**Dynamics of Structures**
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**Module - 02**
**Forced Vibration of MDOF Systems**
**Lecture - 25**
**Forced response of MDOF systems**

Welcome back everyone. So, in today's class, we are going to learn how to find out response of a multi degree of freedom system subject to external force. We have already discussed how to find out the free vibration response of a multi degree of freedom system by decomposing a multi degree of freedoms system into multiple single degrees of freedom system and then combining the response of each single degree of freedom system together total response.

We are going to do somewhat similar for the forced response as well. In this case additionally, we are going to represent force as a vector representation of force at different degrees of freedom. And then, to represent it as a contribution of force in each mode through the diagonalization method that we have learnt. So, let us gets started.

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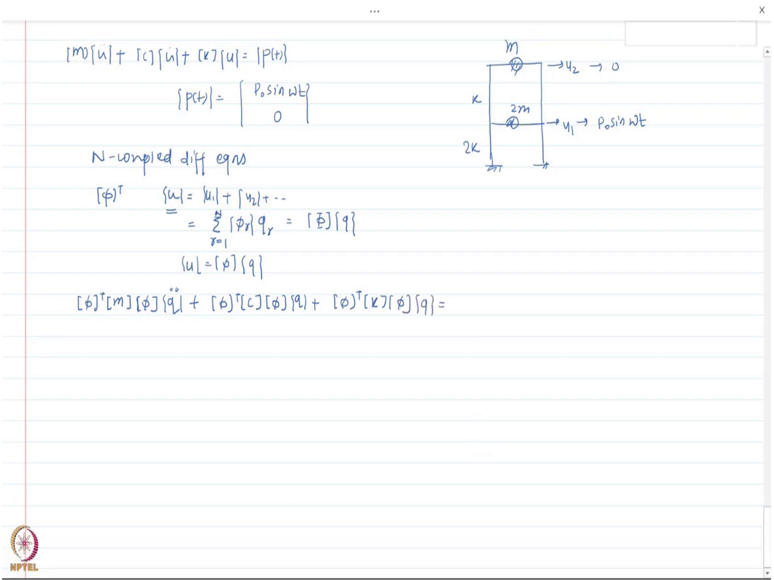

So, till now, we have learned how to set up equation of this form. So, mass matrix times the acceleration vector and then, damping matrix times the velocity vector then, stiffness matrix times the displacement vector and now, this would be equal to a force vector.

Now, remember if we have a multi degree of freedom system and let me take the example of this, which is two-story shear type building we have been discussing till now and this is $2k$ and $2m$ and this is $k$ and $m$.

These are the two degrees of freedom we are considering and the forces that are being applied on these degrees of freedom are basically $P_0 \sin \omega t$ here and there is no force being applied here. So, this is 0. So, then this excitation vector can be written as $P_0 \sin \omega t$ and 0.

So, now, we solve this multi degree of freedom system. Remember there would be N-coupled differential equations, which are basically coupled through the mass matrix damping matrix and the stiffness matrix and like we did for the solution of free vibration response of damped and undamped system.

We are going to uncouple these N-coupled differential equations by multiplying them with modal matrix $\emptyset^T$ and writing down my displacement vector as sum of contribution due to displacements in each mode. So, $u$ would be contribution due to first mode, let us say it is $u_1$ + $u_2$ and so on.

And basically, this is nothing but

$$\sum_{1}^{n} \{\phi_r\} q_r$$

So, what we are going to do? We are going to further write this as modal matrix times modal vector. So, we are going to write $u$ vector as modal matrix times $q$, and pre multiply this equation $\emptyset^T$.

Let us just multiply with $\emptyset^T$, transpose of the modal matrix. So, what we are going to get is basically this equation here, $\emptyset^T m \emptyset$ and then we have the modal accelerations. Then, I would have this quantity here and then the vector of modal velocities.

Then, similarly for the stiffness matrix, I have the shear times that modal coordinate cube, and times $\emptyset^T$ into $p(t)$.

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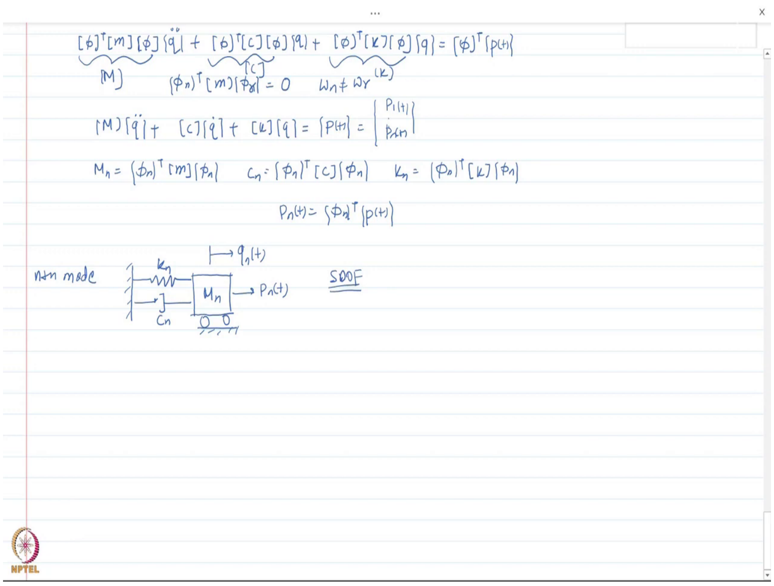


Now, as we know, considering the property the modal orthogonality of the mode, that if the two different modes are multiplied then, let us say this is $\emptyset_n$ times mass matrix times $\emptyset_r$. This is basically equal to 0 if this correspond to different mode.

And same goes for the stiffness as well. And if I have a classical damping matrix the same relationship holds true for the damping matrix as well. So, basically what I get here is diagonal matrices. I get here diagonal matrices, $[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{P(t)\}$.

So, this quantity here is basically your $M$, this quantity here is $C$ and this quantity is $K$, these all three are basically diagonalized mass matrix, damping matrix and a stiffness matrix. So, the element of these matrices basically $M_1$, $M_2$ and $M_n$ and same as $C_1$, $C_2$, $C_n$. We have already seen that what do we get that as, for example, my $M_n=\{\phi^T{}_n\}[m]\{\phi_n\}$.

Similarly, $C_n=\{\phi^T{}_n\}[c]\{\phi_n\}$ and then $K_n=\{\phi^T{}_n\}[k]\{\phi_n\}$. And this is basically a new matrix for which the elements are $P_1$, $P_2$ so on $P_n(t)$, like that where $P_n(t)=\{\phi^T{}_n\}\{p(t)\}$. So, now basically I have been able to do reduce my n – degree of multiple freedom of systems to n – single degree of freedom systems, for which I am going to just show you the spring-mass-damper representation.

So, basically, I have this is spring here, this damper. So, this is for the n<sup>th</sup> mode this is $K_n$ here, this is $C_n$, the mass is $M_n$ and the displacement vector becomes $q_n(t)$ and the force that is being applied on this one is $P_n(t)$. So, basically this is the system the single degree of freedom system that we get by modal decomposition of the multiple degree of freedom system.

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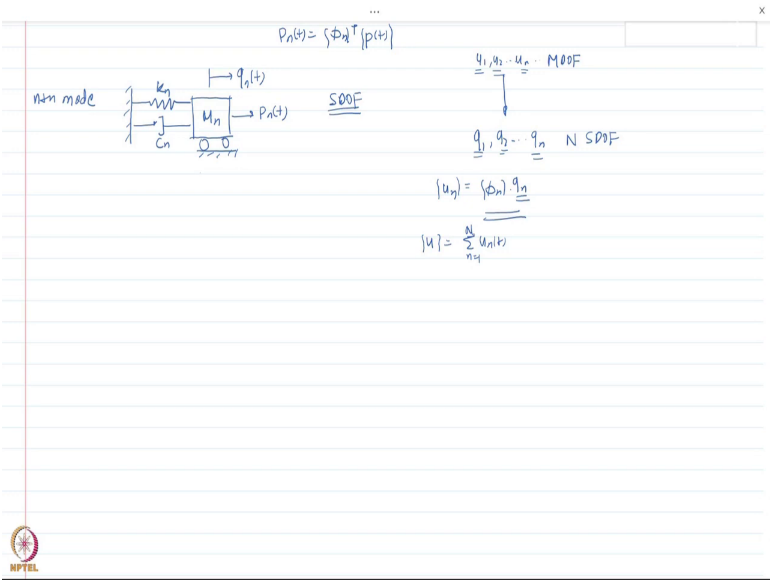


My multiple degree of freedom system which were in terms of degrees displacement at degrees of freedom 1, 2, 3 as $u_1$ and $u_2$, $u_3$ and so on $u_n$ to this is multiple degree of freedom system, in terms of $n$ single degree of freedom system with coordinates $q_1$, $q_2$, $q_n$. Now, remember I would have N such single degree of freedom system.

They are do not correspond to any specific degree of freedom. Remember, the original equations were in terms of degrees of freedom, but here these are modal coordinates, in order for you to convert this modal coordinate to corresponding displacement at the degrees of freedom. So, let us say you are considering $n^{th}$ contribution of the $n^{th}$ mode to the total displacements at each degree of freedom, you would need to multiply the $q_n$, modal coordinate with the shape vector or the mode shape here. Once you multiply that, it will give you the contribution of that mode to the total displacement at each degree of freedom. So, all these equations, when we solve these equations the single degree of freedom system, it gives us the modal coordinate not the displacement at any degree of freedom.

We convert it and once we have for each single degree of freedom or for each mode like this, we can sum it up to get the total displacement at each degree of freedom and we can sum it over all the modes. So, this is how we basically solve our multiple degree of freedom system subject to any external excitation force. And we will take up one example and then see how to get the response. So, let us take the example that we have been doing so far.

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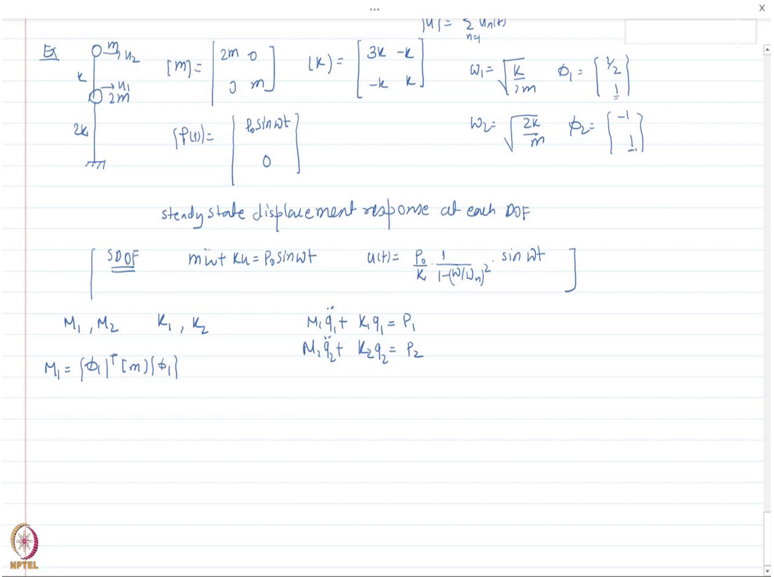


So, what do basically we have here is, this same two story shear frame building, which I am now representing as this lollipop model here. So, this is $k$, this is $2k$, this is $u_1$ and $u_2$ here. Now, we have already derived the mass for this as mass matrix as $2m$, 0, 0, $m$.

My stiffness matrix as $3k$, $-k$, $-k$ and $k$ and on this one – the external force that is being applied is $P_0 \sin(\omega t)$ at degree of freedom 1 and there is no force at degree of freedom 2. So, this is my excitation vector. Now, what I need to find out is the steady state response at each for each degree of freedom, let us say, displacement response at each degree of freedom.

Now, before getting into that, remember when we had a single degree of freedom system in which the displacement variable was $u$ and it was subjected to $P_0 \sin(\omega t)$, for that, the steady state response was, so, let me write down the corresponding SDOF system, if you remember, when you had an equation like this here and do not confuse between these parameters.

So, when you had this kind of single degree of freedom system subject to $P_0$ sin($\omega t$) excitation, the steady state response was

$$u(t) = \frac{P_0}{k}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t$$

So, we are going to utilize this, but now remember we are solving in terms of modal coordinate $q_1$ and $q_2$ here, not $u$ here. So, before solving this equation, the first step that we need to do is to find out the diagonal mass matrix and diagonal stiffness matrix. So, that I can get the generalized mass $M_1$ and $M_2$ for each mode and generalized stiffness $K_1$ and $K_2$ for each mode. So, that I can write down the equation of this form $M_1\ddot{q}_1 + K_1 q_1 = P_1$, which is the normalized excitation for mode 1. Similarly, I should be able to write down $M_2\ddot{q}_2 + K_2 q_2 = P_2$ this equation as well. So, $M_1$, $M_2$ can easily be found out if we write $M_n = \{\phi^T{}_n\}[m]\{\phi_n\}$ and if you remember, we had already derived the mode shape and frequency for this system.

The frequency for the first mode was $\sqrt{k/2m}$ and the mode shape was $\{1/2 \quad 1\}$. For the second mode, the frequency was $\sqrt{2k/m}$ and the second mode shape was $\{-1 \quad 1\}$. We had normalized with respect to the top degree of freedom which is the second degree of freedom here. So, we can substitute those quantities for $M_1$, $M_2$ and so on.

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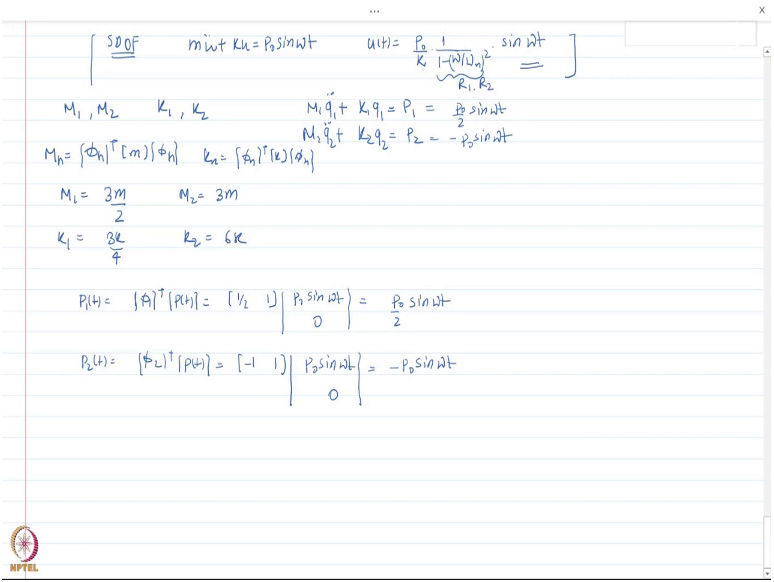


So, let us say $M_n$ is $\{\phi^T{}_n\}[m]\{\phi_n\}$ and $K_n$ is $\{\phi^T{}_n\}[k]\{\phi_n\}$. So, we can substitute it for 1 and 2 and we can get the normalized masses and the stiffnesses as $M_1 = 3m/2$, $M_2 = 3m$ and then $K_1 = 3k/4$ and $K_2 = 6k$. You can do that matrix multiplication and you can check that if these are correct or not.

Once we have found out, let us get the generalized excitation force for each mode. So, first, I want to get what is the $P_1(t)$. So, basically $\emptyset_1{}^T\{P(t)\}$ which is nothing, but $(1/2)\ P_0 \sin(\omega t)$. Similarly, my $P_2(t)$, what the second mode is basically $\emptyset_2{}^T\{P(t)\}$ which is $-P_0 \sin(\omega t)$. So, the two equations that we are solving basically becomes this equal to $\{(P_0\ /2) \sin(\omega t), -P_0 \sin(\omega t)\}$.

So, these are the two equation that we are trying to solve here, and we can utilize this general solution to write down. Now, remember what I am going to do here, this I am going to write as response modification factor for the first mode as $R_1$ and for the second mode as $R_2$. So, that I do not have to keep writing the same ratio again and again.

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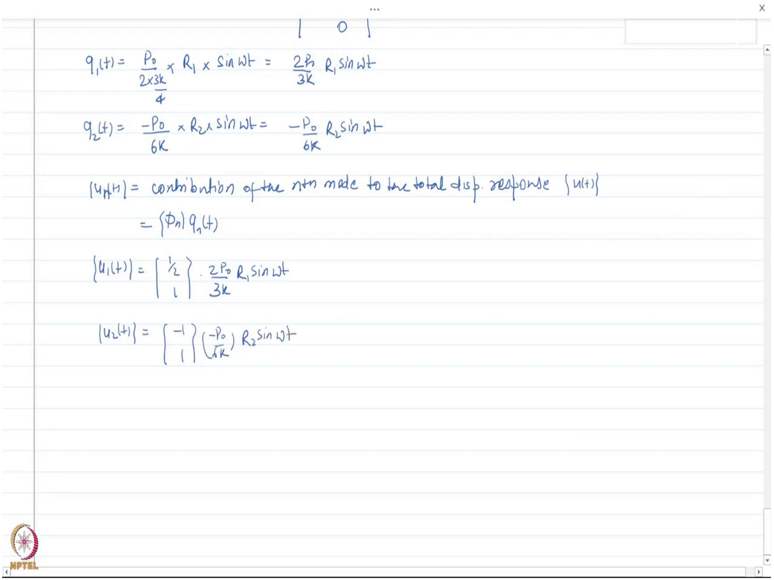


So, let us do that. So, $q_1(t)$ is basically the amplitude of the applied force which is $P_0/2K_1$. Remember, not the stiffness of the story, this is the $K_1$ is stiffness of the generalized modes and I can substitute $K_1$ here as $3k/4$.

Let us say $R_1$ sin $(\omega t)$ and this I would get as $(2P_0/3k)$ $R_1$ sin $(\omega t)$. Similarly, $q_2(t)$ I can obtain as $-P_0/K_2$ and $K_2$ is $6k$ here, times $R_2$ sin $(\omega t)$. So, this I get as $(-P_0/6k)$ $R_2$ sin $(\omega t)$. So, once we get $q_1$ and $q_2$, remember, the next step is basically, find out the combined response. Before we get into that, since we have obtained $q_1(t)$, this is the modal coordinate not the displacement of any degree of freedom.

To get the displacement of any degree of freedom let us consider $u_1(t)$ here, which is basically contribution or if it is called $u_n(t)$ contribution of the $n^{th}$ mode to the total displacement response which is a basically $u(t)$ here. So, $u_n(t) = \{\emptyset_n\} q_n(t)$.

So, let us first get what is the contribution of first mode to the total response and this would give me the contribution at each degree of freedom, the displacement contribution. So, my $u_1=\{1/2, 1\}$ $(2P_0/3k)$ $R_1$ sin $(\omega t)$. Similarly, my $u_2=\{-1, 1\}$ $(-P_0/6k)$ $R_2$ sin $(\omega t)$.

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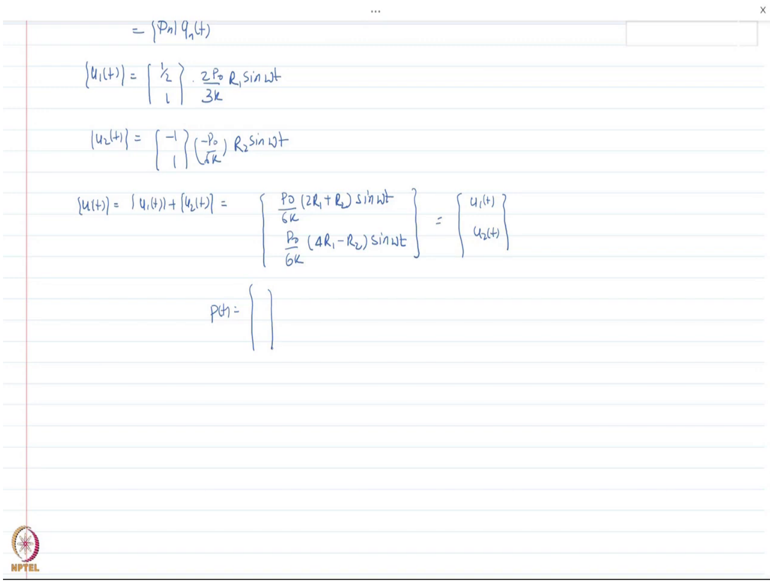


So, the total displacement response we can get as by summing these up, the contribution of each mode that will give me the total displacement at each degree of freedom.

So, that I can write it as $P_0/6k$ $(2R_1 + R_2)$ $\sin(\omega t)$ and $P_0/6k$ $(4R_1 - R_2)$ $\sin(\omega t)$. And these basically corresponds to the displacement history at each degree of freedom 1 and 2. So, we have seen that if the external excitation force is given, we can construct the force vector $P(t)$ and then we can utilize the modal analysis procedure to find out the response at each degree of freedom.

So, this was the only focused on finding out the response at each degree of freedom that is the displacement response. The displacement response, you might also need to find out the forces. So, basically, if you let us say, it could be any type of any multi degree of freedom system.

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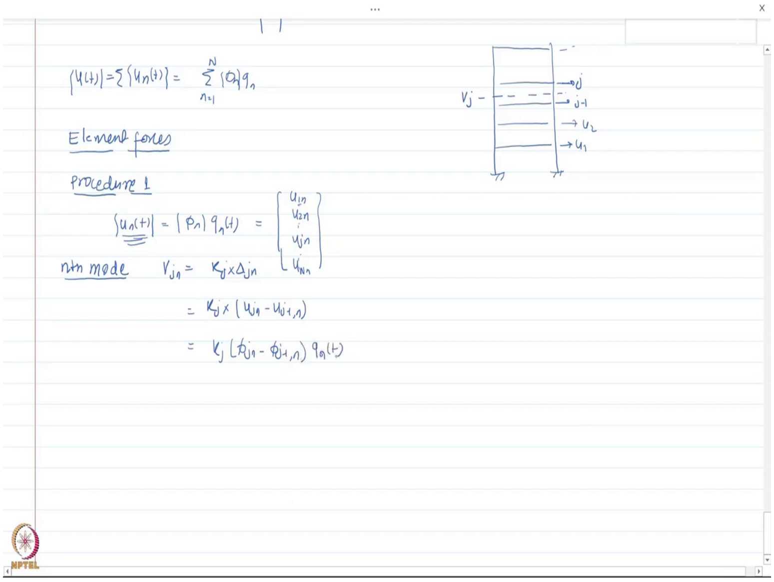


But let us take example of multi-story building and let us say this is the shear type building something like this. So, multiple stories are there here, and $u_1$, $u_2$ the degrees of freedom are like that. So, to find out the forces or the internal forces in the system subject to any external excitation, let us see how we do that. So, remember first we found out.

We have found out how to get the total displacement response using total displacement as sum of individual displacement for each mode, which is nothing but summation N over all the degrees of freedom $\emptyset_n$ and $q_n$ and this was the modal analysis procedure that we had followed. Now, let us see how to get the element forces. So, to get the element forces, there are basically two procedures and we will discuss each of this procedure one by one.

So, in the first procedure, we find out the contribution of the $n^{th}$ mode to the total displacement $u_n(t)$ basically using this, once we find out $q_n(t)$, multiplying this with a mode shape. Once we found out $u_n(t)$, if we know the relationship between the displacement and the internal forces, we can utilize those relationship to find out internal forces.

For example, let us say this is the $j^{th}$ story and this is $(j\text{-}1)^{th}$ story. So, if I define my story shear has in this story as $V_j$ and let us say I am considering $n^{th}$ mode. So, in the $n^{th}$ mode basically story shear is the $j^{th}$ story would be whatever the story stiffness is $k_j$ times the story drift and I will say that this is for the $n^{th}$ mode.

So, it would be $k_j \times (u_{jn} - u_{(j-1)n})$ and these $u_{jn}$ and $u_{(j-1)n}$ we already know from here, because remember $u_n(t)$ is $u_{1n}$, $u_{2n}$, $u_{jn}$ and so on. So, let us say $n$ degree of freedom. So, for the $n^{th}$ mode, this is the contribution of the $n^{th}$ mode to total displacement at each degree of freedom.

So, we can find out this or we can also write this as $(\emptyset_{jn} - \emptyset_{(j-1)n}) \times q_n(t)$. So, this is when we directly know the relationship between the displacements that we have obtained and the forces in the system using this type of relationship.

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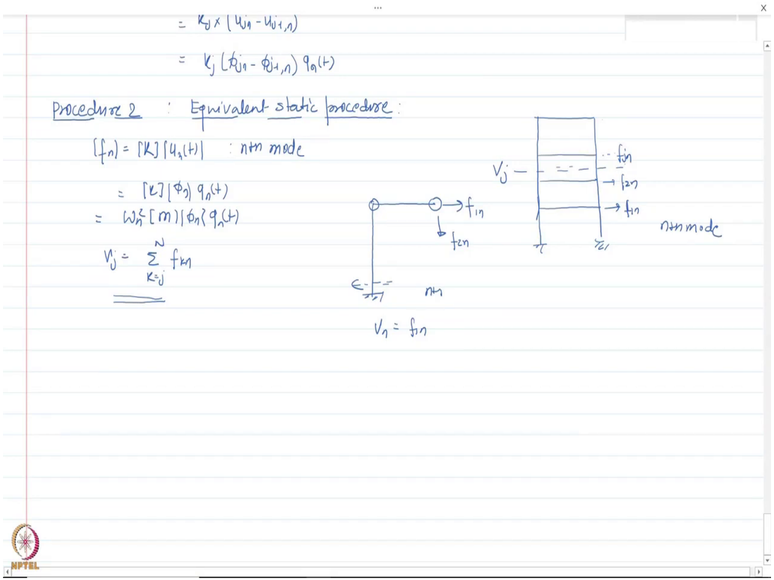

The second procedure that we follow is the equivalent static procedure and in equivalent static procedure we find out what are the equivalent static forces that are acting and then, using the static analysis of the structure, we get the forces or the story shear or the story moment at different level. So, let us say I have a multistory building like this again. So, the equivalent static force is nothing, but the stiffness times the $u_n(t)$.

Now, this is for the $n^{th}$ mode. So, this can be further written as $k$ and this can be written as $\emptyset_n q_n(t)$ and remember from the eigen value equation, this is nothing, but $m\omega_n^2\emptyset_n q_n(t)$. So, we can utilize this and then, we can find out at each degree of freedom what is the equivalent static forces. Let us say here, it is $f_{1n}$, $f_{2n}$ and similarly, at some story here, let us say this is $f_{jn}$ and so on.

So, the equivalent static forces at each degree of freedom in the $n^{th}$ mode. If we have that, finding out the story shear at any level is as simple as summing up all these equivalent static forces above that level. So, this would be let us say $\sum_{k=j}^{N} f_{kn}$.

And remember, I mean we have derived this given the example of multi story building, but the same procedure would also be applicable for any other type of multi degree of freedom system only thing is that this equation might differ. So, let us say, if I have a multi degree of freedom system like this and we can consider different modes of excitation, in that also we can apply. Let us say if the nth mode is that, we can apply f 1 n and f 2 n like that and equivalently we can find out what are the base shear. For example, in this case the base shear would be ok in the let us say $n^{th}$ mode, it would simply be $f_1$ here, if we consider the equilibrium. So, when we apply the equivalent static forces, we do not need to know the relationship between the displacement and the forces in the particular element.

We can just consider the equilibrium of the structure and we can find out forces utilizing the applied equivalent static forces and this can be applied for the examples to find out the forces and the internal forces and the moments in the system. So, we are going to conclude this lecture here today. In the next lecture, we are going to see how to apply these procedures to find out the seismic response of a structure or a multi degree of freedom system.

Thank you.