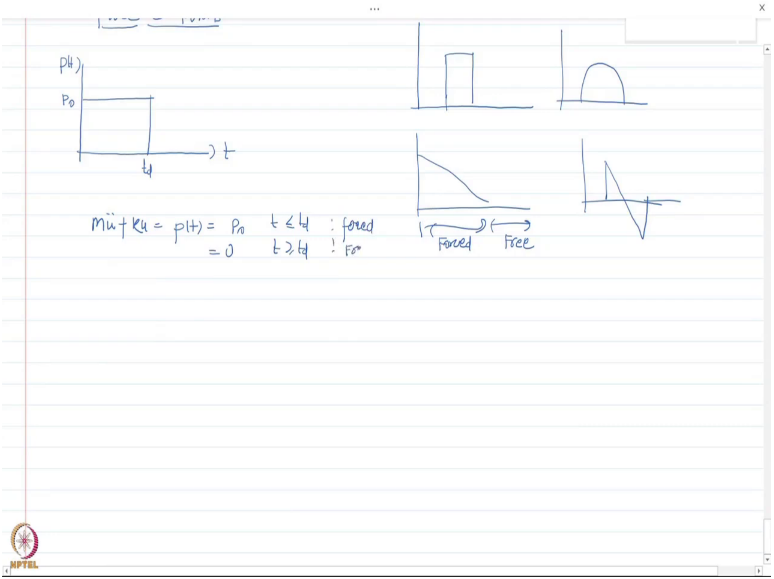

So, I have a rectangular pulse force of amplitude P_0 applied over a time duration of t_d. So, now, there is a new parameter, remember there was a rise time (t_r) in the previous case that we had discussed, now that there is a different parameter it's called t_d.

So, the lode in rectangular pulse is applied suddenly and it is kept constant till the time duration t_d and then it is released so, that force is 0. We have to find out the response of the system subject to this type of loading. So, I can again go ahead and write down the equation of motion as- $m\ddot{u} + ku = P(t)$ where $P(t)$ is basically-

$$P(t) = \begin{cases} P_0 & t \leq t_d \\ 0 & t \geq t_d \end{cases}$$

In terms of solution, we are going to follow the same procedure that we have been doing till now. So, first let us do the forced vibration phase. So, pulse type excitation we would always divide our response in forced vibration phase and free vibration phase. Then we will try to obtain the response of forced vibration phase $(t \leq t_d)$ and free vibration phase $(t \geq t_d)$. So, for this case let us first obtain the response for forced vibration phase and see, what do we get?

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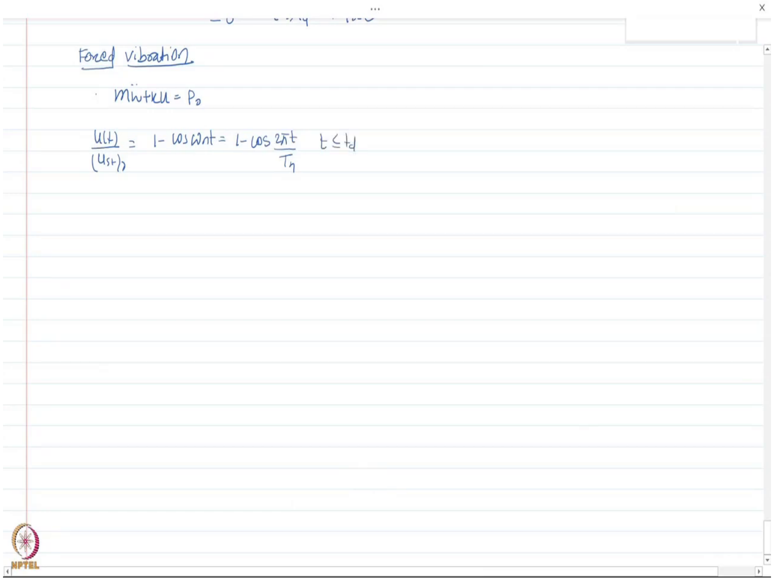

So, for forced vibration the equation that I have is this $m\ddot{u} + ku = P_0$ and we already know the response for this type of loading. We had already derived this is nothing, but a step force. So, the response for this we have already obtained as-

$$\frac{u(t)}{(u_{st})_0} = 1 - \cos(\omega_n t) = 1 - \cos\frac{2\pi t}{T_n} \quad t \le t_d$$

Remember this response is valid only for $t \le t_d$ which is the t_d of the pulse.

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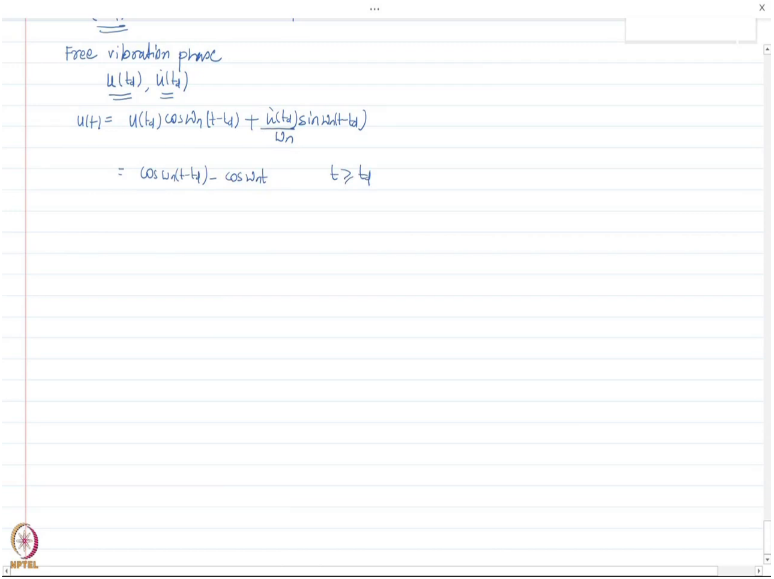

Once this is clear let us obtain the response for free vibration phase $(t \geq t_d)$. Free vibration phase is nothing, but a free vibration where initial conditions as $u(t_d)$ and $\dot{u}(t_d)$ that was imparted to the system due to the forced vibration. So, we are going to utilize these initial conditions and find out the response in the free vibration.

$$u(t) = u(t_d)\cos\omega_n (t - t_d) + \frac{\dot{u}(t_d)}{\omega_n}\sin\omega_n (t - t_d)$$

So, I can go ahead and substitute these values ($u(t_d)$ and $\dot{u}(t_d)$) from the expression (force vibration phase) here, and simplify it to get the final response. So, final response that I get here as actually-

$$u(t) = \cos\omega_n (t - t_d) - \cos\omega_n t \quad t \geq t_d$$

We can further simplify this-

$$\frac{u(t)}{(u_{st})_0} = \left(2\sin\frac{\pi t_d}{T_n}\right)\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \quad t \geq t_d$$

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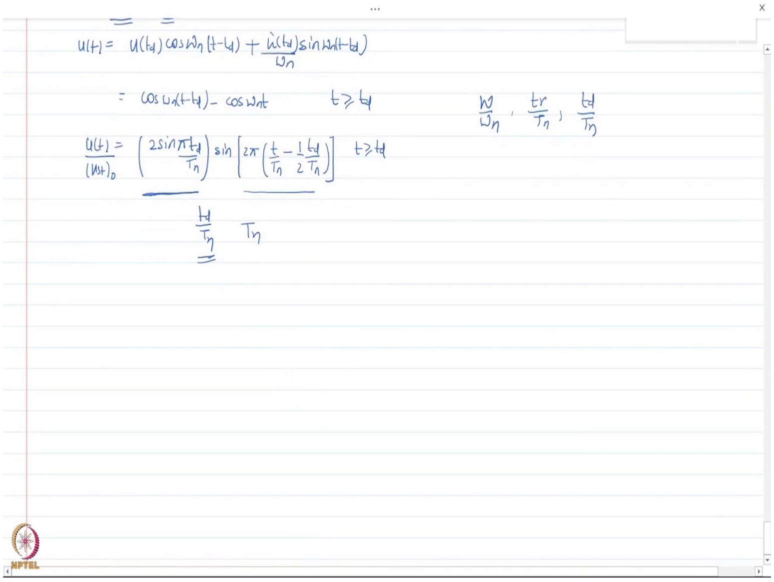

So, this is the response that we get for $t \geq t_d$ and if you look at it $\left(\sin\left[2\pi\left(\frac{t}{T_n}-\frac{1}{2}\frac{t_d}{T_n}\right)\right]\right)$ this is nothing, but this sin function. So, let us call this $\sin\theta$ and it is multiplied with this coefficient $\left(2\sin\frac{\pi t_d}{T_n}\right)$ here which is constant. So, it does not define on time, it is actually a function of t_d/T_n.

So, the normalized response if you look at here, this system actual vibrates again at it is natural frequency ω_n and it depends on the ratio t_d/T_n. So, like for the case when we had the forced vibration of single degree of freedom system subject to harmonic excitation, there the important parameter was ω/ω_n and for a step function with finite rise time or ramp function the parameter was the t_r/T_n and for pulse type motions you will see the important parameter is actually t_d/T_n.

So, basically my system when applied to this pulse oscillates above its static position at time period of oscillation of T_n and we can plot $\frac{u(t)}{(u_{st})_0}$ for different value of t_d/T_n.

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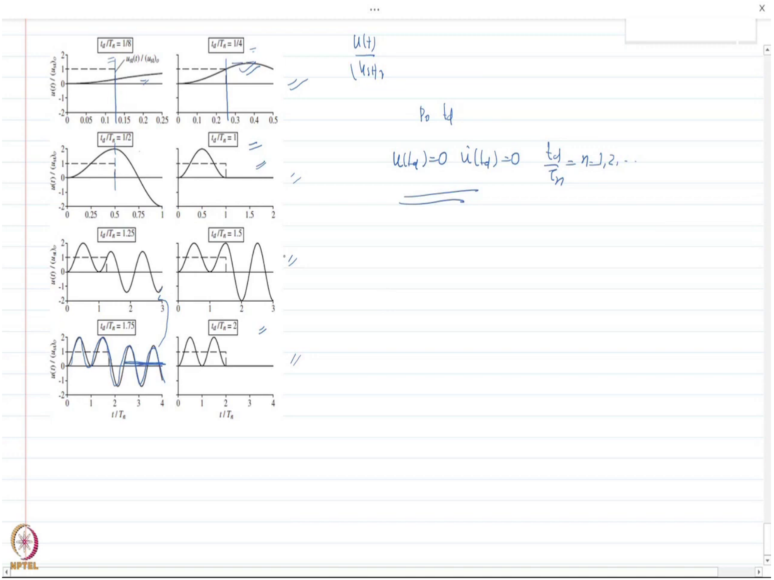

So, if you look at here depending upon the value of t_d/T_n alright we have obtained different response histories here. So, I have obtained $\frac{u(t)}{(u_{st})_0}$ and what I have done here basically combined the response for $\frac{u(t)}{(u_{st})_0}$ the force vibration phase and free vibration phase. These are the curves that we get. And if you look at this carefully depending upon the value of t_d/T_n the maximum can occur during the forced phase or during the free vibration phase like in these two cases.

And this happens when t_d/T_n actually below the value of $1/2$ like in these cases (first two graph) then the peak occurs after the value of t_d. One thing that we just need to focus here that when we applied the load P_0 over the time duration t_d my system actually starts vibrating about a shifted equilibrium position which is P_0/k and when I remove the load it come backs to its original position which is $u = 0$ and then it vibrates about this position.

Even though the amplitude is different, the same type of behavior is observed for different value of t_d/T_n. Except again for these cases where t_d/T_n is actually an integer when that is the case then what happens? At the end of the forced vibration phase, we basically get $u(t_d) = 0$ and $\dot{u}(t_d) = 0$ when t_d/T_n is equal to any integer 1, 2, 3 like that.

So, for those cases (t_d/T_n equal to integer) after the completion of the forced vibration phase, we do not see any response. So, the system just sits there and which can again be interpreted in terms of vibration about equilibrium. If you are having a system that is vibrating and it comes to equilibrium and at that point if you have zero velocity and it is in equilibrium then there will not be any further oscillation because you do not have any velocity or you do not have any initial condition to take the system above that.

Because your force is now zero and if your initial conditions are also here zero, then there will not be any subsequent motion. So, that is why for these cases we obtain behavior which is something like this. So, just to see that how the system behaves subject to different values of t_d/T_n. Now let us come back to finding the maximum value of this $\left(\frac{u(t)}{(u_{st})_0}\right)$.

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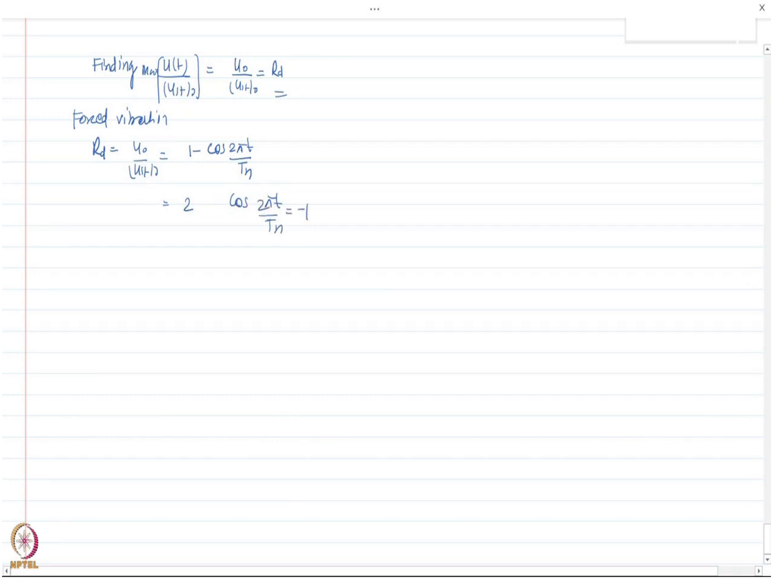

So, finding maximum of this $\left(\frac{u(t)}{(u_{st})_0}\right)$ which is nothing, but peak dynamic displacement (u_0) divided by peak static displacement $(u_{st})_0$ and as previously described that this is basically the response modification factor R_d. Let us obtain this response modification factor R_d for free vibration phase and forced vibration phase. Overall R_d would be maximum of both.

So, let us see what is the value of R_d for forced vibration. The expression is-

$$R_d = 1 - \cos\frac{2\pi t_d}{T_n}$$

So, the maximum value of this function is what? $R_d = 2$, which occurs at $\cos\frac{2\pi t_d}{T_n} = -1$.

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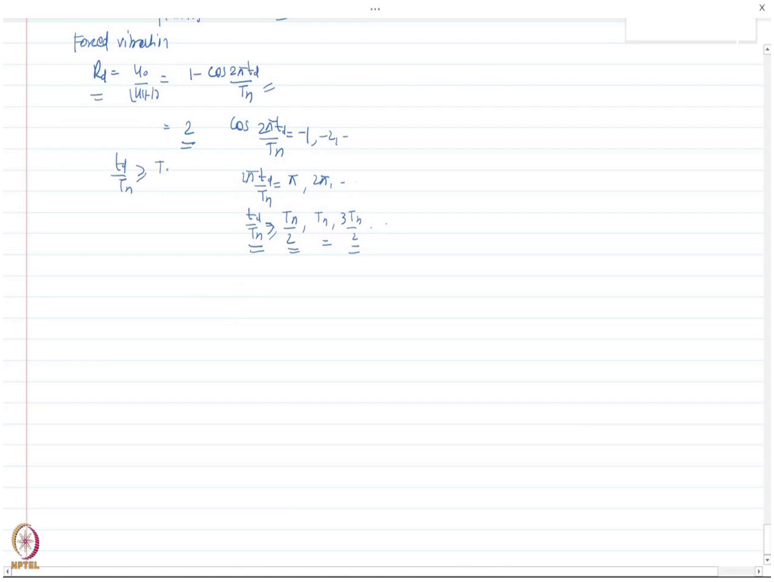

So, it $\cos\frac{2\pi t_d}{T_n}$ could be minus 1 and so on. So, for that condition let us see what is my t_d/T_n. If I substitute $\frac{2\pi t_d}{T_n}$ as π, 2π, 3π and so on. So, my t_d is actually $T_n/2$ then T_n, 3, $3T_n/2$ and so on like that. So, basically the maximum response occurs at these durations and for at least one maximum to occur, can you imagine that my t_d should at least be greater than $T_n/2$.

Can you imagine, R_d would achieve a value of 2 only if t_d/T_n is greater than $1/2$? So, the maximum of this function or the maximum of R_d during forced vibration phase would be 2 only if $t_d/T_n \geq 1/2$.

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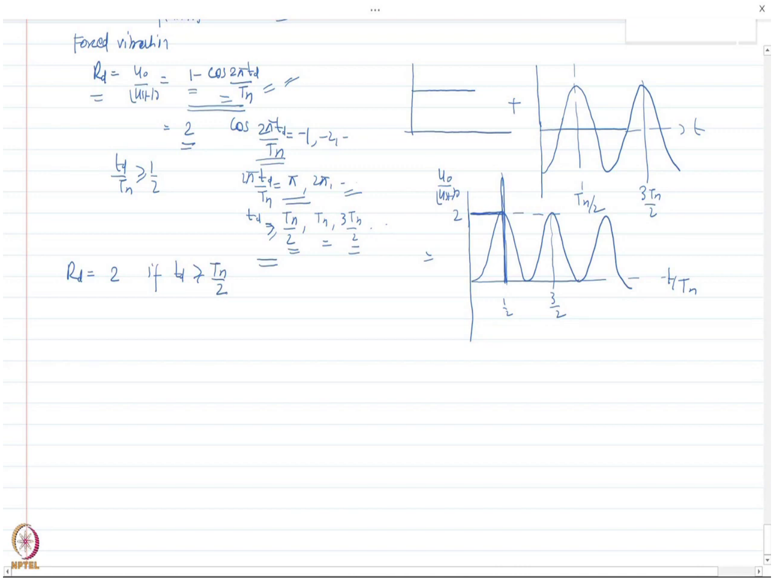

That means, I would have at least one peak of function that I have here. Let us try to understand that graphically as well. So, I have this function 1 and this function $-\cos\frac{2\pi t_d}{T_n}$. So, basically what I am saying? This is 1 here (Refer Slide Time: 50:50) and then I am going to draw $-\cos\frac{2\pi t_d}{T_n}$ let us see how does it look like.

This is a function $\left(-\cos\frac{2\pi t_d}{T_n}\right)$ basically which starts from -1 and it goes something like this. So, this is the first peak that occurs and that occurs at $T_n/2$, the second peak occurs at $3T_n/2$ and so on. and if I sum this up, this whole thing would be shifted by one upwards. So, it would look like something like this. This is the response during the forced vibration phase.

So, let us say this is $u_0 / (u_{st})_0$ on vertical axis. So, this is 2 and this is t/T_n or if you write it as t (in graph of $-\cos\frac{2\pi t_d}{T_n}$) here then it would be $T_n/2$, $3T_n/2$. So, what I see here, this would achieve a value of 2 only if my $t_d/T_n \geq 1/2$.

So, I hope you understand this. this is the expression that I have for R_d and this expression can achieve a value up to 2 provided this condition is satisfied. So, $\frac{2\pi t_d}{T_n}$ as π, 2π, 3π and so on and like that. So, I will write R d is equal to 2 if $t_d \geq T_n/2$. So, the question comes well, what happens if $t_d \leq T_n/2$?

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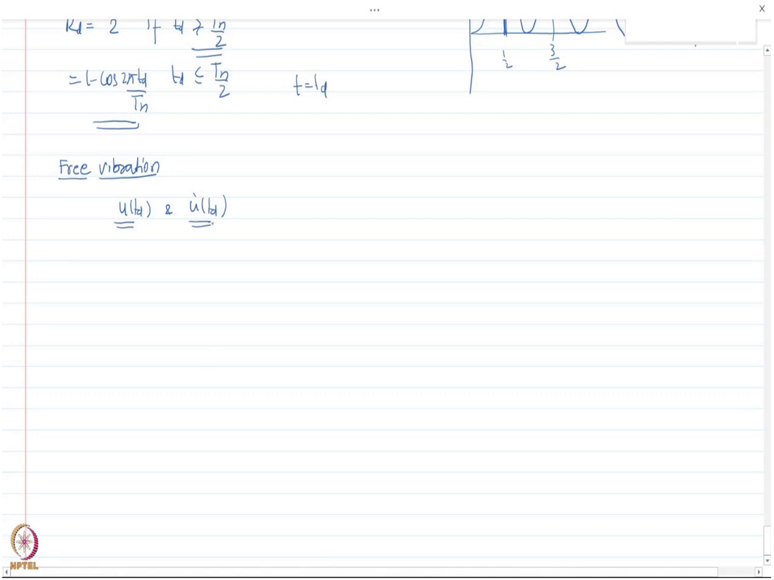

If $t_d \leq T_n/2$, then the maximum response would be during the forced vibration phase whatever the response as at $t = t_d$. Which basically we have $1-\cos\frac{2\pi t_d}{T_n}$. So, it would be just same value and you put the value of t_d here. So, if you look at here if you substitute

${}^{t_d}/_{T_n} = {}^{1}/_{2}$, then it becomes 2, but the duration always might not be large enough so, that you can have at least one peak during the forced vibration phase.

So, I have obtained two conditional expressions for the maximum during the forced vibration phase. Let us now go into the free vibration phase. So, for free vibration phase basically we know that system undergoes free vibration with initial conditions as $u(t_d)$, and $\dot{u}(t_d)$.

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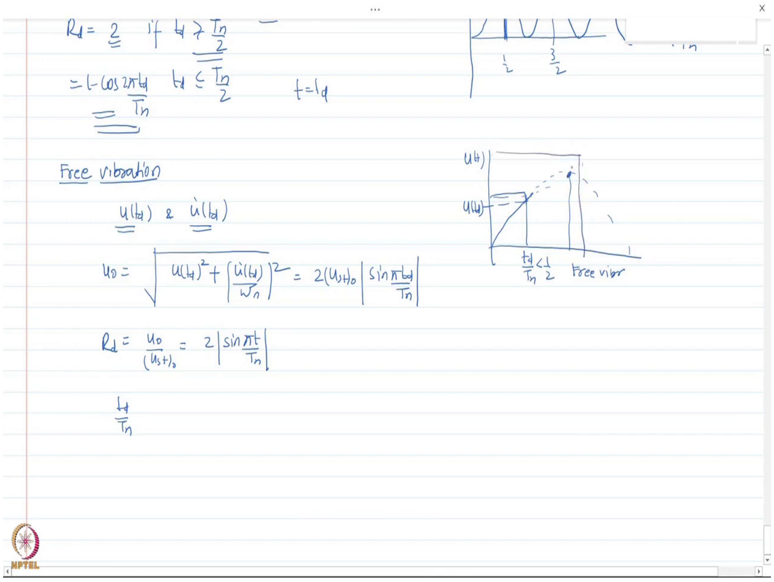

And I know that if a system is undergoing free vibration with these initial conditions, the amplitude of vibration can always be obtained as using this expression here-

$$u_0 = \sqrt{\left[u(t_d)\right]^2 + \left[\frac{\dot{u}(t_d)}{\omega_n}\right]^2}$$

You can find out the value of $u(t_d)$ and $\dot{u}(t_d)$ from the expression $\left[u(t) = (u_{st})_0\left(1 - \cos\frac{2\pi t}{T_n}\right)\right]$ and substitute to obtain the expression for u_0.

$$u_0 = 2\left(u_{st}\right)_0 \left|\sin\frac{\pi t_d}{T_n}\right|$$

So, that my R_d becomes-

$$R_d = \frac{u_0}{\left(u_{st}\right)_0} = 2\left|\sin\frac{\pi t_d}{T_n}\right|$$

So, basically, we saw that during the forced vibration phase depending upon the value of t_d/T_n the maximum could be 2 or maximum could be this $1-\cos\frac{2\pi t_d}{T_n}$.

If $t_d < T_n/2$ Then what happens? In that case my response is still increasing. (Refer Slide Time: 56:20) So, this is let us say $u(t)$ (on vertical axis) this response here is basically your $u(t_d)$ and then there is some velocity here as well, but the maximum response then occurs during the free vibration phase. This is for the case then let us say $t_d/T_n < 1/2$. If this is $t_d/T_n > 1/2$ then this peak would be inside for that case.

So, for that case when $t_d < T_n/2$, the maximum occurs during the free vibration phase and that maximum is actually this $\left(2\left|\sin\frac{\pi t_d}{T_n}\right|\right)$ value. So, if we combine both what do we actually get?

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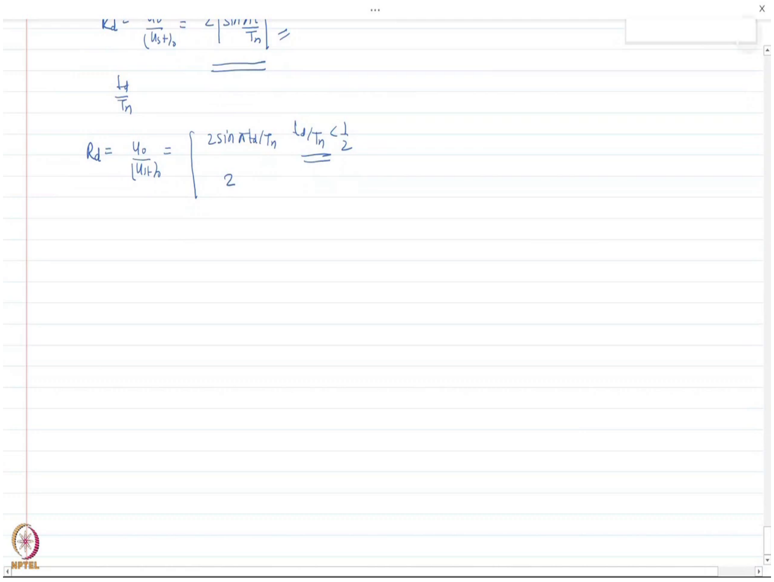

So, if you combine the forced vibration phase and the free vibration phase I can get one expression for R_d which would be equal to

$$R_d = \frac{u_0}{(u_{st})_0} = \begin{cases} 2\left|\sin\frac{\pi t_d}{T_n}\right| & \frac{t_d}{T_n} \le \frac{1}{2} \\ 2 & \frac{t_d}{T_n} \ge \frac{1}{2} \end{cases}$$

So, if $\frac{t_d}{T_n} \le \frac{1}{2}$ then peak occur during the free vibration phase which is $2\left|\sin\frac{\pi t_d}{T_n}\right|$ and if $\frac{t_d}{T_n} \ge \frac{1}{2}$, then the peak occurs during the forced vibration phase which is equal to 2.

And you can go ahead and plot these two functions. So, I am just going to (Refer Time: 57:46) the final figure that we have here and we just copy this. So, this is basically what I am saying, let me write this here.

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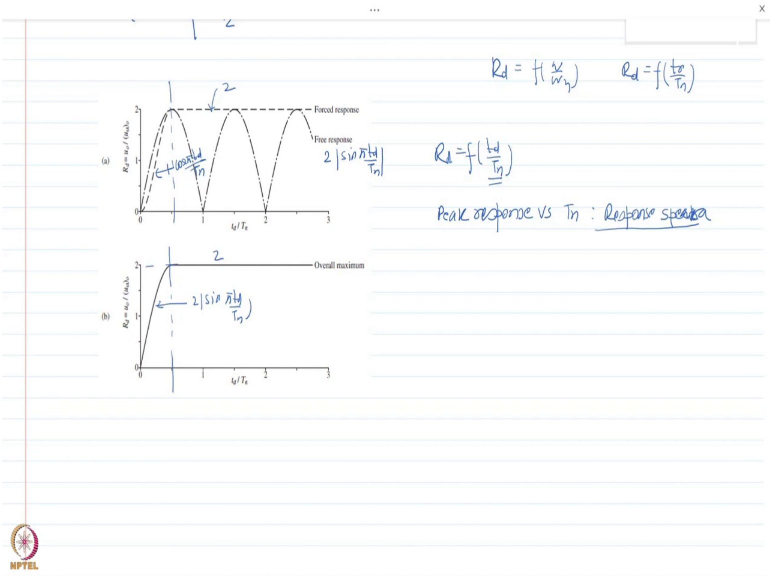

This is forced vibration response (dotted line) and this is for free vibration response (solid line). So, for free response I have $2\left|\sin\frac{\pi t_d}{T_n}\right|$. Forced response is when it is $\frac{t_d}{T_n} \le \frac{1}{2}$, this value is basically here which is $1-\cos\frac{2\pi t_d}{T_n}$ and if it is $\frac{t_d}{T_n} \ge \frac{1}{2}$ then it is 2.

So, if you take the envelope of both these functions you get basically the overall maxima. In this one this is governed by the free response $\left(\frac{t_d}{T_n} \le \frac{1}{2}\right)$ and this function is basically $2\left|\sin\frac{\pi t_d}{T_n}\right|$ and this $\left(\frac{t_d}{T_n} \ge \frac{1}{2}\right)$ is basically the value equal to 2. So, if you look at it, we are able to obtain the $R_d = f\left(\frac{t_d}{T_n}\right)$ and this is similar when we have obtained R_d for a harmonic excitation as $f\left(\frac{\omega}{\omega_n}\right)$ or for a ramp loading we had obtained this as function of $f\left(\frac{t_r}{T_n}\right)$.

In this case we are obtaining as $f\left(\frac{t_d}{T_n}\right)$ in which t_d is the load duration and T_n is period of the system. This is the plot of peak response corresponds verses some function of T_n or some multiple of T_n, it is called a response spectra.

So, plot of peak response of a single degree of freedom system versus its time period T_n is called response spectra and it is a very useful tool that is used by designers in industry. For example, not everybody is going to obtain the differential equation then go through all those calculations and get the peak response.

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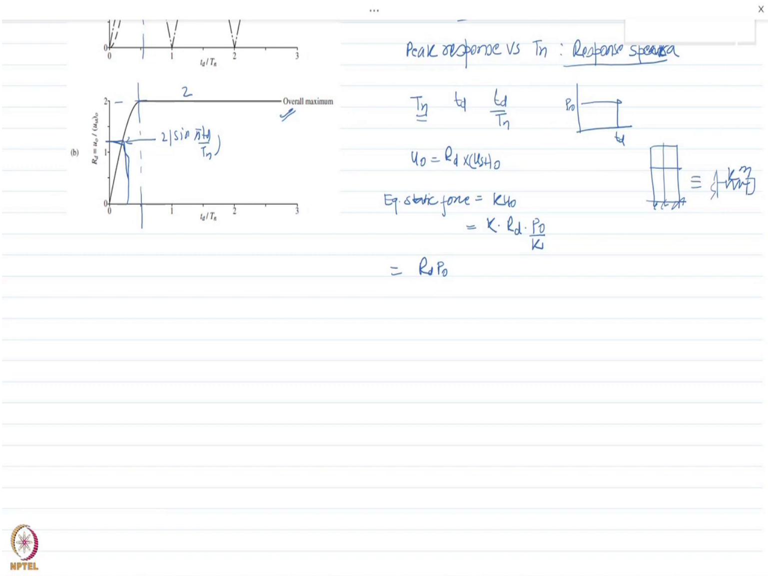

So, let us say this chart is given to us for a rectangular pulse loading, then you have a structure of time period T_n and you know that a load duration rectangular pulse is being applied with a load duration of t_d. So, you can find out the t_d/T_n. So, basically P_0 and t_d is known to you and T_n is also known to you.

So, if those parameters are known to you, you do not have to solve any differential equation. You can find out the maximum value of u_0 as $R_d\left(u_{st}\right)_0$ and R_d can be directly obtained from here (graph). So, let us say for any given value of t_d/T_n you can just go here (graph) and then find out what is the value of R_d. If you know u_0 then you can also find out the equivalent static force in the structure as ku_0 assuming that it is represented as a single degree of freedom system.

So, I am saying if you can represent this building as a single degree of freedom system something like this with *k* and *m*, then for that case the static force is *k* times the peak dynamic displacement (ku_0). Which we can further obtain as $kR_d(u_{st})_0$ and $(u_{st})_0 = P_0/k$.

So, the equivalent static force is basically $R_d P_0$. So, with this the discussion on rectangular pulse duration actually concludes. We are going to extend the same procedure to find out the response for other type of pulses in the subsequent classes.

Thank you very much.