Dynamics of Structures

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Free Vibration of MDOF Systems
Lecture - 22
Mode shapes and frequencies

Hello everyone. So, till now in Multi Degree of Freedom System we have seen how to set up the equation of motion. Now, the next step is to learn methods to solve those equations of motion to get the response of a multi degree of freedom system. When we considered single degree of freedom system, we saw that there is only one way in which a single degree of freedom system can respond.

However, a multi degree of freedom system which is made up of several degrees of freedom can respond in multiple ways, and the total response actually is combination of different type of responses that the structure can uniquely respond in. And, we are going to look into that.

So, we are going to also learn about mode shapes, which basically reflects or which basically describes one of the ways in which a structure can deflect. And, we are also going to learn about frequencies. Now, a multi degree of freedom system would have multiple frequencies, so we will see that how to actually find out those frequencies. So, let us get started.

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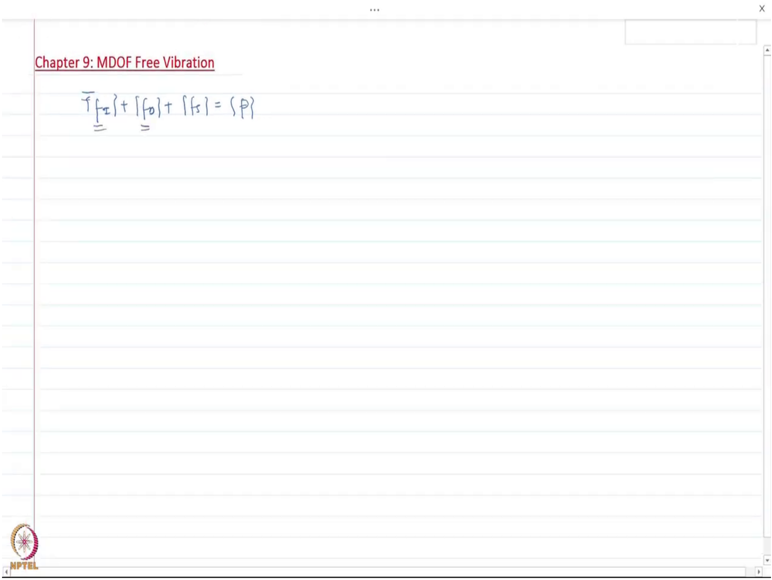

We have already studied how to set up the equation of motion of a multi degree of freedom system, and we saw that now the forces are represented in terms of vectors and the displacement are represented in terms of vectors and there are mass matrices and stiffness matrices. So, as against to a single degree of freedom system we set up the equation of motion for a multi degree of freedom system in terms of these forces.

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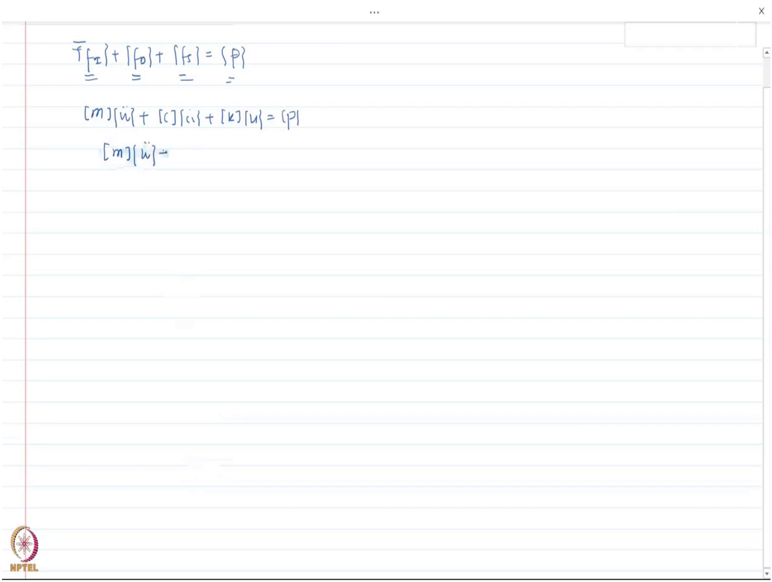

And we saw that how we can get the mass matrix, the damping matrix and the stiffness matrix. Let us see, once we have obtained our equation of motion as this

$$[m][\ddot{u}]+[c][\dot{u}]+[k][u]=0$$

. How do we solve this equation of motion ok?

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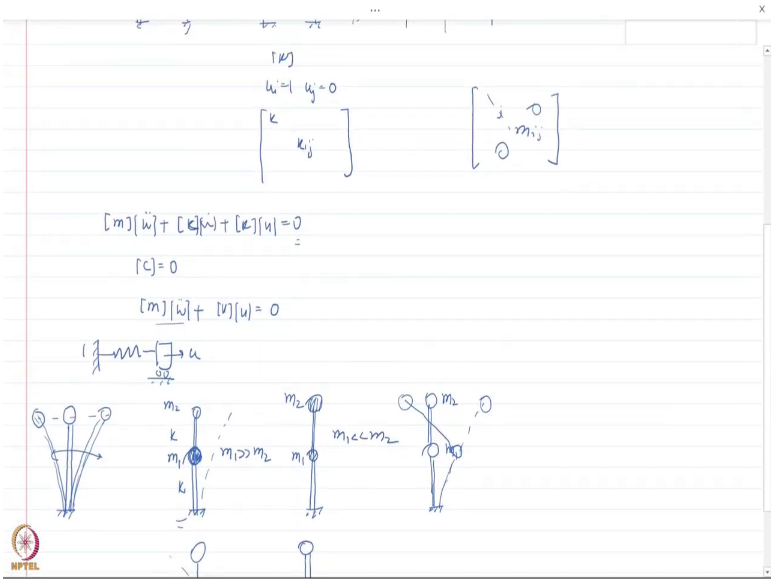

Can we employ the same analytical solution that we have derived for single degree of freedom system to solve the multi degree of freedom system? And if we can, then how do we do that? So, first thing that we are going to talk about is basically free vibration. And, let us first consider undamped free vibration, so right damping matrix is 0 and there is no external force, so I am considering that 0.

So, my equation of motion becomes mass times acceleration vector and then K times displacement vector to be equal to 0. So, we are going to discuss about the solution of this equation, but to get into that first we need to discuss some basic characteristic of vibration for a multi degree of freedom system.

$$[m][\ddot{u}]+[k][u]=0$$

Now, we know that for a single degree of freedom system; let us say, let me just write here. I had only one degree of freedom u, and there was only one possible way in which a single degree of freedom system could move. So, by definition I needed only one degree of freedom to represent the default position with respect to its initial equilibrium position.

However, let us consider a two degree of freedom system something like this. So, the such that I am going to consider different two degrees of freedom such that, the total mass remains the same. However, the distribution is different little bit. So, let me draw in the first let me draw the one degree of representation.

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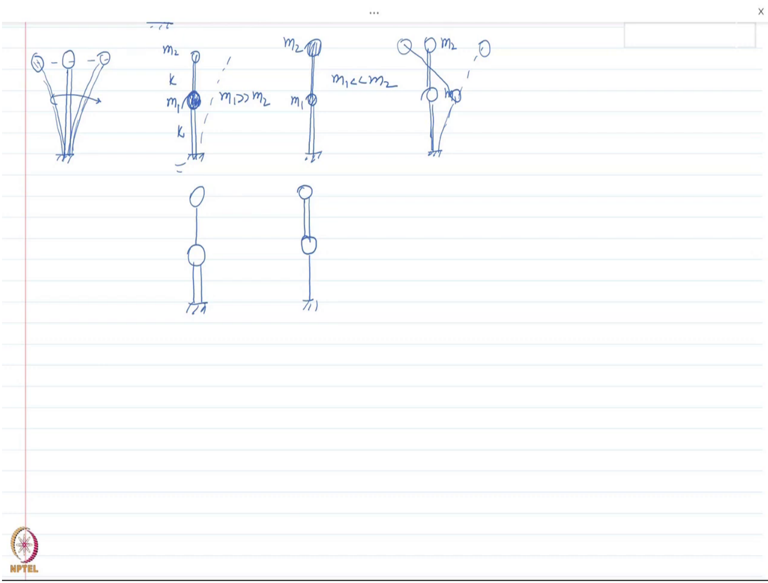

This is the one degree of representation of a system. Now, let us consider, so there is only one way it can deflect right, we saw that you only consider deflection like this. So, if it is a deformable consider deflection like this. So, we know how it would deflect; however, let us consider system like this in which I have 2 masses, which are joined first from the support to the first mass I have considered the same stiffness where stiffness K and stiffness K.

However, mass m_1 and m_2 and m_1 is much greater than m_2 here. In the second case though, let us consider different representation. So, again in this case I have m_1 m_2 and, in this case, m_1 is much smaller than m_2 here. And, in the third case I am going to consider m_1 and m_2 which are comparable to each other.

As you can imagine, physically if you look at it there are multiple ways or in this case if it is a two degree of freedom system, there are two ways in which this structure or this system can deform. One would be simply like this; however, if this mass is very heavy then it is likely that the movement of this mass is going to dictate the overall response.

Now, compare this to the second system that I have drawn here, in this case if the mass m_2 is much larger than mass m_1, then overall response is going to be dictated by the movement of mass m_2. But, in general reality lies somewhere in between, where mass m_1 and m_2 are comparable and basically the deformation or relative deformation of both m_1 and m_2 actually going to dictate the overall response.

And for two degree of freedom system, it can either respond like this or it can also respond like this. So, there are only two ways in which this structure can deform in two dimensions. So, for multi degree of freedom system, the total mass and the stiffness are not the only parameter, the distribution of mass and stiffness also plays a role.

And we saw that for the distribution of mass here, I can do the same thing where I keep the masses same, but I consider different stiffnesses. And, in this case also you we can imagine that depending upon the stiffness between the masses and the support, the overall response is going to change.

Now, in previous chapters we saw, that we used to assume a deflected shape of a continuous system and utilize that which we called as a shape function and utilize that to reduce this to single degree of freedom system. But that was an approximate method, and we assume that that the deflected shape can be represented by static deflected shape or some other function.

However; through the analysis of multi degree of freedom system, we wish to find out what would be the exact deflected shapes of a multi degree of freedom system. And, to get that we are going to do or solve the equation of motion that we have just written here.

Now, so as you can imagine the total response of a multi degree of freedom system depends on the distribution of mass and the distribution of stiffness. And it can deflect in multiple ways depending upon the number of degrees of freedom. So, the total response, so let us say.

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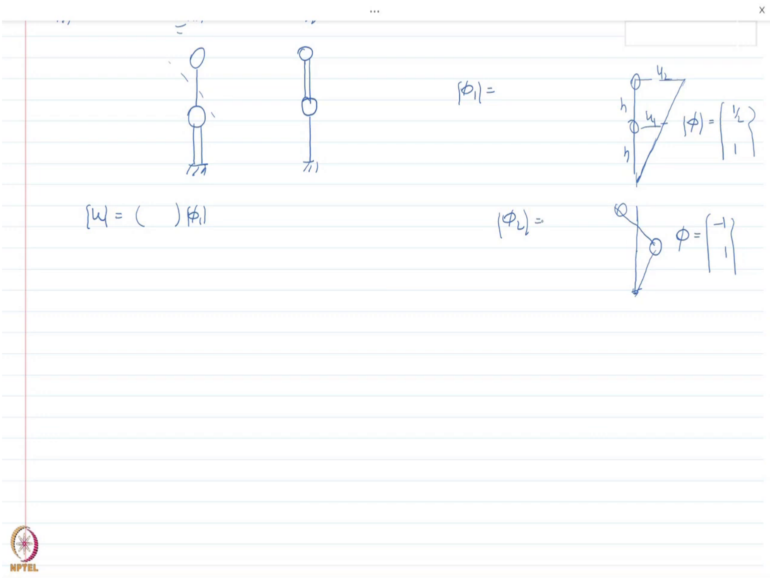

I have the total response as vector [u] can be written as linear combination of different deflected shapes.

$$[u] = ()[\phi_1]$$

So, I am using ϕ_1 to represents a deflected shape. This is similar to the shape vector that we had used in previous chapters. So, let us say I have a deflected shape which is something like this, linearly increasing with height.

So, if the height is h here and height is h, then this deflected shape can be written as half and 1, where this is the first degree of freedom and this is the second degree of freedom. Similarly, if I have a deflected shape which looks like something like this, I can write my phi as let me say -1 and 1. So, these are some of the possible ways in which a system can deflect.

Now, this is a two degree of freedom system. You could have multiple degree of freedom system and there could be different shapes. So, these are the possible ways in which a structure can deflect, but we still do not know that which deflected shape is going to be the close or going to determine the overall response. So, what do we do? We consider all deflected shape and we say that the total response is actually going to be sum of all such deflected shape.

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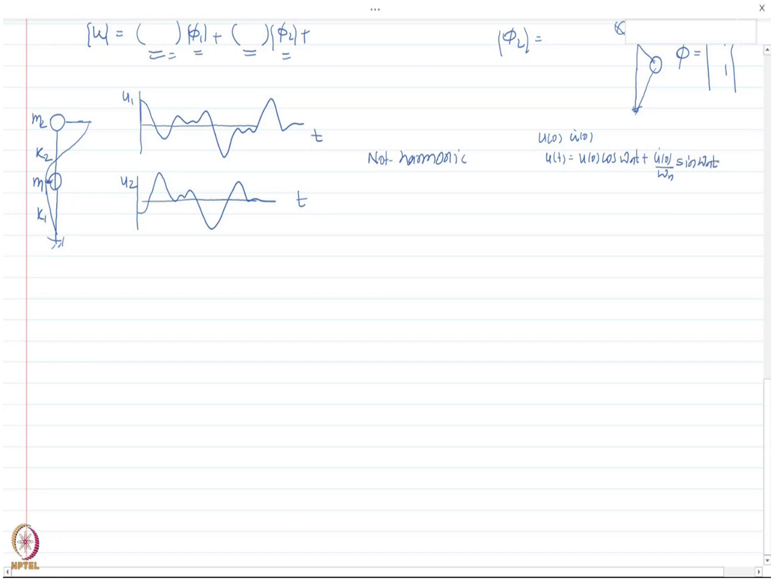

A linear combination. And these factors actually determine the relative contribution of each of these mode shapes to the overall response. So, this is how I imagine my multiple response of a multiple degree of freedom system. And we are going to see, how to find out these shapes exactly and how to find out these factors. So, let us start with an example of a two degree of freedom system.

Let us say this is m_1 m_2 and this is K_1 K_2 and let us say it has been given some vibration, initial vibration. So, it might be given some initial displacement. Let us say it looks like something like this. So, this is the initial displacement at these two degrees of freedom. Now, if we give initial displacement to these two masses and then plot the resulting motion for these two masses, we are going to see that in general the motion of these masses is not harmonic.

Similarly, for this, so this is u_1 plotted against time and this is u_2 plotted against time. So, if I give some initial displacement to a multi degree of freedom system and let it vibrate, the response of each degree of freedom in general is not harmonic.

Now, compare that to single degree of freedom system in which when we provided initial condition u_0 and , we said that the resulting motion would be harmonic and it was given as

$$u(t)=u(o)\cos w_n t+\frac{\dot{u}(0)}{w_n}\sin w_n t$$

but that is not the case here. So, in general the resulting motion is not harmonic.

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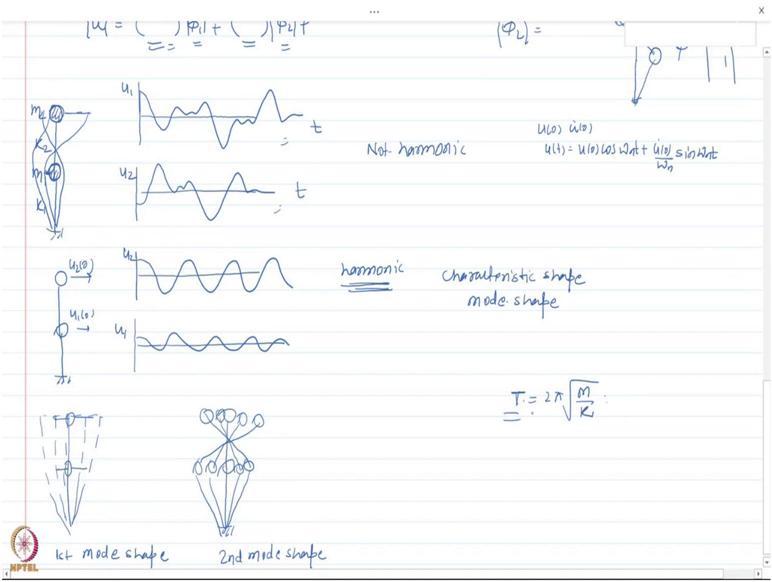

Plus, as these two masses m_1 and m_2, vibrates we will see that the structure or these two degrees of freedom system here, the shape of these two degrees of freedom system is going to change with time. So, after some time it may look like something like this or it may also look like something like this.

So, the shape is actually going to change. But there is a possible combination of initial displacement given to these degrees of freedom, such that if these two degrees of freedom system. Given initial displacement in such a fashion, there exists a combination of $u_1(0)$ and $u_2(0)$ such that the resulting motion is actually going to be harmonic.

So, let me redraw it with initial displacement. This is going to be harmonic, and u_1 is also going to be harmonic. So, there exists few combinations, such that, if the motion is initiated by providing those initial displacement to each degree of freedom, then the resulting motion of each degree of freedom is actually harmonic.

Those combination of degrees of freedom or, those combinations of initial displacements are actually called characteristic shape or also called mode shape of a multi degree of freedom system. Now, for two degree of freedom system, we would have two such characteristic shapes or mode shapes such that if it is given, the initial displacement with those proportions then it would have resulted motion at the degrees of freedom which would be harmonic in nature, plus it would maintain its shape.

So, for example this two degree of freedom system, if I give initial displacement like this ok, it is going to vibrate such that it maintains its shape. So, if you can look at here, the relative at any time T, the relative shape of the structure is not changing. Of course, that overall amplitude is changing, but the relative shape is not changing.

And if I have an n degree of freedom system, then I would have n such characteristic shape. For example, this is the first type first characteristic shape, or first mode shape, second mode shape could be something like this.

I can have something like this, then something like this and then. So, each of these the shape is actually not changing, so this is second mode shape. So, let us just recap, we said that in general if you initiate the motion, the free vibration motion of a multi degree of freedom system by providing initial displacement to different degrees of freedom, the resulting motion at each degree of freedom would not be harmonic and the structure is going to change its shape at each time instant.

However; there exists few characteristic shapes with which if the initial displacements are assigned proportional to those characteristic shape, then the resulting motion is going to be harmonic plus, the structure is going to maintain its shape. Those characteristic shapes are called mode shapes or and a n-degree of freedom system would have n such mode shape or characteristic shapes. Alright, now, we defined for a single degree of freedom system the natural time period as

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The question comes, how do we define or analytically we said that it is time taken to complete one cycle of motion or vibration. Now, as I said in general, the resulting motion here is not harmonic.

So, then how do we define this time period for a multi degree of freedom system? So, for a multi degree of freedom system, the time period is not defined for the overall response but, it is defined for each mode shape.

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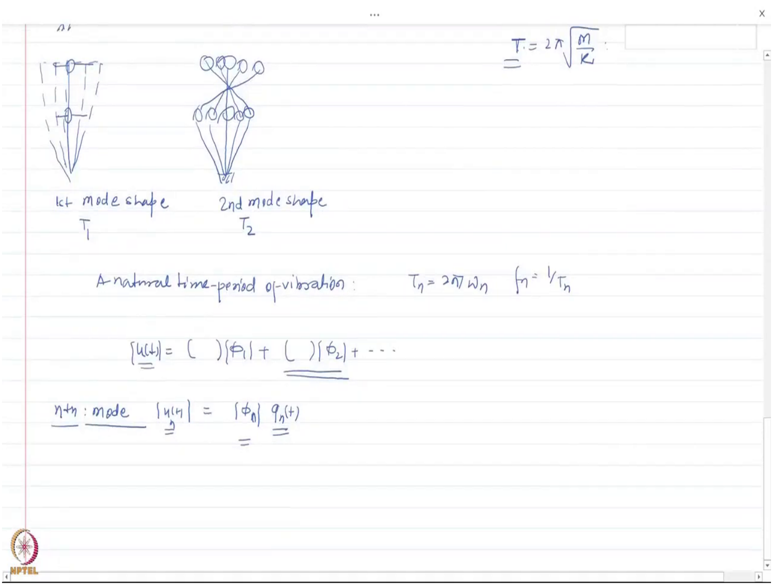

So, the time period we defined for each mode shape. So, let us say I have T_1 here for the first mode shape and then T_2 here is the second mode shape. And, T_1 or let us say a natural time period of vibration is basically the time taken to complete one cycle of vibration in a particular mode. So, T_1 T_2 T_3 like that for a multi degree of freedom system. Now, as we have previously studied, this relationship is still hold that T_n would be $T_n = \frac{2\pi}{w_n}$ or frequency would be $1/T_n$.

So, we have defined or we have discussed that how a structure can deflect in multiple ways. In general, if we initiate the motion by providing initial displacement and velocity or whatever the resulting motion at each degree of freedom is not harmonic, but there exists some shapes according to which if the structure is assigned initial deflection and allowed to undergo free vibration; then in those cases the resulting motion would be harmonic and the structure would it maintain its shape while vibrating in those characteristic shape. Those are called mode shapes.

So, as we have discussed, the total response of a multi degree of freedom system can be represented as some linear combination of each mode shape. So, let us say I have n degree of freedom system, so it would be some linear combination of first mode shape, then second mode shape like that.

Now, the free vibration response of a multi degree of freedom system, the total response u(t) ok in the n_{th} mode let us say, this is the n_{th} mode, can be written as the mode shape. So, I am just considering one of these mode shapes.

Let us consider the contribution of the n_{th} mode shape to the total response by writing it as $u_n(t)$ and that would be equal to ϕ_n the mode shape times the time variation $q_n(t)$. And that we have previously also discussed.

$$\{u_n(t)\} = \{\phi_n\} q_n(t)$$

That, if the deflected shape of a structure is represented through some shape function then the total response can be represented as a product of that shape function times some time variation function that represents the evolution in time.

Now, as we have said, this ϕ_n is actually constant in time does not vary with time. So, the time variation is actually represented by $q_n(t)$, and this $q_n(t)$ as we have said that if it is vibrating in one of its mode shape the resulting motion would be harmonic.

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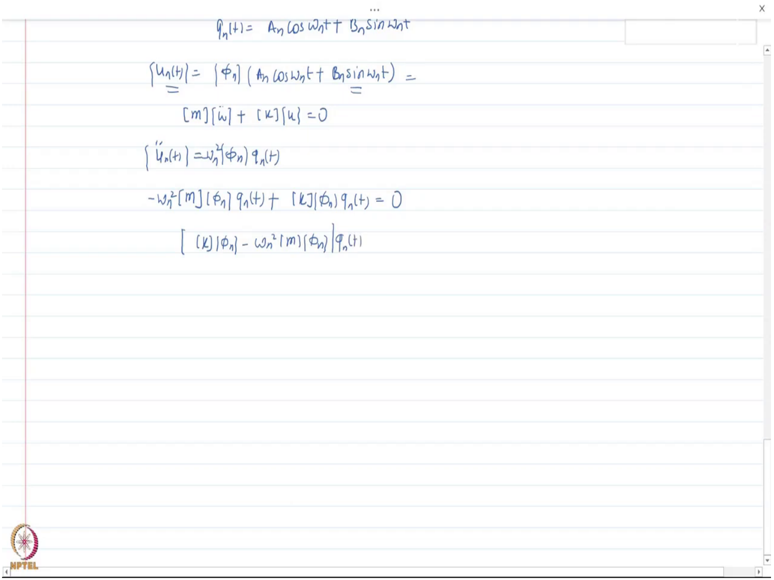

So, can I say that

$$q_n(t) = A_n \cos w_n t + B_n \sin w_n t$$

This is the general equation of motion for the vibration or general equation of motion for the harmonic vibration. And, this is for the n_{th} mode.

So, the response contribution in the n^{th} mode can be represented as shape factor for the n^{th} mode or mode shape times $q_n(t)$ which we have written as

$$\{u_n(t)\} = \{\phi_n\}\left(A_n \cos w_n t + B_n \sin w_n t\right)$$

and our equation of motion for free vibration is this.

$$[m]\{\ddot{u}\} + [k]\{u\} = 0$$

So, what we are going to do here? We are going to take this and substitute in this equation of motion.

Now, if I double differentiate this term here, let us see what do we get. So,

$$\{u_n(t)\} = w_n^2 \{\phi_n\} q_n(t)$$

So, I am going to substitute that here;

$$-w_n^2[m]\{\phi_n\} q_n(t) + [k]\{\phi_n\} q_n(t) = 0$$

$$\left[[k]\{\phi_n\} - w_n^2[m]\{\phi_n\} \right] q_n(t) = 0$$

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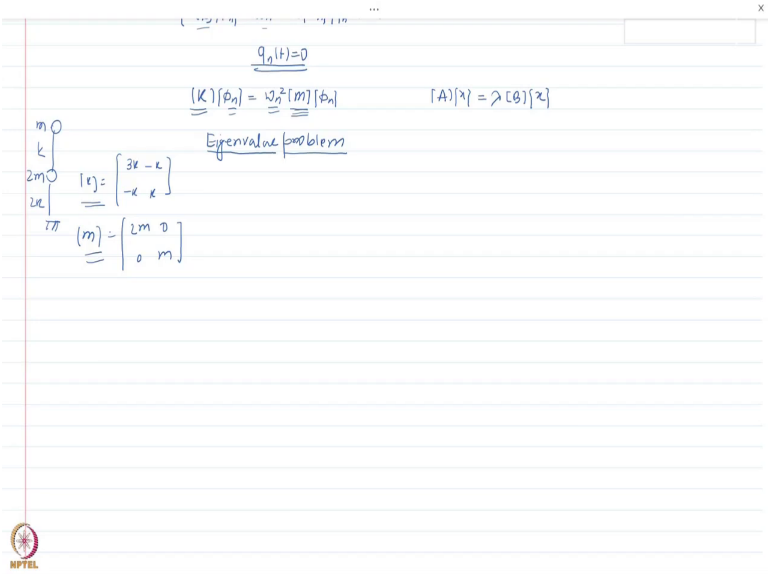

Now, this equation tells us that either $q_n(t) = 0$, which means that there is no motion and which is called like you know trivial solution, so we are not bothered about that. We are the only thing that is going to give us meaningful solution in this one when we consider

$$[k]\{\phi_n\} = w_n^2[m]\{\phi_n\}$$

If you look at this carefully, this is of the form a vector times

$$[A]\{x\} = \lambda[B]\{x\}$$

This is called eigenvalue problem, and you may have come across this kind of problems in your advanced numerical methods course or mathematics course or matrix algebra course.

So, this is basically eigenvalue problem. Now, in this equation my K matrix is known to me, my m matrix is known to me. So, remember if I have a like you know multi degree of freedom system something like let us say this, K, 2K I know how to get my K matrix right which in this case would be

$$[k]=\begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$$

$$[m]=\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

We saw that how to get these matrices. So, in general if the structural properties are given my K matrix is known to me and my m matrix is known to me. The unknown parameters are actually w_n^2 and the mode shape ϕ_n. And through this eigenvalue problem we are going to solve for these unknown parameters.

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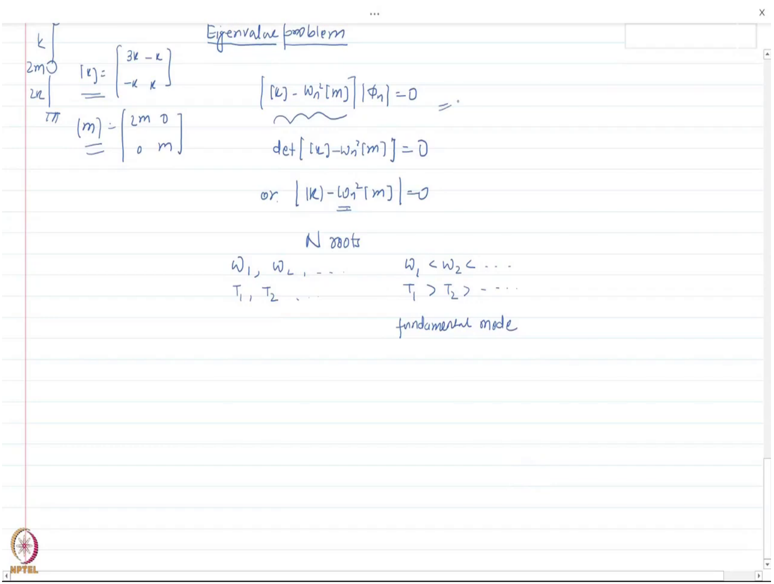

So, basically let me rewrite that

$$\left[[k]\{\phi_n\} - w_n^2[m]\{\phi_n\}\right] = 0$$

Now in this case, this would have a non-trivial solution if this term here is equal to 0. So, this is like solution of n simultaneous equation of motion. And, because this is homogeneous one solution could be the $\phi_n = 0$, that is the trivial solution. So, for non trivial solution we need to have this equal to 0 or specifically not this actually, but the determinant of the coefficient

$$w_2 = \sqrt{\frac{2k}{m}}$$

So, you can either write determinant of this or you can just use in this equal to 0. This is how we get the solution to an eigenvalue problem. Now, if let us say these are n homogeneous equation, then this would give me, remember the constant here is w_n^2 not w_n, so this give me an algebraic equation of n[th] power of this constant here, which would be w_n^2 to the n[th] power.

So, it would have N roots which would be real and positive, because my K matrix and m matrix are usually real and positive definite and symmetric, so in that case I would have N roots here. So, we can solve those type of equation of motion and we can find out the frequencies; w_1, w_2 so on. And, typically we arrange it like w smallest frequency first or if you want to write in terms of time period, we will have time period T_1 T_2 and we arrange it so the largest time period mode is first. This is just a convention.

The first mode is also called the fundamental mode. And, once we have found out all these roots w_1, w_2, remember it will give you $\begin{vmatrix} (3k - 2mw^2) & -k \\ -k & k - w^2 m \end{vmatrix} = 0$, w_2^2, but then you can take this square root and find out w_1, w_2, and so on. So, once you get that you can substitute it back to this equation.

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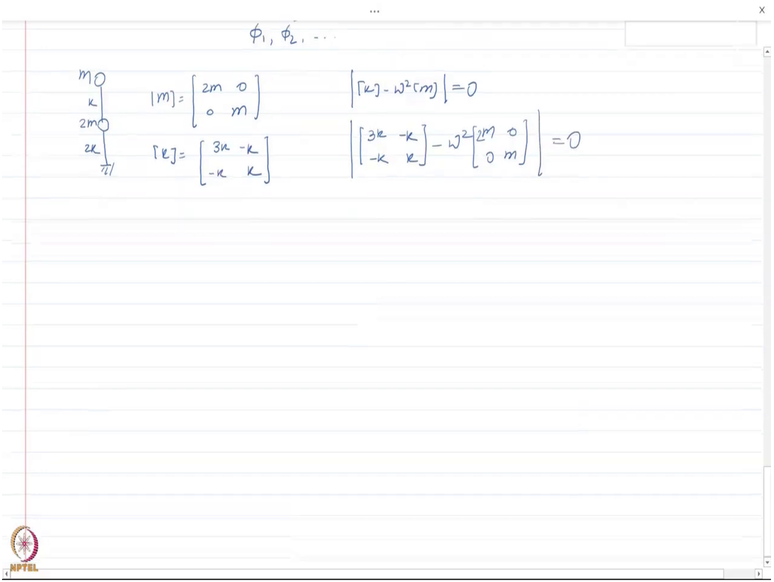

And you can find out the shape function ϕ_1 , ϕ_2 I was more specifically the mode shape here, the shape vector ϕ_1 , ϕ_2 and so on corresponding to each frequency. And that, through this procedure we can find out the exact mode shapes and frequency of a multi degree of freedom system.

And let us take one example to see how do we do that. So, I am going to take example of just what we have seen above. I have a two degree of freedom system with storage stiffnesses as K, 2K and K and this is 2m and m. So, the mass matrix I had assembled as

$$[m]=\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

$$[k]=\begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$$

So, let us see what are the modal frequencies and modal shapes for this two degree of freedom system. So, our equation was

$$\left|[k]-w_n^2[m]\right|=0$$

So, let us substitute that and see what do we get. So, I have here

$$\left|\begin{bmatrix}3k & -k\\ -k & k\end{bmatrix}-w^2\begin{bmatrix}2m & 0\\ 0 & m\end{bmatrix}\right|=0$$

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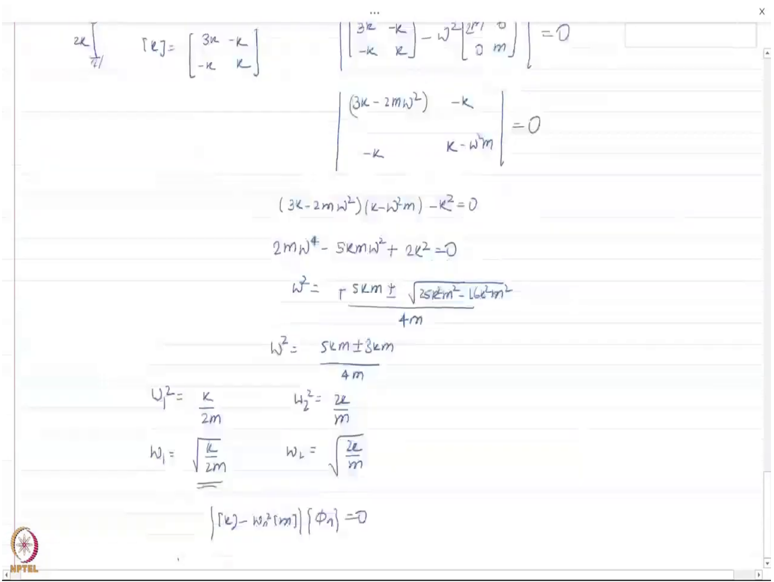

So, I basically get as

$$\begin{vmatrix}(3k-2mw^2) & -k\\ -k & k-w^2m\end{vmatrix}=0$$

So, the equation that I get is

$$(3k-2mw^2)(k-w^2m)-k^2=0$$

So, this would give me a quadratic in w^2, and let us see what that equation is.

$$2mw^4-5kmw^2+2k^2=0$$

$$w_1 = \sqrt{\frac{k}{2m}}, \quad w_2 = \sqrt{\frac{2k}{m}}$$

So, we have got the two frequency of this two degree of freedom system, once we have that let us substitute back in the eigenvalue equation.

$$\left[[k] - w_n^2[m]\right]\{\phi_n\} = 0$$

So, let us first get the first mode shape. So, I can going to substitute

$$w_1 = \sqrt{\frac{k}{2m}}, \quad w_2 = \sqrt{\frac{2k}{m}}$$

Once I substitute that let us see what do we get. So, basically I would have these expressions inside and if I substitute $w_1 = \sqrt{\frac{k}{2m}}$

$$\begin{bmatrix} 2k & -k \\ -k & k/2 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = 0$$

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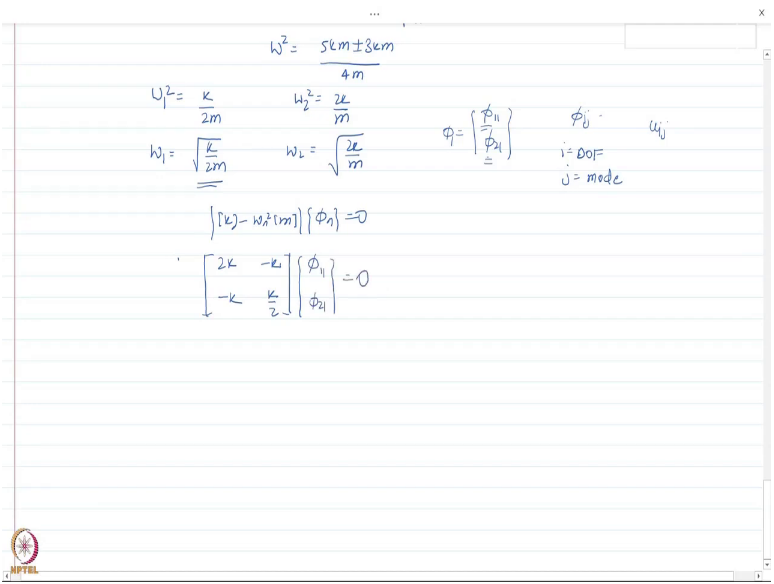

So, remember, if you have ϕ_{ij} the i corresponds to degree of freedom and j corresponds to the mode. And the same would be true for other response vectors as well like u_{ij} and so on.

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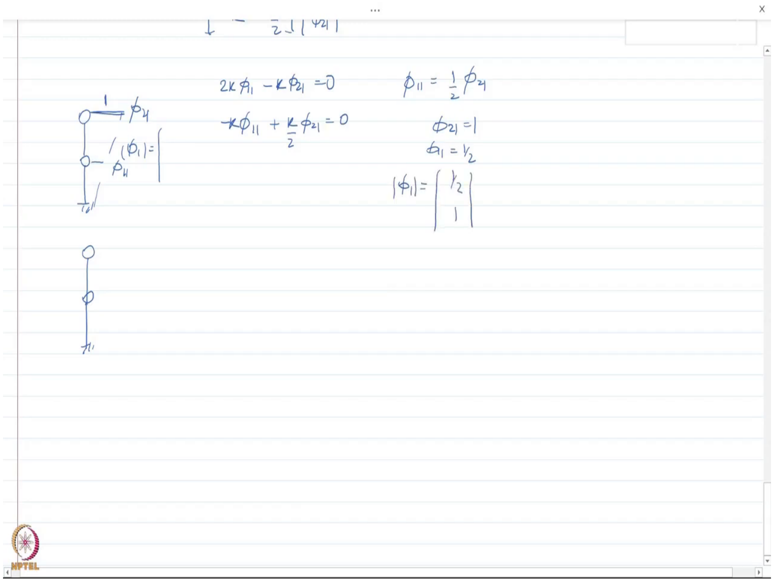

The equations we get is,

$$2k\phi_{11} - k\phi_{21} = 0$$
$$-k\phi_{11} + \frac{k}{2}\phi_{21} = 0$$

Now, if you look at carefully both of these equations are one and the same. Second thing is, you are going to notice you can only get solution for this vector in terms of some multiplicative constant.

For example, your ϕ_{21} or let us say ϕ_{11} is nothing, but

$$\phi_{11} = \frac{1}{2}\phi_{21}$$

So here, because the homogeneous equation it can be infinite solutions. So, you can only get solution for this mode shapes which are in terms of some multiplication factor, and that is because, when we represent the shape factor or a mode shape, these are always the relative location of each degree of freedom.

So, what we do typically in these cases to get the mode shapes? We normalize these mode shapes somehow such that, we assume or we find out the individual elements of these mode shape. So, one of the ways in which we can normalize and especially if it is a representation of a multistorey building, is that we can normalize it with respect to the top storey degree of freedom. So, let us say we assume it to be 1 which for , remember for ϕ_1 this quantity is ϕ_{11} and ϕ_{21} this is.

For ϕ_2 , again the same quantity would be ϕ_{22} and ϕ_{21} for the second mode. So, let us first deal with the first mode. So, if I assume my ϕ_{21} equal to 1 my ϕ_{11} become half. So, my first mode shape I get as

$$[\phi_1] = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

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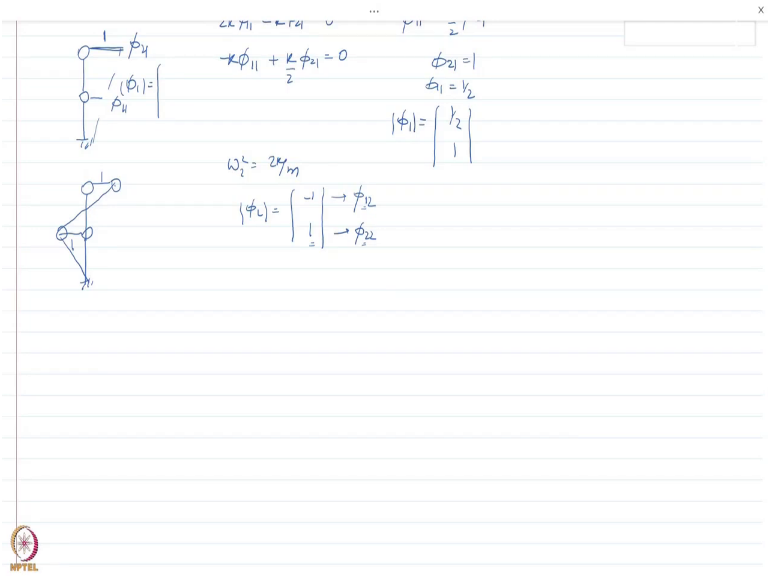

Similarly, I can substitute my

$$w = \sqrt{\frac{2k}{m}}$$

we will see I can get my second mode shape as

$$[\phi_2] = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

In the second mode shape we are normalizing with respect to the second coordinate.

So, in both cases I have normalized it with respect to the top storey modal coordinate. And there are different ways in which you can normalize it. You can normalize it with respect to and some specific degree of freedom ok or you can also normalize it with respect to some other method, we will see that later.

So, using this procedure we can get the mode shape and the frequencies of different modes of a multi degree of freedom system. Now remember, here in a multi degree of freedom system we would be doing lot of matrix algebra and it would be very easy if we learn or if we employ some of the techniques and the properties of this mass matrices, stiffness matrices, mode shapes you know, so that it becomes or it makes the job easier for us to solve some of the equation of motion.

So, let us learn some of the specific properties and some special matrices that we would be using to actually analyze multi degree of freedom system using matrix algebra.

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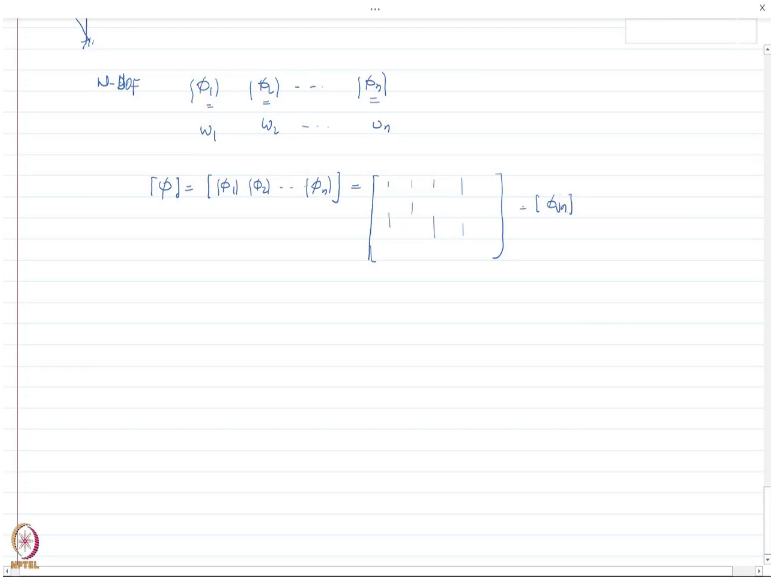

Now, remember for a n degree of freedom system I have said that I would have ϕ_1, ϕ_2 that represents the mode shapes of each mode, corresponding to that I would have the frequency for each mode shape as well. What I can do? I can combine all these mode shape vector, remember right now each of them are a column vector. I can write a mode shape matrix for which each column is actually one of the mode shapes vectors like this.

$$[\phi] = \left[\{\phi_1\}\{\phi_2\}......\{\phi_n\}\right] = \left[\phi_{jn}\right]$$

So, the overall matrix is basically representing columns that are the mode shape of a n degree of freedom system. This is called modal matrix.

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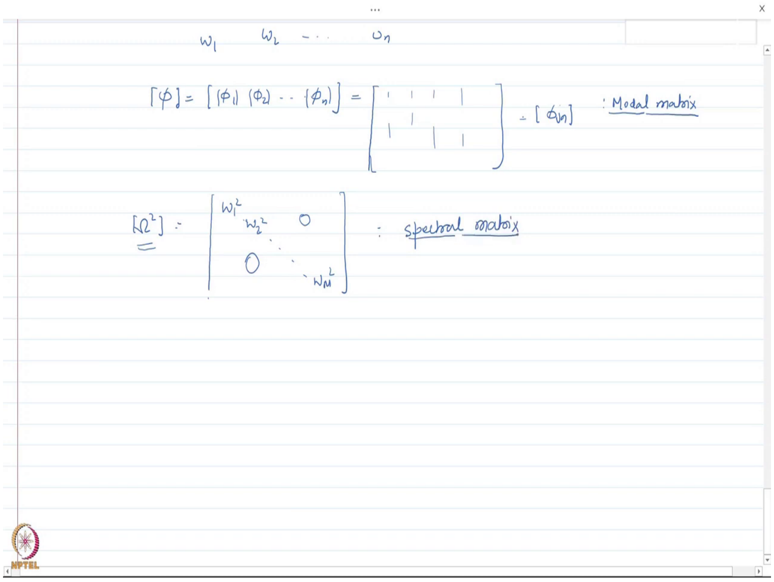

Similarly, I am going to define a diagonal matrix represented through this quantity here. In this each diagonal element is actually the frequencies of that individual mode and the off-diagonal term are 0.

$$\left[\Omega^2\right]=\begin{bmatrix} w_1^2 & 0 & 0 & 0 \\ 0 & w_2^2 & 0 & 0 \\ 0 & 0 & w_3^2 & 0 \\ 0 & 0 & 0 & w_4^2 \end{bmatrix}$$

And remember, I am considering frequency square because my eigenvalue is actually frequency square. The λ in that eigenvalue equation is actually frequency square. So, I am going to define a matrix which for which the diagonal terms are the frequencies square of each mode shape and this is called spectral matrix.

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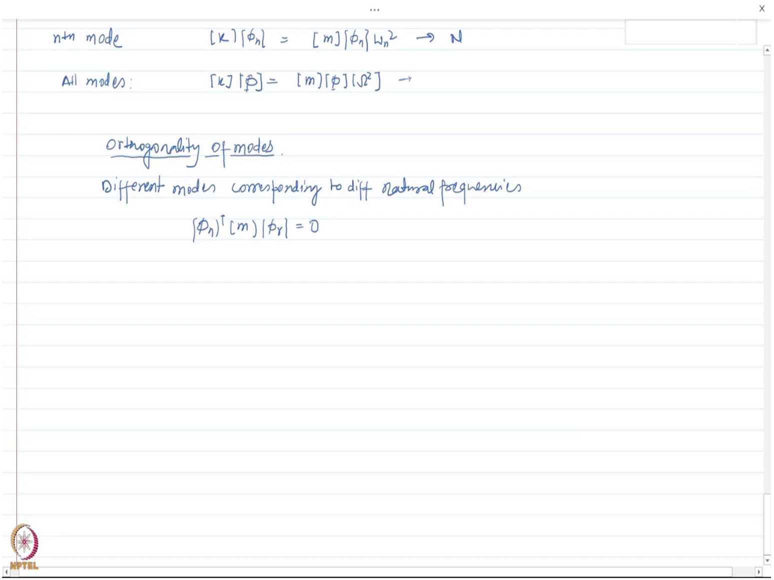

Now, let us get back to our eigenvalue equation which was

$$[k][\phi_n] = [m][\phi_n]w_n^2$$

Now, this is the equation, the eigenvalue equation for the nth mode.

We can combine all the modes together. So, remember this is for the nth mode for the n degree of freedom system I would have n such equations, n such separate equations. We can combine all those equation in a single equation so that, I can write for all modes matrix

$$[k][\phi] = [m][\phi][\Omega^2]$$

And this basically represents the same thing. This represents this would have if you expand it n such characteristic equations for n degree of freedom system ok. So, this is just a compact way of writing all the characteristic equations, the eigen equation eigenvalue problem for all the modes of a multi degree of freedom system.

Now let us look at a very important property of the mode shapes, which is going to help us immensely in solving the equation of motion for a multi degree of freedom system. That property is called orthogonality of modes. Now, the orthogonality of the modes states that the different modes corresponding to different natural frequency can be shown to satisfy the orthogonality condition which is basically, if I take a mode shape n and take a transpose of that and multiply it with the mass matrix times the mode shape of another mode. This would be equal to 0 and the same condition can be shown for the stiffness matrix as well.

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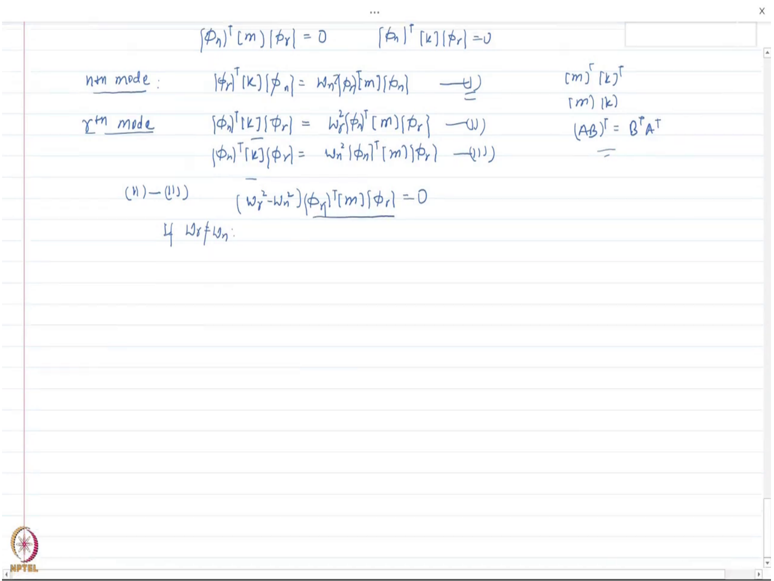

That, if I take a stiffness if I take the mode shape transpose of a mode shape n multiply with the stiffness matrix times another mode shape, again this would be equal to 0. And mathematically we can prove this. Let us see how we can do that. Remember that, I can write for my nth mode ok for the nth mode I can write my eigenvalue equation as

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

This is for the nth mode. Similarly, for the rth mode, which is a mode that is different from the nth mode, I can write down the similar equation,

$$[k][\phi_r] = w_r^2[m][\phi_r]$$

Now, what we can do here, we can multiply any of this equation with the transpose of other mode. So let me say, let me multiply this

$$[\phi_n]^T[k][\phi_n] = w_n^2[\phi_n]^T[m][\phi_n] \quad \text{........(1)}$$

$$[\phi_r]^T[k][\phi_r] = w_r^2[\phi_r]^T[m][\phi_r] \quad \text{........(2)}$$

Then, I can take basically the transpose of one of the equations. So let us say, I take the transpose of this equation on the both side remember, if a matrix is symmetric like m or K matrix, if you take the transpose of this, it would be same ok.

And, if you take transpose of 2 matrix, it would be transpose of the second matrix times the transpose of the first matrix. So, I am going to employ these rules here ok. So, if I take transpose of the equation 1, I would get as

$$[\phi_n]^T[k][\phi_r] = w_n^2[\phi_n]^T[m][\phi_r] \quad \text{........(3)}$$

So now, if I take equation 2 minus equation 3,

$$(w_r^2 - w_n^2)[\phi_n]^T[m][\phi_r] = 0$$

For $w_r \neq w_n$,

$$[\phi_n]^T [m][\phi_r] = 0$$

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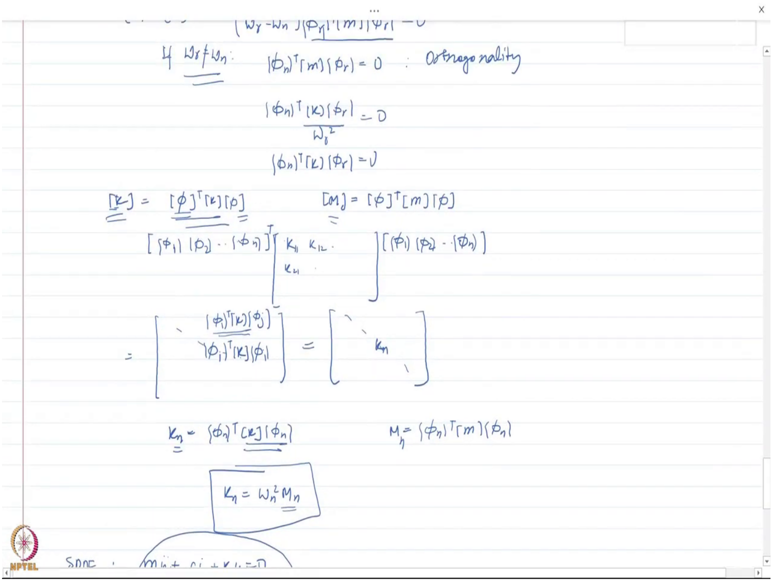

This is basically the orthogonality condition. Now, we have written this for mass matrix. We can derive the same expression for the stiffness matrix as well, just by writing this expression as

$$\frac{[\phi_n]^T [k][\phi_r]}{w_r^2} = 0$$

So, this would effectively become

$$[\phi_n]^T [k][\phi_r] = 0 \text{ for } w_r \neq w_n ,$$

So, this orthogonality condition is a very powerful basic conclusion that we utilize to solve our equation of motion, we will just see after discussing this.

Now, if these conditions are satisfied, let us define two square matrices like this. A square matrix K which is the stiffness matrix. K is defined as the shape factor or the modal matrix transpose times the stiffness matrix. So, this is capital K here.

$$[K]=[\phi]^T[k][\phi]$$

Now, look at this quantity here and similarly we are going to define the capital mass matrix as well, which is basically

$$[M]=[\phi]^T[m][\phi]$$

Now, if you look at carefully here, what do you have? The modal matrix which I have previously written as

$$[K]=\left[\{\phi_1\}\{\phi_2\}......\{\phi_n\}\right]^T\begin{bmatrix} k_{11} & k_{12} & . \\ k_{21} & . & . \\ . & . & . \end{bmatrix}\left[\{\phi_1\}\{\phi_2\}......\{\phi_n\}\right]$$

Now, if I multiply these elements here what you will see? That, for any i[th] j[th] term what I will get as in this case,

$$\begin{aligned} m_1\ddot{u}_1+k_1u_1&=0 \\ m_2\ddot{u}_2+k_2u_2&=0 \end{aligned}$$

So, this is any i[th] j[th] term. Now, if it is a diagonal term then I can write it as $[\phi_i]^T[k][\phi_i]$ and any off diagonal term, I can write it as $[\phi_i]^T[k][\phi_j]$. Where $i\neq j$. Now, if you look at carefully using orthogonality property, this i and j wherever they would be different those terms would be equal to 0.

so only the diagonal terms are going to exist here. And those diagonal terms can be written as

$$K_n = \{\phi_n\}^T [k] \{\phi_n\}$$

So, this matrix is actually a diagonal matrix and the each element of this diagonal matrix can be represented as $K_n = \{\phi_n\}^T [k] \{\phi_n\}$. Similarly, for the mass matrix as well, I would get a diagonal matrix and the n^{th} element of the diagonal matrix can be written as

$$M_n = \{\phi_n\}^T [m] \{\phi_n\}$$

And both these are obtained utilizing the condition of orthogonality of the mode shapes. So, K_n is now known to me, so the n^{th} element of these diagonal matrices is now known to me and they are again related by the same expression

$$K_n = w_n^2 M_n$$

So, the diagonal terms of these matrices K and M are related using this equation right here $K_n = w_n^2 M_n$. So, now the question comes why we are getting into all these diagonal matrices, why do we need condition of orthogonality and everything. So, to explain that ok, let us consider first a two degree of freedom system.

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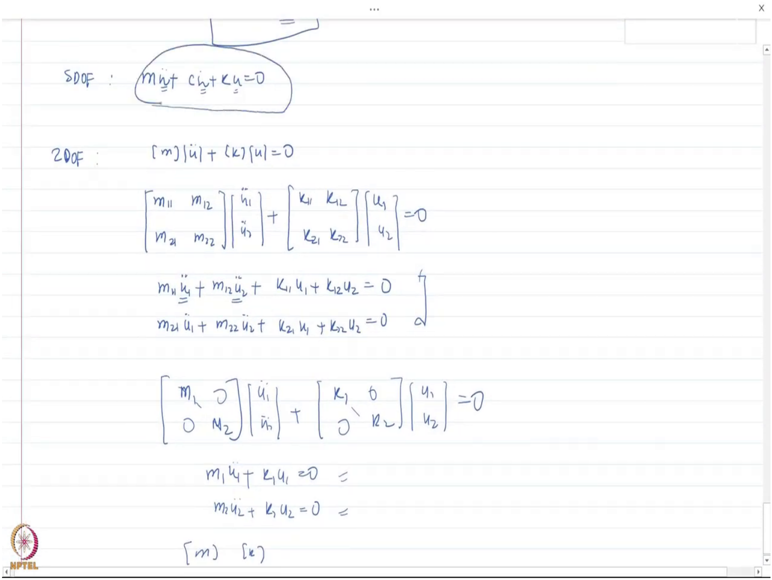

Remember, for single degree of freedom system the equation of motion was

$$m\ddot{u} + c\dot{u} + ku = 0$$

So, this was a second order linear differential equation and I could directly solve it. I had only one variable u ok and I needed to solve it u as a function of T. But now, if I have a two degree of freedom system like this let us say

$$[m]\{\ddot{u}\} + [k]\{u\} = 0$$

In general, I would have elements in the mass matrix as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

So, when I write down it will give me two simultaneous equation right. Which is basically

$$m_{11}\ddot{u}_1 + m_{12}\ddot{u}_2 + k_{11}u_1 + k_{12}u_2 = 0$$

$$m_{21}\ddot{u}_1 + m_{22}\ddot{u}_2 + k_{21}u_1 + k_{22}u_2 = 0$$

The question is how do we solve these two differential equation? These are two simultaneous differential equation and these are coupled. The first equation has term in u_1 u_2 both and the same for the second.

So, I cannot directly solve this differential equation. Had they been uncoupled differential equation, then it would have been lot easier for me to directly solve this and some terms of some parameter, but that is not the case here. But, just imagine, if I have diagonal matrices which only have diagonal element and then u_1 u_2. So, let us say this is

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

Now, if the mass matrix and the stiffness matrix are diagonal matrix, then what do I basically get,

$$m_1\ddot{u}_1 + k_1u_1 = 0$$
$$m_2\ddot{u}_2 + k_2u_2 = 0$$

And now the equations are uncoupled right. So, if so this if I have a n degree of freedom system, I get n coupled differential equations which are coupled through the mass matrix and stiffness matrix.

And, if somehow, I can diagonalize the mass matrix and the stiffness matrix then I can uncouple those equations of motions, and then I can solve n differential equation individually like single degree of freedom system. For which we already have standard results available, and that is what we are trying to achieve by utilizing the condition of orthogonality of the motion and by you like you know constructing matrices in which the elements the mass and the stiffness elements are diagonal.

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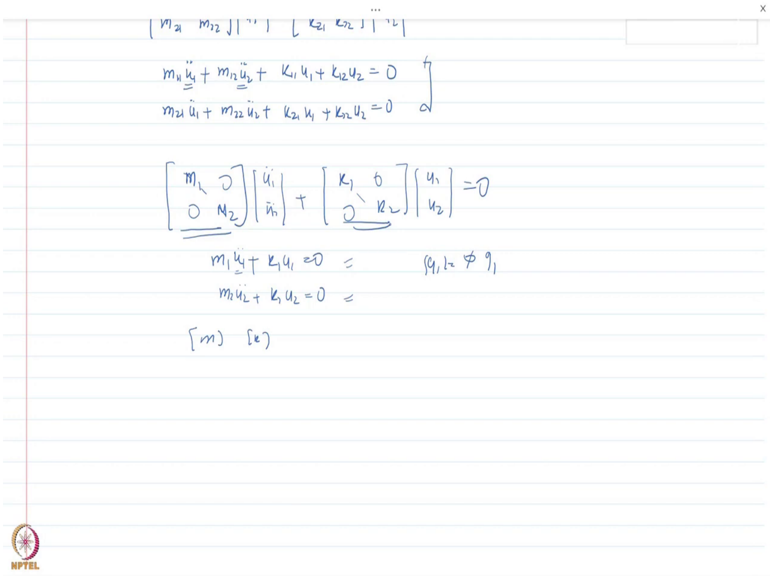

The idea is if you have mass and the stiffness matrix diagonal then I can get n uncoupled equation and I can solve the multi degree of freedom systems, as n single degree of freedom systems and n differential equations.

And for which the analytical results are already available to me. So, with this I am going to end this chapter here. In next chapter we are going to see how to actually utilize the mode shapes ok and the frequencies to actually get the overall response of a multi degree of freedom system.

Thank you very much.