Seismic Analysis of Structures
Prof. T.K. Datta
Department of Civil Engineering
Indian Institute of Technology, Delhi

Lecture – 15
Frequency Domain Spectral Analysis

In the last 6 lectures we discussed about the response of single degree of freedom and multi degree of freedom system for a specified ground motion.

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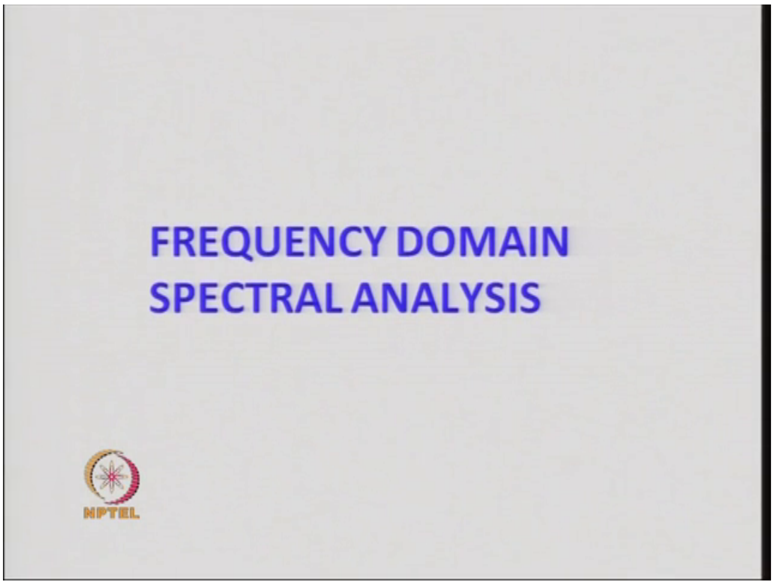

Now these analysis is useful for checking the structures, which have already been designed for earth quake and for cases where an earth quake is specified which is likely to occur in future. This is also used many a time for performing a non-linear analysis of the structures in time domain; however, most of the structures are to be designed for future earth quake and since the future earthquake is not known therefore, we try to model it or represent it in different ways to different forms of modeling or representing earth quake where discussed in the input.

There we talked off the response spectrum of earth quake and the power spectral density function of earth quake. Response spectrum of earth quake has a particular shape and that has been verified from the past earth quakes and we propose that the future earth quakes that will take place that would have a similar kind of shape in so far as the response spectrum is concerned, and from that we developed what is known as the design response spectrum and these design response spectrum is widely used for designing the structures for earth quake.

So, that becomes a very good input for analyzing the structures for future earth quake and we will take up the response spectrum method of analysis of the structures subsequently. In this sequence of lecture we will discuss the other input that is the power spectral density function of earth quake, that will be applying to structures for analyzing the structures considering earth quake as a random process that is we assume that the future earth quake is a random process it is not known therefore, it is a quite rational to model it as a random process rather than a deterministic process. When we model the earth quake as a random process we discussed in connection with the input or earth quake inputs that we describe it in the form of the power spectral density function of the earth quake, and we also describe what is known as a coherence function. This coherence function takes care of the phase lag or the time lag between excitations that occur at different supports which are at spatially long distance.

So, with the help of this coherence function and the power spectral density function, one can analyze the structures for random ground motion rather you should say that.

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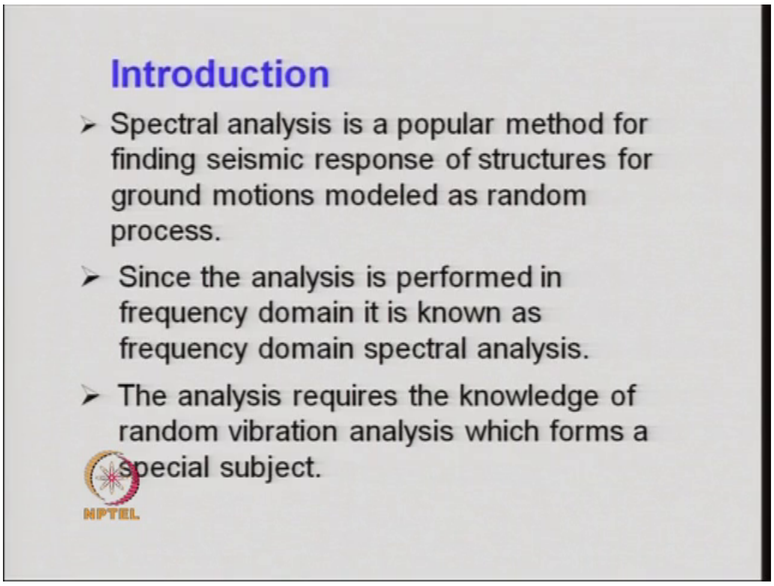

The ground motions are modelled as a random process and the technique by which we analyze the structure with the power spectral density function as input and the coherence function the input is known as the spectral analysis. Since the entire analysis is done in frequency domain it is better called a frequency domain spectral analysis. It is a very popular method for finding size seismic response of structures for ground motions modeled as a random process.

The analysis requires the knowledge of random vibration analysis which forms a special subject of its own and therefore.

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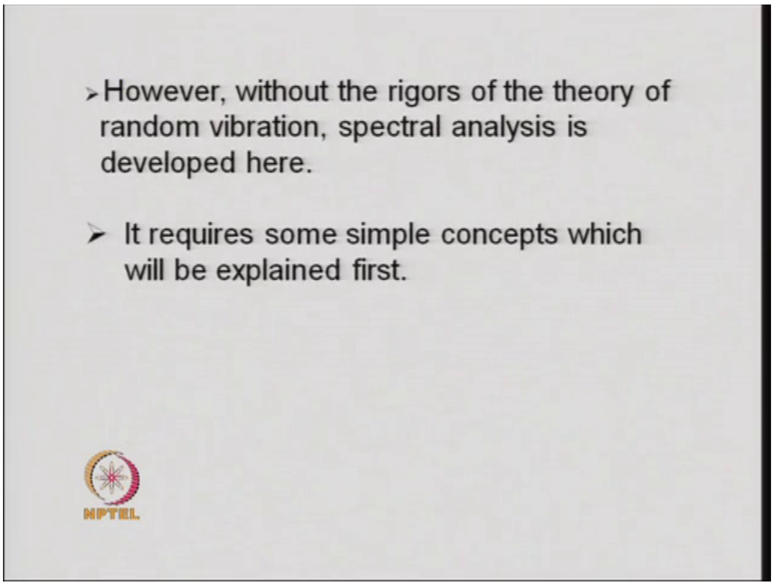

we will not be able to cover these entire random vibration analysis over here we will take up only a portion of the random vibration analysis which is known as the spectral analysis, and using that spectral analysis we will try to obtain the response of the structures for future earth quake modelled as a random process. The spectral analysis is quite elegant simple to understand and provides us a very good estimate of the peak response and the root means square response of the structure; however, the spectral analysis when we are using we have to have some simple concepts before we explain the spectral analysis technique. The a 2 most important concepts that are required for understanding the spectral analysis are the concept of the power spectral density function of earth quake or may be for any random process, it may be the response x t of a structure it may be a bending moment of the structure, it may be a shear force of the structure since the input is random process therefore, they also would constitute a random process.

So, the concept of power spectral density function not only of the earth quake ground motion, but also the power spectral density function of displacement power spectral density function of acceleration, power spectral density function of bending moments they must be we will understood. Then comes the coherence function whenever we have got 2 random processes for example, if we have a displacement of a structure x 1 t at the say top of a frame and say x 3 t at the first 4 level a between these 2 random processes there will be a lack of correlation or there is a phase gap or a time lag.

So, because of this phase lag or the time lag or the lack of correlation, we have a cross power spectral density function defined for these 2 random processes and this is the concept of these cross power spectral density function with the help of a coherence function that also we introduced in discussing the seismic inputs. So, let us a look at these concepts again starting with the definition of the power spectral density function, and some other concepts that will be required somewhat in the regard of the random vibration analysis; however, we will not go through all the rigors of the random vibration analysis over here, we will simplify our spectral analysis technique by assuming the earth quake process not only to be stationary, but also ergodic that is ergodicity is a assumed in order to simplify the solution process and in understanding the concepts that we are now will be talking of. And you already have undergone the concept of the ergodicity in connection with the seismic input to structures.

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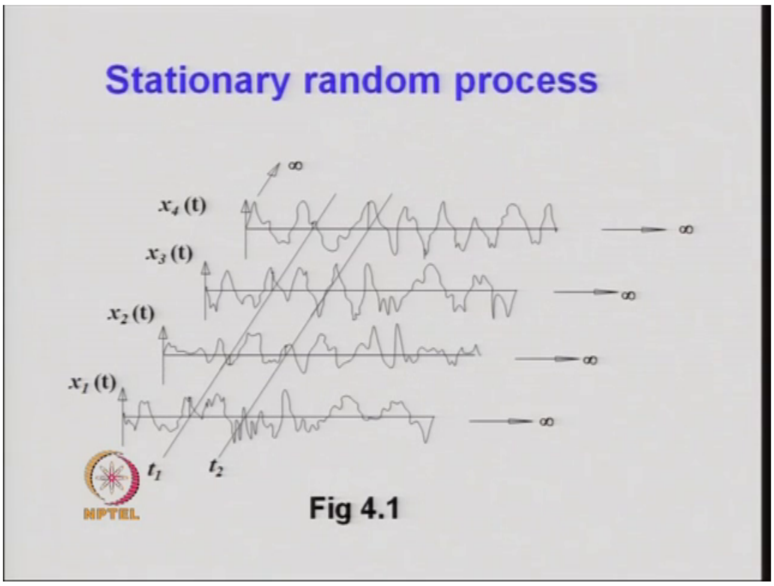

Fig 4.1

Now, let us take a collection of records of the ground motion and they are available at a particular site, which is prone to earth quake. So, from the past records one can collect as many number of a ground motion records as possible. Ideally if we wish to define a random process we must have infinite number of records and each record should be of infinite duration. However, in reality we cannot have a infinite number of the records nor we can have a infinite duration of the ground motion or for that matter any other time history. So, we have always a finite number of collection of the ground records and the duration of earth quake is generally of the order of 30 seconds 40 seconds or so on.

So, whatever data we get with the help of that, we try to look at a property of the random process. Now one important thing that I mentioned in connection with the seismic input is that whenever we are talking of a random process then we are not only interested in only one record, but an ensemble of records, and I also mentioned that the anything irregular is not random. In order to describe a random process you must have a number of earth quake records or number of time histories of the particular variable. Also these kinds of variables are called a parametered random variable for example, x t over here is a parameter random variable t is the parameter and the meaning of the parametered random variable is that at any instant of time t x t itself is a random variable.

So, if we look at these figure we see that at time t 1 the x t 1 can assume different values and if we have a number of such records, then x t 1 itself is a random variable which can assume any value depending upon the number of records that you have got and so, these values we can get if I draw a line across this ensemble. Similarly x t 2 is a random variable and it can assume any value depending upon the number of records that we have and we can in the same fashion define x t 3, x t 4, x t 5 all of them will be the random variables and they are these random variables are connected by a function of time t therefore, it is called a parameter random variable.

Now, if I consider the value of x t 1 over the ensemble, then if I average them; that means, take an average of all the values that x t 1 can take across ensemble then we will get an ensemble average at time t 1. Similarly we can get an ensemble of average at time t 2, t 3, t 4 so on. Now if it is found that these ensemble averages are the same or more or less are the same, then we can say that these ensemble average is independent of the time shift. Time shift means if we define t 2 by t 1 by tau designating the time shift then whatever be the time shift the ensemble average will remain the same; that means, the ensemble average that will work out at x t 1 and say at x t 5 both of them will have the same value.

So, we phrase it in this fashion that the ensemble average is independent of the time shift, similarly one can obtain the variance.

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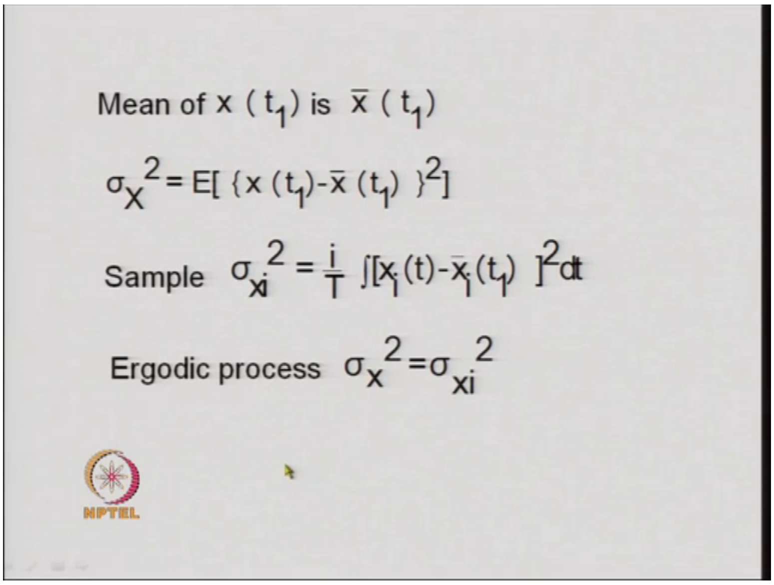

So, these variance would be described as the x t 1 values across the ensemble and the average value of the ensemble at t 1. So, we deduct this from x t 1 and take a square of that and expectation means the averaging of this quantity. So, this is called the variance that you have also discussed before. So, if we find that the variance of the ensemble at t 1 at t 2 at t 3 and t 4 if all of them are more or less the same, then we can call this also as a process having a variance as in variant of the time shift.

Now, for our analysis generally we define a process or a random process as a stationary process of order 2 or second order stationary process, if we find that the mean ensemble mean and the ensemble variable they are independent of the time shift. And we generally for engineering purposes at least for structure engineering calculations we satisfy our self with the second order stationarity; however, if one has to really call a process to be a stationary process then it should be an nth order stationary process n being the order; that means, instead of sigmas x square it should be sigma x cube or sigma x 4, sigma x 5 higher the number higher would be the order of the stationarity.

Then if we consider a single sample out of it then for that single sample we can work out a variable for the sample. So, this variance of the sample will be simply the values at different time minus the average value and we take a square of that and 1 by t and in t squaring the integrating the square of this quantity, that gives the sample variance. In an ergodic process we find that sigma x square that is the ensemble variance is equal to the variance of any sample time history; that means, if I take out any sample time history from this ensemble this will also give a the same variance across time.

Now, if you have such an ideal process then that process is called an ergodic process. It is a really hard to get an exactly an ergodic process in reality, but for solving many problems in random vibration we assume the property of ergodicity because the assumption of ergodicity simplifies the analysis procedure. So, in our case also we will assume that the process the earth quake process or the future earth quake is not only stationary, but it is also ergodic.

Now, once we assume that then we will find that many of the concepts that will be subsequently describing they become simpler to define; however, those concepts can also be defined in the rigor of random vibration analysis. So however, here will not do that in order to understand in a simple manner we will assumed a process to be an ergodic process.

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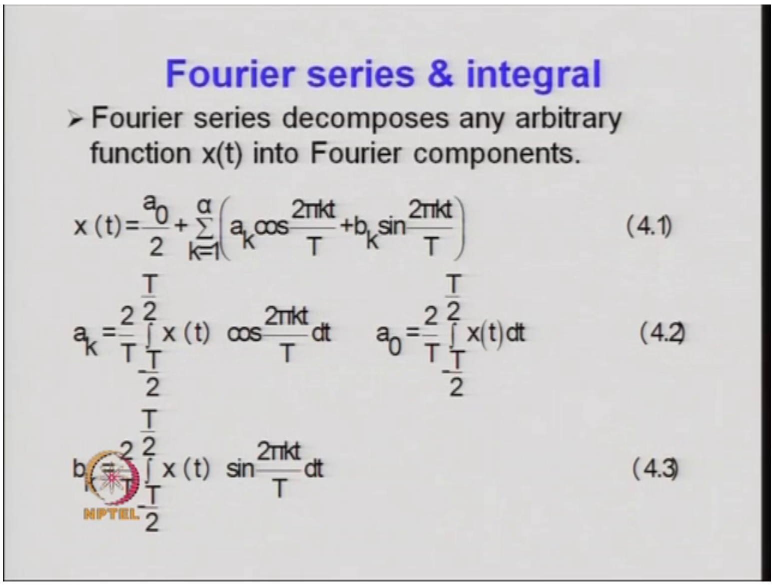

This Fourier series and Fourier integral we have discussed before in connection with again seismic input and there we have used that Fourier series for finding out the Fourier spectral as well as defining the power spectral density function of a ground motion and. So, therefore, I will not be again repeating it in (Refer Time: 21:30), but the salient features let me again reiterate over here. So, that you get a recapitulation.

If I take a single time history then it can be broken up into a number of harmonics or in other words if we sum up of the number of harmonics then we get a time history in irregular time history x t, and a K and b K they are obtained from this relationship if x t basically is a function which is integrable then one can get the values of a K and b K in the closed form. if it is not an integrable function then we go for a numerical integration.

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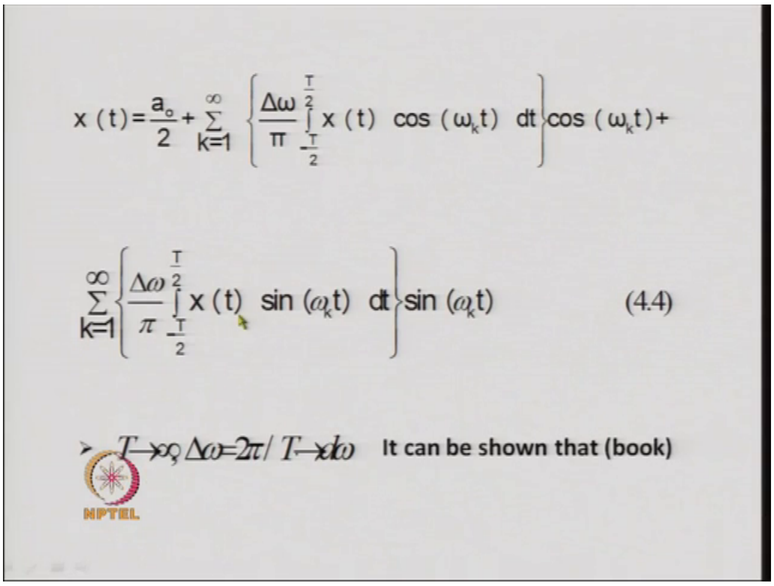

In order to perform this numerical integration effectively we now use FFT that is fast Fourier transform and in order to get back the time history of the signal or time history of ground motion in this case, we can obtain this time history from the frequency components of the time history if we use IFFT.

So, the pair of FFT and IFFT that we use for Fourier synthesizing a time history of ground motion. Now how one can obtain a Fourier integral from the Fourier series that is described over here, if we simply substitute 1 by T in the previous equation that is in this equation this 1 by T if it is substituted over here, then you will find that it becomes delta omega by pi here also we will get a delta omega of the pi and if we assume that t is tending to infinity then delta omega converts to d omega and the summation sign can be replaced by an integration sign.

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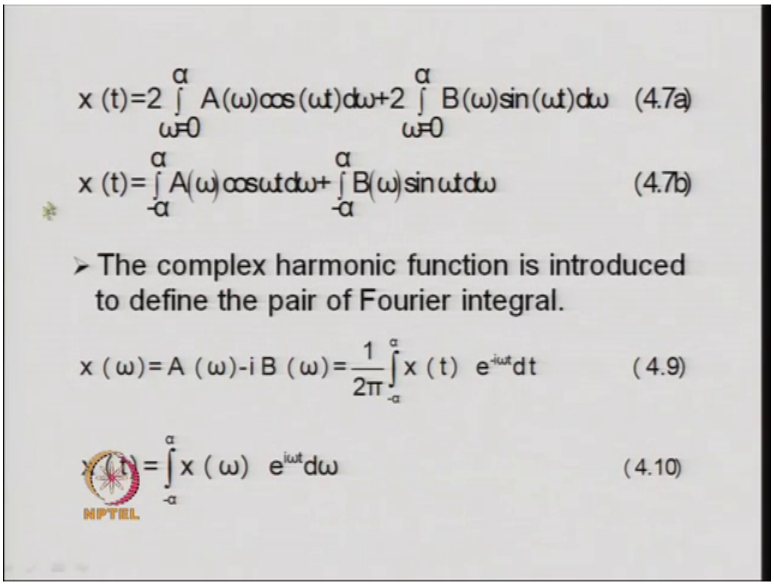

So, that is what is shown over here x t can be shown to be equal to this particular form 2 times omega is equal to 0 to alpha A omega cross omega t d omega plus B omega sign omega t d omegas like this. Now, one can remove this 2 if we instead of integrating from 0 to infinity, if we integrate from minus infinity to plus infinity. Now with these 3 definition of the x t over here, where A omega and B omega they are obtained from this expression that is a K and b K that is this a K and b K they take the form of A omega and b omega, then we try to define a quantity x omega as A omega minus i B omega and if we substitute for A omega and B omega from the equation of a K and b K this a K and b K they become A omega and B omega if we substitute this integrations into this equation and do a algebraic manipulation, then x omega turns out to be 1 by 2 pi minus infinity to plus infinity x t to the power i omega t.

So, that is how the first part of the Fourier integral has is derived that is x omega is given by this integral. The second part that is if we used to find out x t given x omega that is the frequency contents of the ground motion, then we use this IFFT that is inverse Fourier transform and it can be easily proved like this that if we take x omega e to the power i omega t d omega these integral if we wish to write down then we have A omega minus iB omega multiplied by cross mega t plus i sin.

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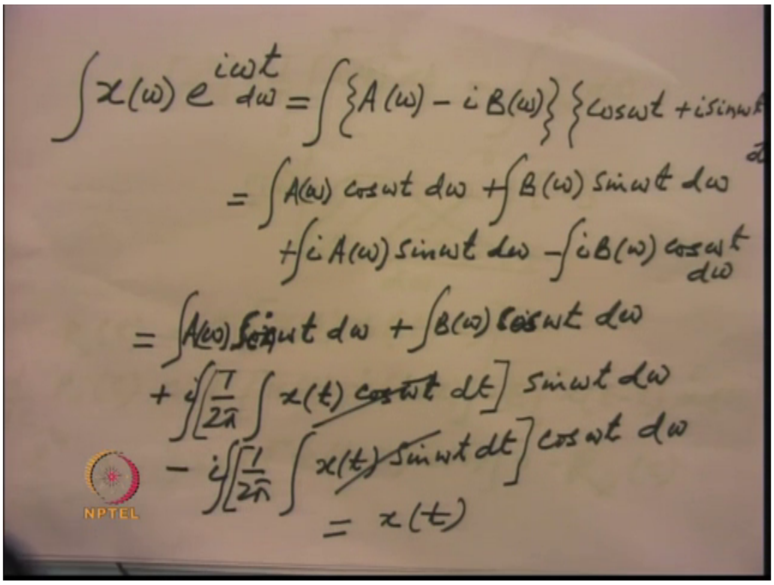

Omega t that is e to the power i omega t is replaced by cross omega t plus i sin omega t and if and then we also keep d omega over here and when you multiply this 2 then we have these quantity, this quantity that is B omega sin omega t A omega cos omega t and then also these 2 quantities that is i A omega sin omega t minus i B omega cos omega t d omega.

Now, this we keep as it is this two and this two in this if we substitute for A omega and b omega by the equation of a K and b K that I had shown you before if we substitute that then you will find that these 2 terms they cancel finally, we have A omega sin omega t d omega plus B omega cos omega t d omega this is the these are the quantities that remain and this is nothing, but equal to x t because we have defined this like this x t has been defined by equation 4.7 b as A omega cos omega t d omega plus B omega sin omega t d omega.

So, we can prove that x t becomes equal to the IFFT of x omega and they form what is known as the Fourier integral pair. So, the Fourier integral pair that we use in Fourier synthesizing any particular time history, that can be straight away derived from the Fourier cities.

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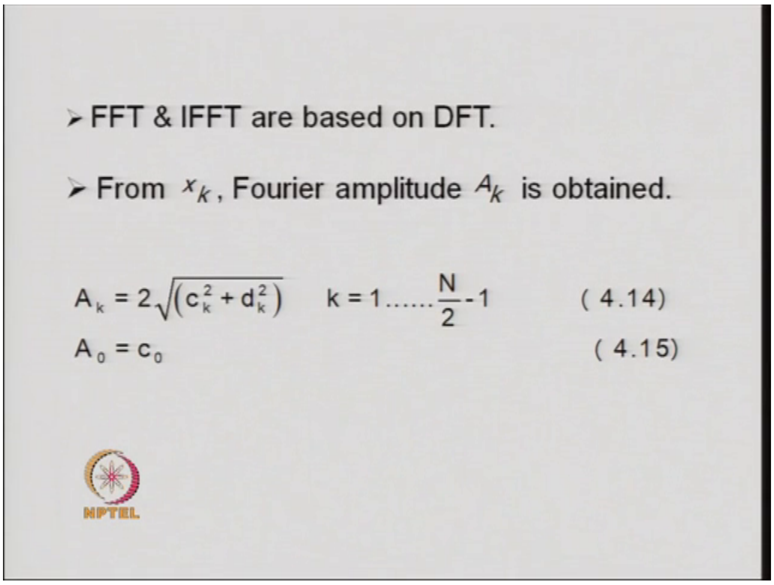

Then when we try to use FFT and IFFT, we perform that integration with the help of a summation sign or we numerically calculate this integration and that format is known as the discrete Fourier transform and inverse discrete Fourier transform and that is what is used in the computer programs for obtaining the Fourier components of any signal or of any time history.

The Fourier amplitudes a K end of phase etcetera that has been discussed before.

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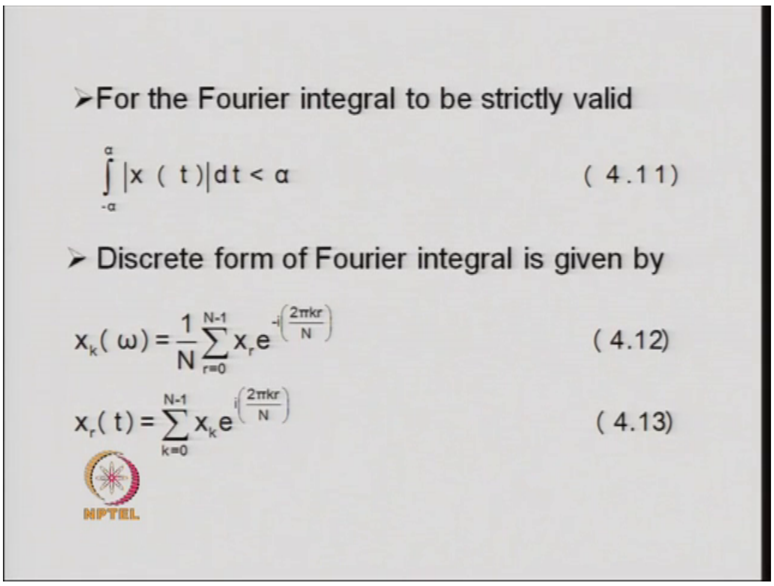

So, I am not going to again repeat that only one important thing that I wish to mention over here is that Fourier integral that we have shown you before, that Fourier integral is valid only under this condition that is absolute value of x t integrated over the entire duration must be less than infinity; that means, that quantity must be a finite quality and in reality with this particular condition is made because our time history records are not infinite, they are all finite time history records. Therefore, if I wish to make it infinite that is if you wish to make the integration for minus infinity to plus infinity, then we have to add on zeros to that and that way in reality this particular condition is satisfied.

Now, if you are using the discrete Fourier transform then that Fourier integral is expressed with the help of these summation signs and this has also been discussed a before in connection with the seismic input also we discussed about the Parsavels theorem in order to find out the mean square value.

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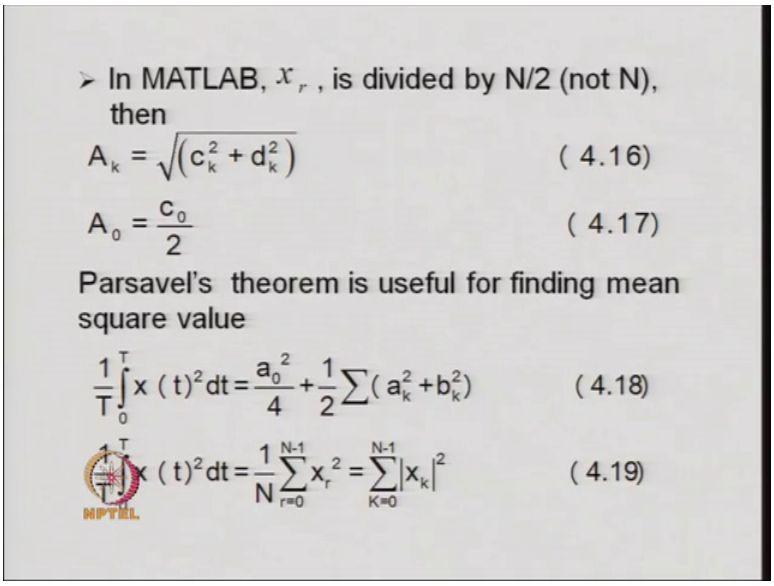

So, mean square value of the process of our x x t s square integration of 8 from 0 to T divided by 1 by T that gives you the mean square value of a particular time history that can be shown to be equal to the some absolute square absolute value square of the complex quantities that we get after we obtain a FFT of a time history.

So, and if they are added together then we get the same mean square value. So, that is what the Parsavels theorem is and how from that.

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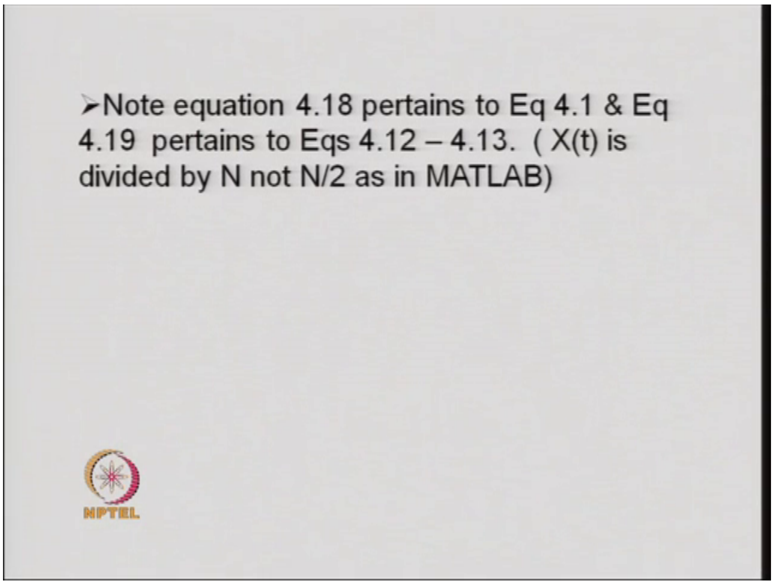

We use we obtain the power spectral density function that also we have discussed before that is if we use mat lab then the whatever the values we get as x omega, that is the complex number a plus I plus I b then we take n by 2 plus 1 values first n by 2 plus 1 values from that a plus I b quantities. And then we take the absolute square of these complex quantities and we have got n by 2 plus 1 such quantities that we plot against omega, then divide each ordinate by d omega and we get bars small bar diagrams and the small bar diagrams small bars will be again equal to n by 2 plus 1 in number, then join the center of those bars in order to get the power spectral density function.

So, all these things has had been discussed before. So, I do not want to again repeat it, only I wish to mention over here that we are able to define the power spectral density function of ground motion in this particular fashion, because we assumed the process to be an ergodic process. And therefore, we looked at only one single time history record and from there we are constructing its power spectral density function and that power spectral density function represents the entire stationary process, because the area under the power spectral density curve is again equal to the mean square value that also we have discussed before.

So, since a stationary process is characterized by a unique mean square value therefore, the power spectral density function constructed in this fashion provides a good estimate of the input that will go as an input for the random vibration analysis or the spectral analysis in particular over here spectral analysis of structures.

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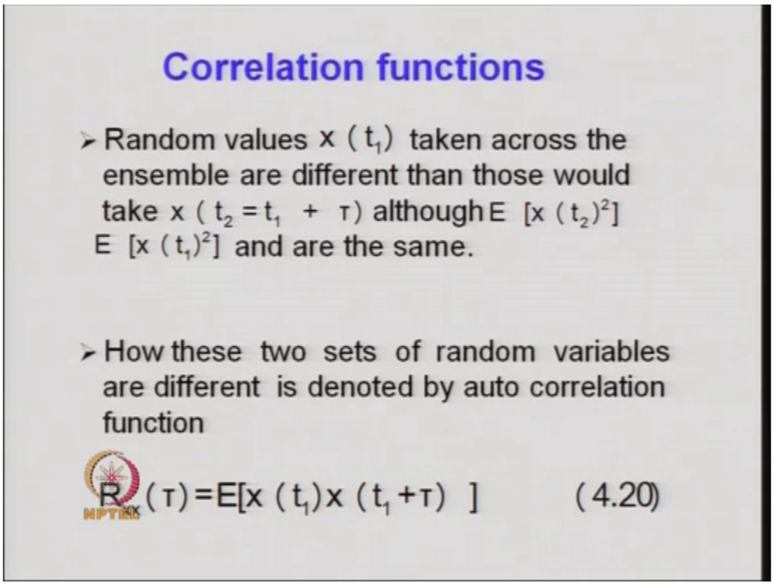

Next we this also we mentioned before, but again let me reiterate it if we have got a random variable x t 1 and then we defined another random variable x t 2, t 2 defined as t 1 plus tau, tau is the time shift between t 1 and t 2 then although there would be the expected value of x t 2 square and expected value of x t 1 square they are same, but they would differ in so, far as the phase lag is concerned. That is there will be phase lag between these 2 time histories and in order to denote these phase lag we use what is known as the auto correlation function.

These autocorrelation function is a function of tau that is the separation time between x t 1 and x t 2 in the ensemble, let me show you again in the figure what are those that is this t 1 and t 2 and tau is this distance. So, greater the value of tau more is the time lag between 2 random variables and autocorrelation between the random variables x t 1 and x t 2, they are a function of tau that is a time separation. So, we write own expected value of x t 1 multiplied by x t 2 in place of t 2 we write t 1 plus tau.

So, this is a definition of the R x x tau. So, what we do in obtaining the r x (Refer Time: 36:49) tau we take a particular time shift and describe 2 random variables in time multiply the ordinates of these 2 random variables over the ensemble and then take an average of that, that is what is called the R x x tau that is the autocorrelation function and it depends upon the value of tau.

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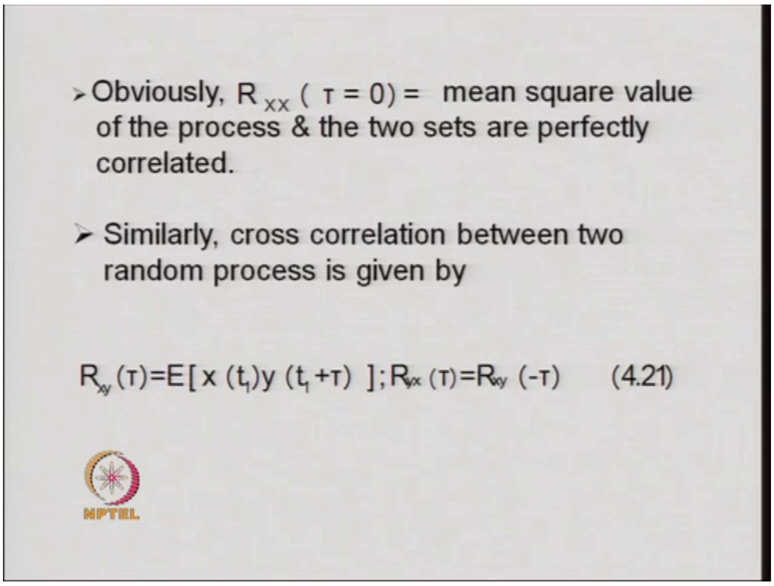

If rau is equal to 0 then you can see that the R x x tau simply becomes is equal to R x x tau becomes equal to the expected value of x t square. So, or R x x tau becomes equal to the mean square value. In this fashion we can also define the cross correlation between 2 random processes say x is a random process and y is another random process then we call R x y tau as a cross correlation function between x and y over a time separation which is called tau and the one can define the R x y tau in the same fashion that we have done for R x x tau, in this case we take at t 1 at time t 1 the values of x and for the y process we take the values at t 1 plus tau.

So, at that particular time we take the values of y and then multiply them over the ensemble and take an average. So, that is how we define R x y tau and it could be easily shown that R x y minus tau is equal to R y x tau. So, that is a property that can be easily proved.

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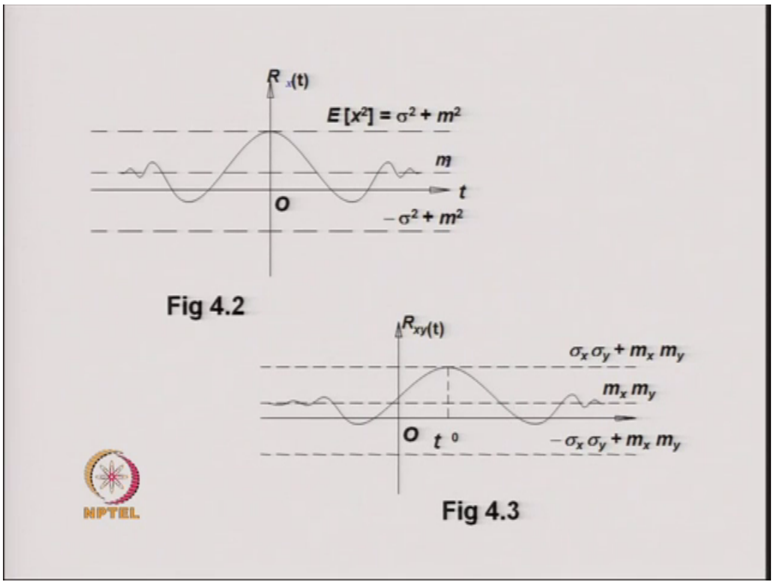

Figures 4.2 to 4.3 they show the plots of R x tau and R x y tau with tau. The figure 4.2 shows the variation of the autocorrelation function r x tau with tau and it is seen that the value of the R x tau becomes maximum at a tau is equal to 0 as it would be expected and the value of the R x tau a tau is equal to 0 is equal to sigma square plus m square where sigma square is the variance of x and m square is the square of the mean value.

For 0 mean process the value of the R x tau at tau is equal to 0 becomes equal to variance or is equal to the mean square value as we as we can see it from the definition of r x tau the r x tau or the autocorrelation function drastically falls down as tau increases. And it is seen that after a value of tau the value of r x tau oscillates about the mean value n for 0 mean process this oscillation takes place at the axis tau axis and these a fluctuation is of very small order.

So, as the tau increases, then there is a lack of correlation between 2 random variables x t 1 and x t 2 and for large value of tau the correlation between these 2 random variables may be negligible. Similarly we can see the variation of R x y tau v tau in figure 4.3, unlike the case of the autocorrelation function R x y tau does not become maximum at tau is equal to 0. But at some other value of tau that is tau is equal to tau 0 and the maximum value is equal to sigma x multiplied by sigma y plus m x into m y where sigma x is the standard deviation of the random process x and sigma y is the standard deviation for the random process y and m x and m y are the correspond corresponding mean values the after the R x y tau reaches maximum then it again falls down drastically as tau increases and it fluctuates about the value of m x into m y and this fluctuation again is of very small order. So, thus we can see that for large value of tau, the correlation between the process x or the stochastic process x and the stochastic process y can be ignored or we can say that that is very becomes very negligible.

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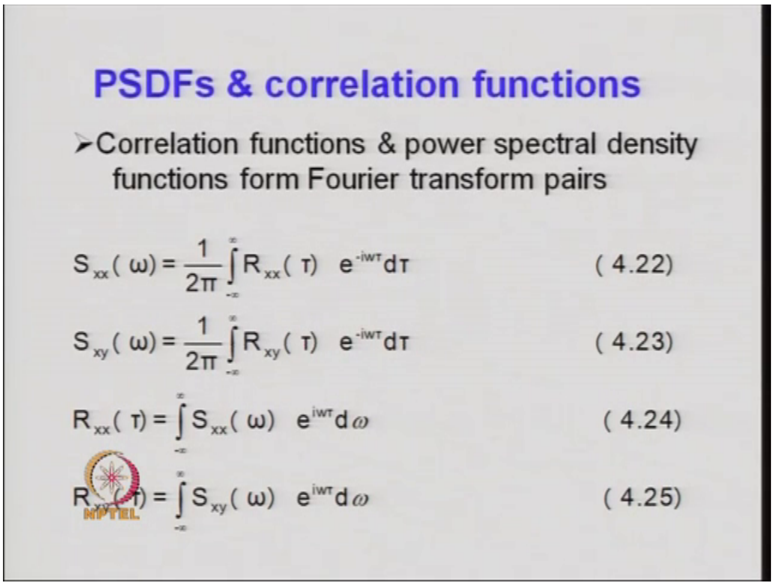

Now, the correlation function and the cross power spectral density functions they form the Fourier transform pair. So, that is what basically you are going to use now, what we would say that previously in the case of seismic input we said, that the cross power spectral density function is equal to the power spectral density function of the ground motion multiplied by a coherence function. Now this particular thing defined the cross power spectral density function between 2 excitation, but now we will define it slightly in a different way although the meaning remains the same that is the S x x tau can be shown to be equal to the inverse Fourier transform of S x x omega that is the inverse Fourier transform of the power spectral density function of the process and the power spectral density function of the process is the Fourier transform of or FFT of R x x tau.

So, given R x x tau one can in principle compute the power spectral density function of the process, similarly given the power spectral density function of a random process one can obtain its autocorrelation function. So, in this particular fashion one can also construct the power spectral density function from the ensemble of records. So, if you have an ensemble of records one can obtain the autocorrelation first and then take a Fourier f Fourier transformer of that FFT that will give us the definition or the power spectral density function. We need not then assume ergodicity and try to find out the power spectral density function from a single time history that we discussed before.

Similarly, if we have R x y tau defined for a 2 processes then taking a Fourier transform of that or using FFT on this R x y tau, we get the cross power spectral density function. So, then we need not again define the cross power spectral density function by stating that cross power density function between 2 support excitations is equal to the power spectral density function of the ground motion multiplied by the coherence function. So, we need not try we need not define in that fashion, but we can define it in this particular way through if we wish to define it through an ensemble; that means, then we are not no more again using what is known as the ergodicity property.

So, the cross power spectral density function and the cross correlation functions and the power spectral density function and the autocorrelation function they form the Fourier transform pair.

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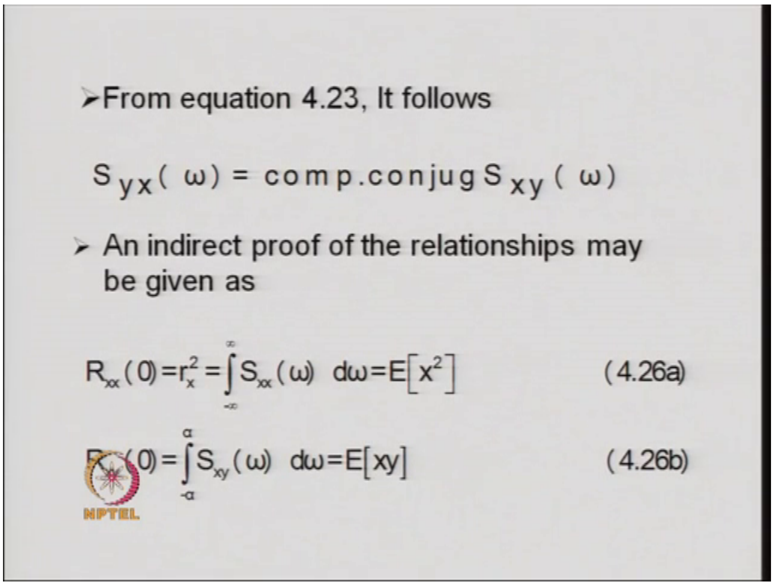

So, that is used in many a cases when will be a deriving the spectral analysis technique. Another important thing that I should mention over here is that S y x omega; that means, S x y omega is the cross power spectral density function between x and y, similarly one can have a cross power spectral density function defined between y and x. So, that is called S y x omega you will be is nothing, but complex conjugate of S x y omega.

So, if we know S x y omega you can just take a complex conjugate of that and that would give you the value of S y x omega this also can be proved an indirect proof to show that the power spectra density function and the autocorrelation function the cross power spectral density function and the cross correlation function they are they form the Fourier transform pair we can use this indirect relationship that R x x at tau is equal to 0 we know that the value is equal to R x x square. That is the mean square value of the process and if we substitute in this Fourier transform, the value of tau to be equal to 0 or rather in this equation tau to be equal to 0 then these becomes simply is equal to S x x omega d omega and we know that the area under the power spectral density function curve is equal to the mean square value.

So, this is an indirect proof of the Fourier transform pair that R x x and S x x, they confirm to. Similarly R x y 0 would be the integration of s x y omega d omega and the integration of this s x y d omega will give you the expected value of x y. So, in this particular way one can indirectly prove that the cross power spectral density function and the power spectral density function they and the autocorrelation function and the cross correlation function form the Fourier transform pairs.

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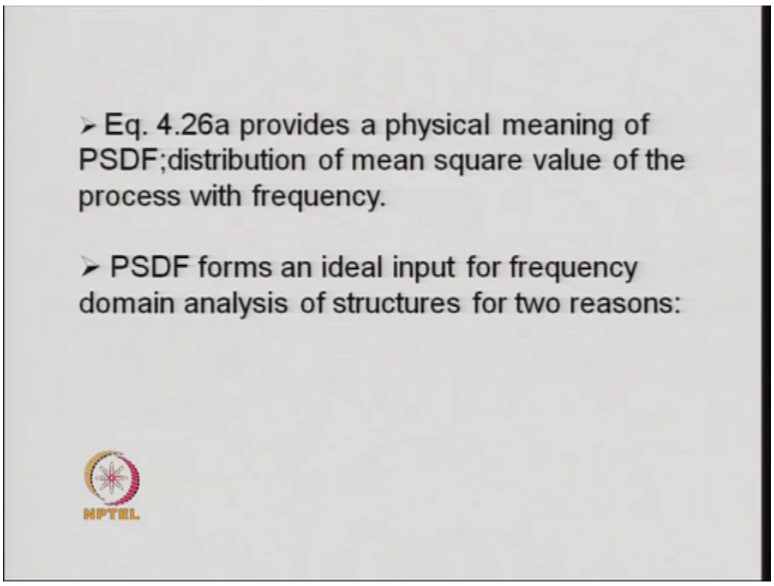

Now, the PSDF the power spectral density function forms an ideal input for frequency domain analysis of structures, for 2 reasons these reasons are firstly.

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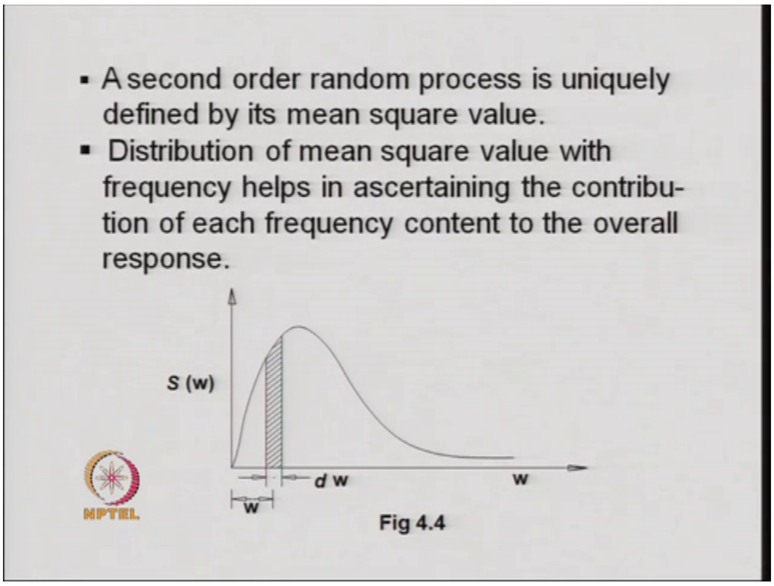

a second order random process is uniquely defined by its mean square value, that is the definition of the second order stationarity which we assume to be sufficient and for our structure analysis. So, therefore, if we have the mean square value known or as a unique value known for the 2 random process, then the random process is defined to us.

Next since a we are wanting to find out carry an analysis in frequency domain, then it is better to have not only the knowledge of mean square value of the process, but how the mean square value is distributed over the frequency and that comes from the definition of the power spectral density function itself because we know that the area under the power spectral density function curve is equal to the mean square value. So, in other words we can say that the power spectral density function is a distribution of the mean square value with frequency.

So, in a frequency domain analysis now we expect that at each frequency, what is the contribution of the power spectral density function to the mean square value. If that goes as an input then we can get an output in a similar fashion that is we can get a power spectral density function ordinate at a particular frequency contributing to the total mean square value of the response. So, with this particular concept in mind we develop the spectral analysis technique or frequency domain spectral analysis technique.

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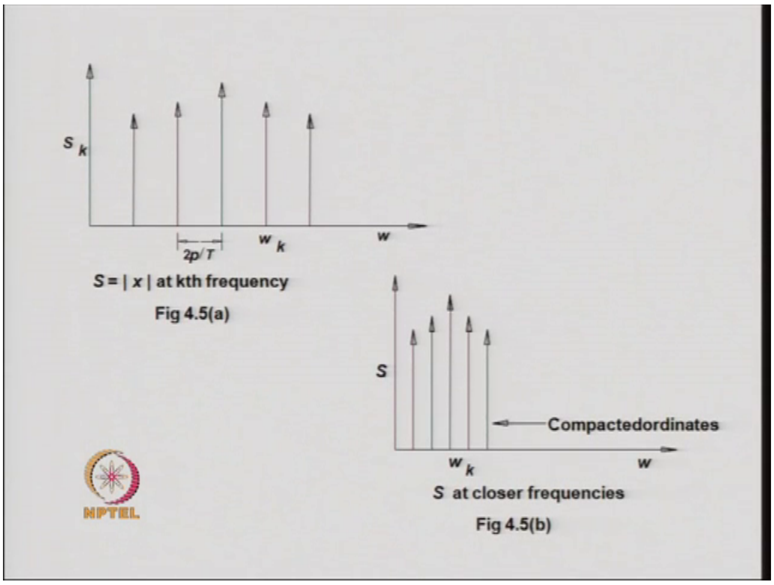

How we obtain the power spectra density function curve that I describe before. So, if you if you can have the effect is of the time histories and plot the amplitude squares in.

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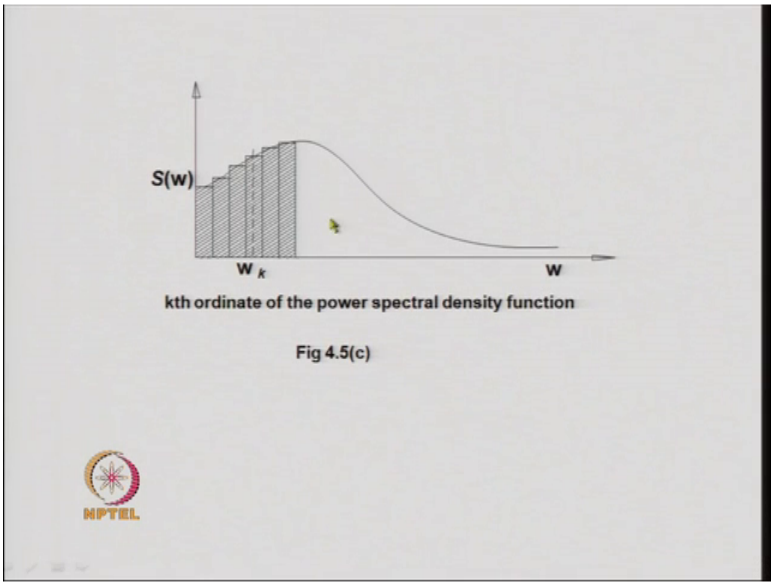

Close enough range and then divide it by d omega then these are the different bars and you will have n by 2 plus 1 number of bars over here and through the center of this bars if you draw a line that gives you the value of s omega denoting that the area under the curve is equal to the mean square value of the process.

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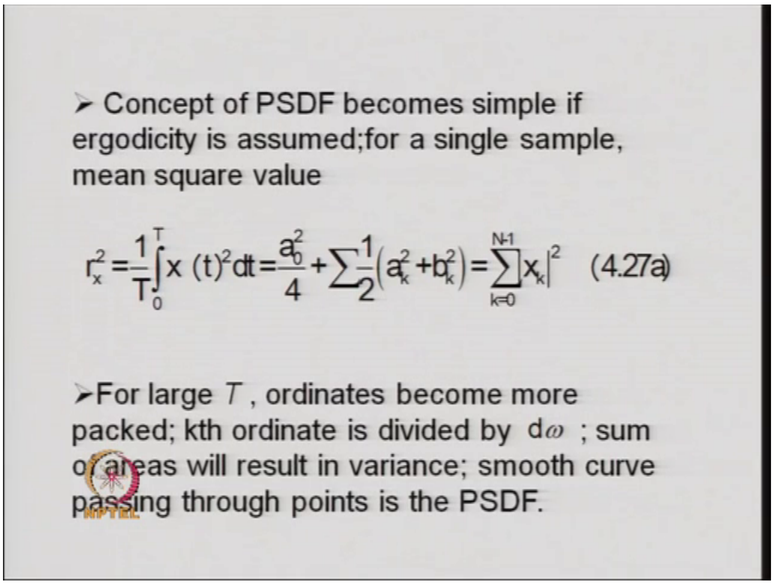

So, in this fashion you get the value of the or you can the obtain power spectral density function provided we assume the process to be a ergodic process. So, I am not going to again repeat the same thing that I had repeated or discussed before.

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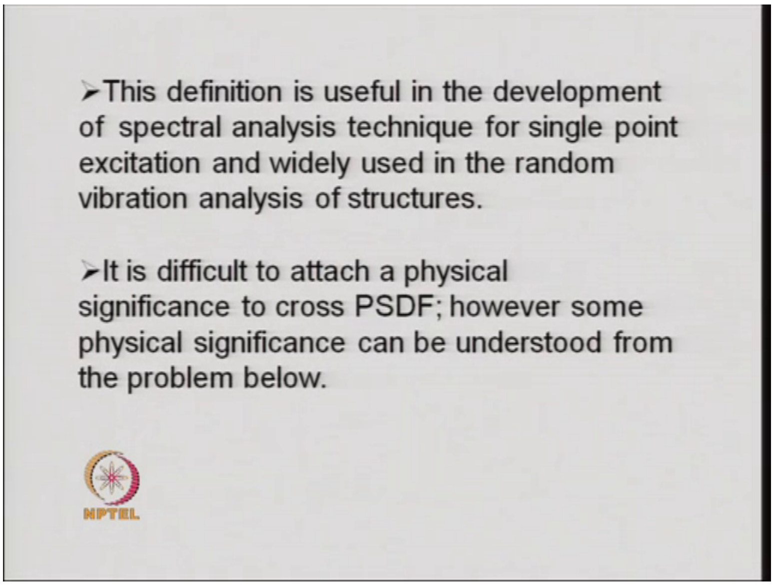

Only thing which again I wish to mention over here is that there is a physical meaning which is attached to the power spectral density function, that is the area under the power spectral density function curve is equal to the mean square value or alternatively we can say the distribution of the mean square value with frequency is a power spectral density function.

So, that gives a physical sense to the spectral analysis; however, we cannot give such a physical interpretation to a cross power spectral density function as we can give it for or the power spectral density function.

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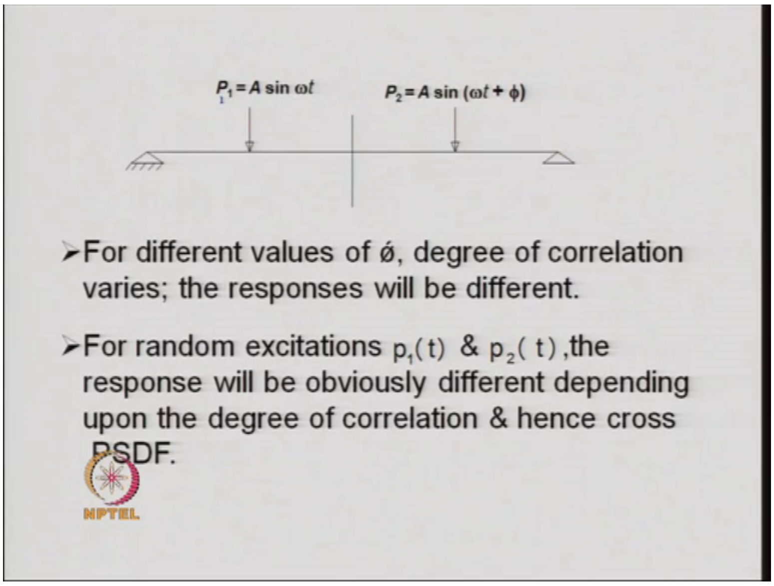

so, but some idea about the physical significance of the power spectral density function would be clear from this figure, if in this simply supported b in if we have got a 2 loads dynamic loads with the phase difference of phi, then as phi goes on changing the degree of correlation between P 1 and P 2 they would be changing and the moment this phi is changed we will find the bending moment at the center that goes on changing with the value of phi.

So, thus the degree of correlation is very much related to the response of the system. So, that is the physical meaning that we can attach to the cross power spectral density function of 2 random processes. So, in that case P 1 will be a random process P 2 will be another random process, but having a phase lag and in this 2 random processes since they are not perfectly correlated they will have a distinct overall contribution or influence on the response of the structure therefore, these need to be considered along with the power spectral density function of the two processes of as an input for spectral analysis.