**Seismic Analysis of Structures**
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**Lecture – 16**
**Frequency Domain Spectral Analysis (Cont.)**

In the previous lecture, we discussed about the random process, then discussed when it becomes a stationary random process, then we also defined the mean variance of the ensemble, then how a stationary process can be simplified as an ergodic process, then we looked at the auto correlation function cross correlation function.

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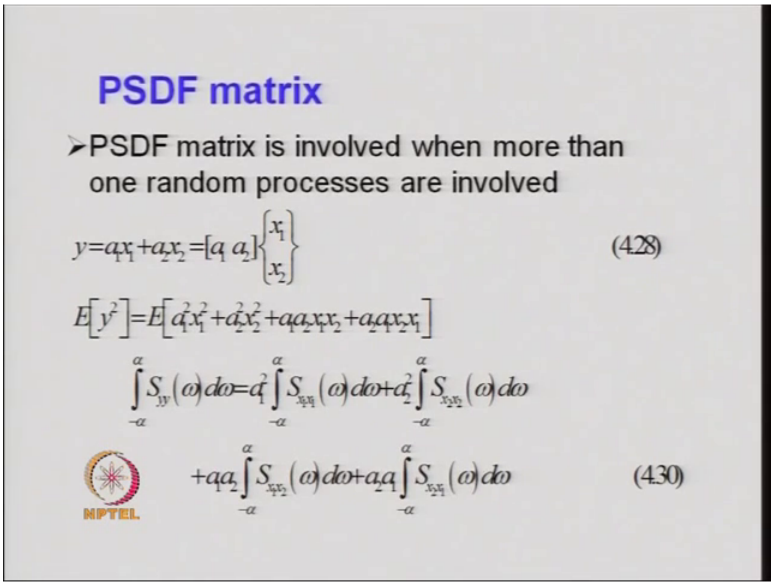


Then the Fourier relationship or Fourier transform relationship that exist between the power spectral density function and the autocorrelation function that is they form the Fourier transform pair that was discussed, then the significance physical significance of the power spectral density function physical significance of the cross power spectral density functions were discussed. Today let us look into when we have got more than one random process, then how do we define them because in a particular structure, you may have more than one excitations and each excitation could be random process.

The output; similarly could be more than one and then we have to define these output and the relationship that exist between the outputs that also we may have to look in. Therefore, the essential relationships that exists between a set of input random process and a set of output random processes that must be understood clearly here, in this slide you can see that y is a random process consisting of the 2 random processes, there weighted by a 1 and a 2 that is y is equal to a 1 x 1 plus a 2 x 2 and that can be written in a matrix form as a 1 a 2 x 1 x 2 by equation for point 2 a, then if you are wanting to find out the mean square value or the variance of y, then the variance of y can be given by the expected value of a 1 square x 1 square plus a 2 square x 2 square plus a 1 a 2 x 1 x 2 and plus a 2 a 1 x 2 x 1.

So, the last term is purposefully written like that that is a 1 a 2 x 1 x 2 can be also written as a 2 a 1 x 2 x 1 since instead of writing 2 a 1 a 2 x 1 x 2, we put it in this fashion, then since the mean square value or for the variance for 0 mean process can be written as the area under the power spectral density function curve; therefore, E expected value of y square E is a replaced by minus infinity to plus infinity S y y omega d omega, similarly; a expected value of a 1 square x 1 square that can be written as a 1 square integration of minus infinity 2 plus infinity S x x omega d omega.

And that way you can write down all the terms within the expectation in the above equation as the quantities that you show in equation 4.30 and you can see that the last 2 terms they are S x 1 x 2 and the other one S x 2 x 1 that is the cross power spectral density function between x 1 x 2 and cross power spectral density function between x 2 x 1 and that they are they form the complex conjugate relationship that we have discussed before; therefore, if there is a single power spectral density function or cross power spectral density function x 1 x 2 known, then one can find out the cross power spectral density function S x 2 S x 1 by simply taking the complex conjugate of S x 1 S x 2.

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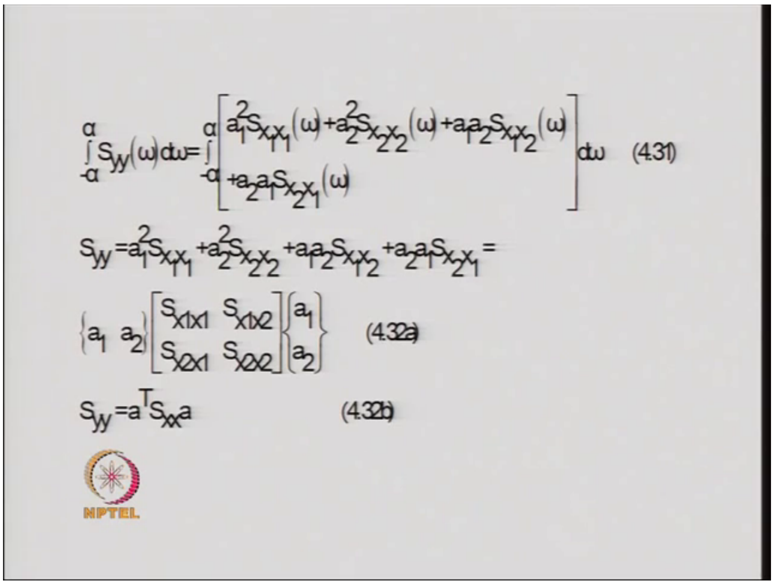


Now, thus integration on the right hand side; now take these form that is we can take out the integration minus infinity to plus infinity outside and we can write down the entire thing within the integration that is equation 4.31 and when we compare the left hand side to the side right hand side, we get simply this relationship that is S y y that is power spectral density function of y is equal to a 1 square into power spectral density function of x 1 plus a 2 square of power spectral density function S x 2 plus a 1 a 2 of into cross power spectral density function between x 1 x 2 and a 2 a 1 into power cross power spectral density function between x 2 and x 1 and this can be written in a matrix form given in equation 4.32 a and if we say that S x x is the matrix that is shown above that is S x 1, S x 1, S x 2, S x 2 form the diagonal elements and S x 2, S x 1 and S x 1, S x 2 they form the 2 off diagonal elements of S x x, then the equation 4.32 can be written in brief notation as S y is equal to a T S x x a.

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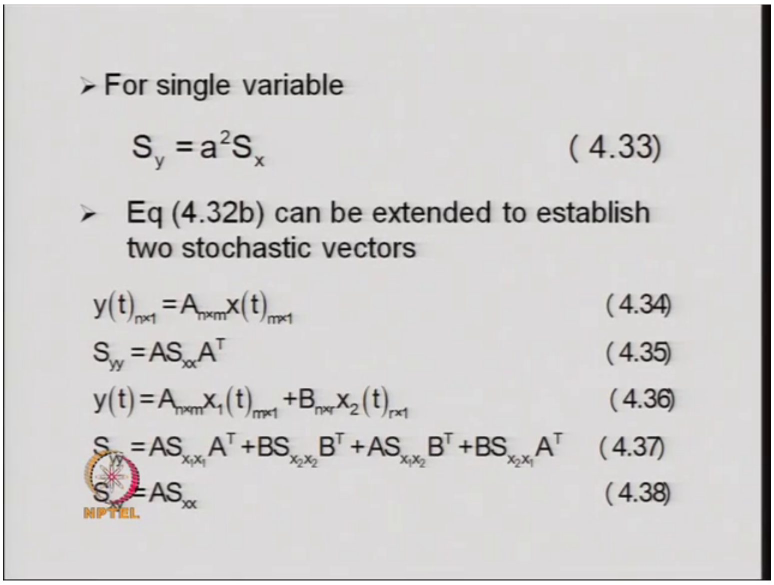


So, if we have a equation of the form that y is equal to a matrix a into x then one can find out the value of S y from the previous relationship that we have discussed before. The specific condition when we write down y is equal to a into x for that S y will be simply is equal to S square into S x that follows from the previous equation. This equation which we have derived before that is S y y is equal to a t into S x x a that can be generalized for a vector a relationship that is the relationship that exist between 2 vectors y and x and say y is related to vector x through matrix a, then one can write down S y will be is equal to a into S x x into A T, where S x x will be a matrix of size m by n and S y y will be a matrix of size n by n. The diagonal terms of S y matrix will be S y S y 1 y 1, S y 2 y 2, S y 3 y 3 so on and of diagonal terms will be S y 1 y 2 S y 1 S y 3 and on the other half of the diagonal it will be S y 2 y 1, S y 2 y 3 so on.

So, we have a matrix S y y similarly S x x would be a matrix and the terms or the elements of the matrix will be similar to S y y matrix and the matrix the coefficient matrix a that exist as a prefix to x t vector that comes from the left hand side A into S x x A T where A T is the transpose of matrix a. So, this is a basic relationship that exist between 2 sets of random variable connected by a coefficient matrix and if the power spectral density function of the matrix of the random variables x t that is x t contains is a vector and contains x 1 x 2 x 3 so on and then we construct a power spectral density function matrix for this vector of random processes, which will be called as S x x and if that is given to us then we can found out the power spectral density function matrix for the y vector that is the y vector consisting of a number of random processes y 1 y 2 y 3 and so on.

Now, this can be further extended or generalized 2 equation 4.36 that is a vector of random processes is a summation or other weighted summation of 2 random vectors x 1 and x 2 by the coefficient matrix a and b, then the power spectral density function matrix of y is given by A S x 1 S x 1 A T plus B S x 2 S x 2 B T plus A S x 1 S x 2 B T that is the cross power spectral density function x 1 and x 2 when we are considering then a comes as a prefix and B T is post multiplied. Similarly for x 2 x 1 the b is a matrix which is a prefix to S x 2 S x 1 and A T is a post multiplied.

Now, the next relationship that exist are between the input and output say for example, if we take the equation 4.34, then in this equation the input is x t and the output is y t, then the cross power spectral density function between the input and output that is S x y is written as A into S x x. So, that be easily proved from the relationship that one can show over here in this with the help of this simple equation.

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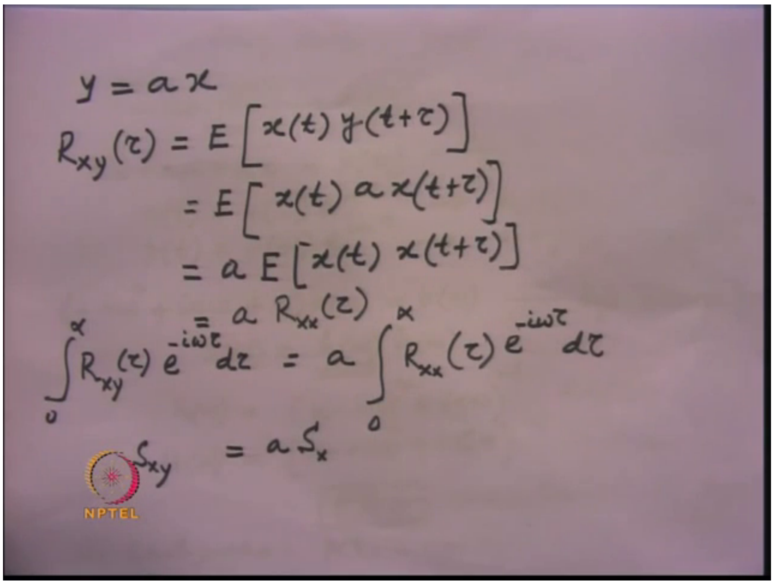


So, if we write y is equal to a x that is a is a simple one constant x is a random process y is another random process then one can write down R x y tau this is the cross autocorrelation a cross correlation function between x and y tau that can be written as expected value of x t into y t plus tau and if we substitute for y t plus tau from this equation then it becomes expected value of x t into a x t plus tau taking out a we have a into expected value of x t into x t plus tau and that is nothing, but a into R x x tau.

Now, if we take a Fourier transform of both sides then we have R x y tau E to the power minus i omega tau d tau and on this side we will have R x x tau E to the power minus i omega tau d tau. Now this integration of course, will be equal to minus infinity to plus infinity not 0 minus infinity to plus infinity and this we know that is equal to S x y because in the last lecture, we have seen that the cross power spectral density function and cross correlation function, they form a Fourier transform pair similarly autocorrelation function and the power spectral density function they form the Fourier transform pair therefore, this quantity this integration becomes equal to S x and this integration becomes equal to S x y.

So, we can see that S x y that is the cross power spectral density function between the input and the output this is the input and this is the output is equal to a into S x.

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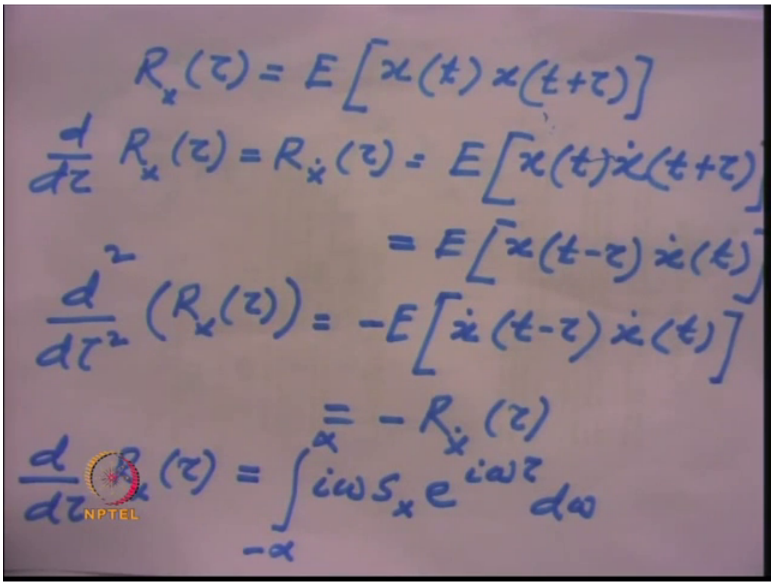


So, this can be generalized in this particular form which is given in equation 4.38 in the 4.38, we see that S x y is written as matrix a into S x x. So, if we know the basic relationship between the input vector and output vector, then we can find out the cross power spectral density function matrix from the relationship given by equation 4.38.

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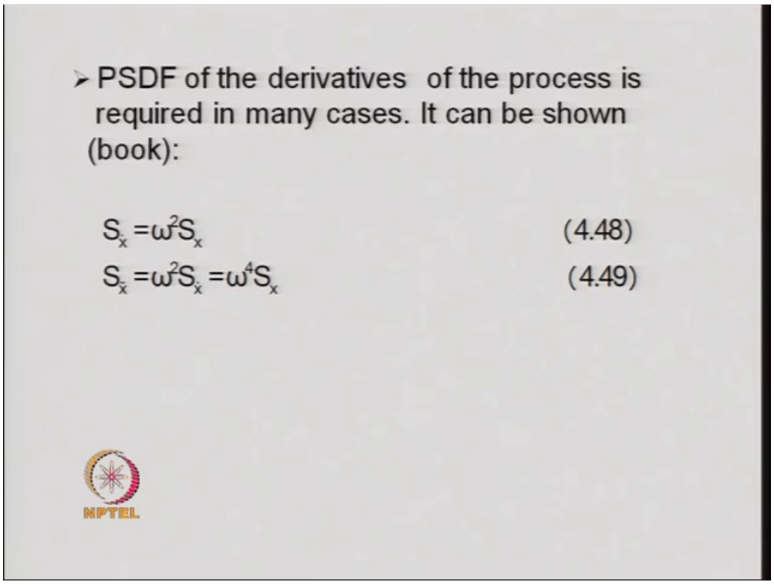


Next we have to know in many cases about the power spectral density function of the derivatives of the process for example, if we know the power spectral density function of x that is S x as a matrix of power spectral density function of a random processes set of random processes x 1, x 2, x 3, etcetera, then we may had to find out the S x dot that is a power spectral density function of the velocity and the power special density function of the acceleration.

So, that can be derived again from the basic relationship that is shown over here for example, if we take a single variable that is if we take x as a random process then the autocorrelation function of the process are x tau that will be equal to expected value of x t and x t plus tau that we have already seen if I differentiate it with respect to tau then it becomes R x dot tau and these differentiation can be written as expected value of x t and multiplied by x dot t plus tau because tau is existing over here therefore, we differentiate this term and the these becomes x dot t plus tau.

Now, since in a stationary random process the characteristics of the process that is the mean square value or the power spectral density function or R x tau that remains invariant with the time shift, then we can interchange tau over here then the first term, we can write down as x t minus tau and second term then becomes x dot t. Next we differentiate this once more that is d 2 d tau 2 into R x tau that becomes is equal to minus expected value of x dot t minus tau this is differentiated once more and this remains x dot t this is not differentiated because it does not contain the term tau and this becomes by definition minus R x dot tau because if the autocorrelation function of velocity will be expected value of x dot t minus tau into x dot t or x dot t into x dot t plus tau whatever we wish to define.

So, we see that d 2 d tau 2 into R x tau or of R x tau that is equal to minus R x dot tau. Now if I differentiate R x tau with respect to tau then we get i omega S x E to the power i omega tau d omega integration is from minus infinity to plus infinity because of the fact that R x tau as such is equal to the Fourier inverse Fourier transform of S x that is the power spectral density function and the autocorrelation function of the process they form the Fourier transform pair that we discussed before therefore, R x tau is equal to S x into E to the power i omega tau d omega.

Now, if I differentiate it with respect to tau then we get i omega into S x E to the power i omega tau d omega, the first differentiation of R x tau that gives me an expression like this.

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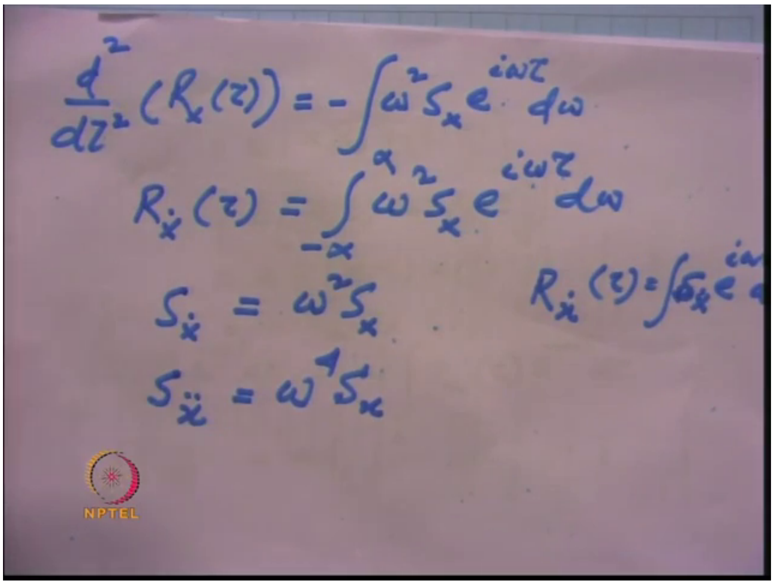


Similarly if I differentiate it once more, then it becomes minus omega square because i square omega square will be equal to minus omega square minus omega square S x e to the power i omega tau d omega. Now if I take this expression and the previous expression over here that is d 2 d tau 2 R x tau is equal to minus R x dot tau this and this if we equate, then it becomes R x dot tau equal to omega square S x e to the power i omega tau d omega minus infinity to plus infinity.

Now, since R x tau itself can be written as S x dot E to the power i omega tau d omega that is the autocorrelation function of velocity and the power spectral density function of velocity, they are related to Fourier transform pair, then if we compare this and this because both of them are equal to R x dot tau, then you come to this basic relationship S x dot is equal to omega square S x. Similarly one can find out S x double dot is equal to omega to the power 4 into S x, thus, if we know the power spectral density function of a random process, then one can find out the power spectral density function of the velocity of the process and the acceleration of the process using this basic relationships and they are used in solving many structural engineering problems.

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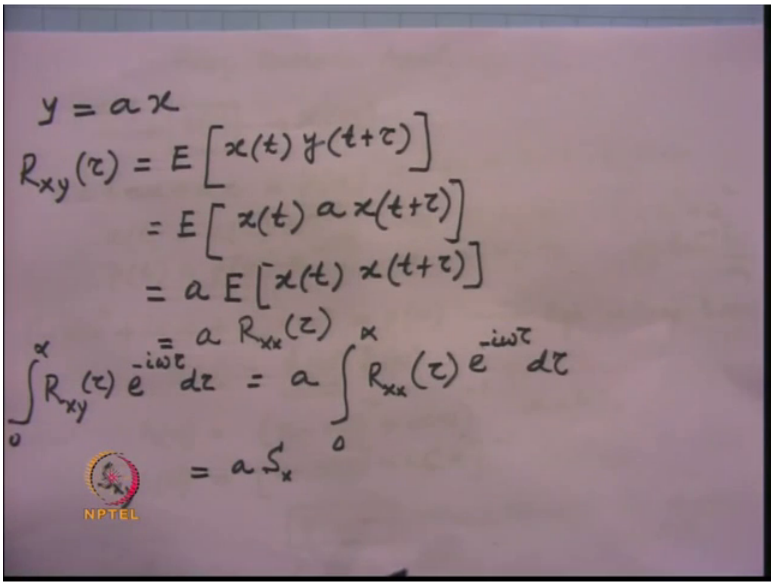


Now in this 4.48 and 4.49 in these equations this basic relationship between the power spectral density function of velocity and acceleration with the power spectral density function of the displacement they are shown.

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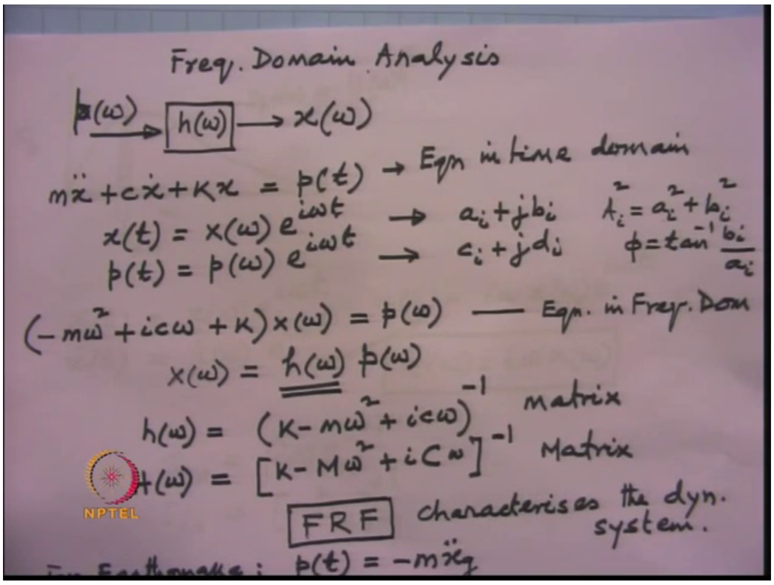


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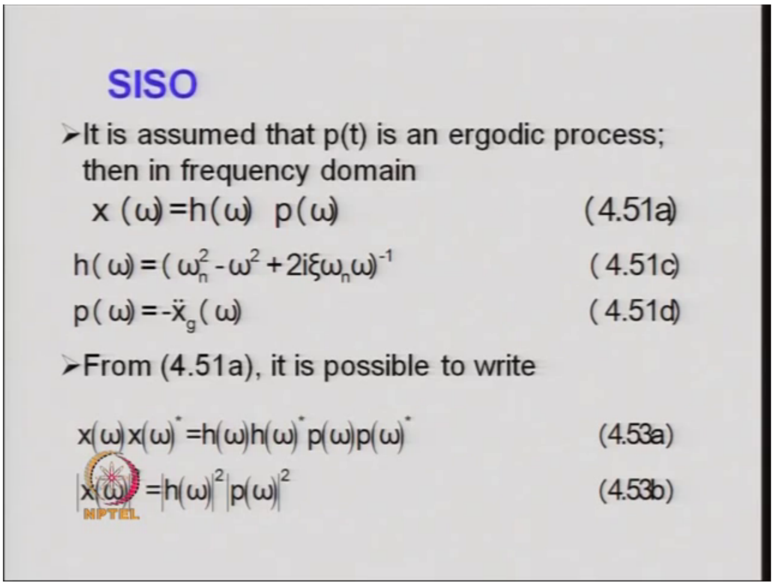


With this background, now let us get into the SISO that is single input single output and in the single input and single output we have only a single degree of freedom system in which the excitation is say p t and the output from the process is x t. So, that is shown here in this particular figure this is a single degree freedom system and the p t is the input and x t is the output the frequency component of them are shown over here and the characteristics of the single degree of freedom system is represented by is frequency response function h omega.

Let us try to recall what is this h omega that is frequency response function, if we write down that equation of motion for a single degree freedom system, it is like this and when we have earthquake or the support excitation in particular, then p t becomes equal to minus m x double dot g that is this this is a definition of m x double dot g. Now since we are trying to define the entire thing in frequency domain, then we write down x t to be is equal to x omega E to the power i omega t that is we are Fourier synthesizing in other words using the FFT algorithm then p t as equal to p omega E to the power i omega t if we do this and substitute this into this equation, then we get this basic relationship in frequency domain for the single degree of freedom system that is x omega the frequency contents of the output that is related to the frequency contents of the input that is the load p omega by this equation or x omega can be written as equal to h omega into p omega where h omega is nothing k minus m omega square plus i c omega inverse of that.

Now, this h omega is called the frequency response function of the single degree of freedom system; similarly if we have a multi degree freedom system, then one can extent from this the definition of the frequency response function matrix of the multi degree of freedom system which will be defined as capital h omega is equal k minus m omega square where here k with the k matrix m is the mass matrix and c is the damping matrix and inverse of this entire thing is known as the frequency response function matrix of the system.

This is also known as FRF of the system frequency response function of the system and if we wish to characterize any dynamic system be it a single degree or multi degree freedom system then it can be completely characterized by this FRF that is the frequency response function either in the form of a matrix or as a single quantity for a single degree freedom system and the quantities over here in this they are generally of the in a complex form each element is in a complex form and if we in particular; say look at this, this, this terms x omega term will be in the form of a i plus j that is the imaginary notation in to b i and the amplitude ith amplitude is given by a square plus b square and the ith phase will be given by tan inverse b by a that we discuss before in connection with your Fourier series analysis.

So, therefore, the any function x t can be broken up into a number of the harmonics harmonic has an amplitude and a phase angle all this things we discussed in detail while discussing about the FFT and the Fourier synthesis of the time histories and using that particular concept one can find out the frequency response function of a single degree freedom system or a multiple degree system by a frequency response function matrix.

So, this is highlighted over here with the help of this diagram provided, we are able to characterize this of a system then one can find out the response of that system to any external excitation which can be represented in the form of its frequency contents. Now with this fundamental thing in mind, let us look into how we try to solve a single degree freedom system for a random excitation.

Now if we wish to write down simply the frequency component of the response x omega, then it is equal to h omega into p omega. So, that basically we have seen or proved before in the case of the ground motion or the support motion p becomes equal to minus m x double dot g therefore, p omega becomes here is equal to minus x dot x double dot g omega provided we divide the left hand side and right hand side of the equation to motion by m and the definition of h omega becomes equal to omega n square minus omega square plus twice i psi omega n omega because k by m would become equal to omega n square and that c by m will turn out to be this.

So, this becomes the frequency response function of a single degree freedom system where n becomes equal to unity. Now you using equation 5; 4.51 a 1 can find out the absolute value square of x omega that is we write down x omega multiplied by x omega star that is the complex conjugate of that. So, multiplication of this 2 quantities become absolute values square of x omega that will be equal to h omega absolute square and p omega absolute square. So, the relationship that exist between the absolute squares of the responses at each frequency is related to the multiplication of the absolute value square of the frequency response function and the absolute value squares of the excitation at each frequency.

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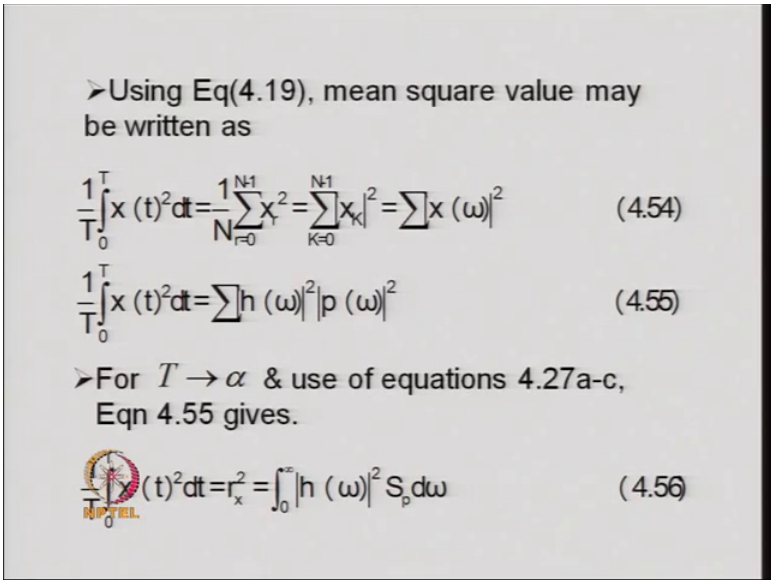


Next we come to again the Parseval's theorem which states that the mean square value of the process is nothing, but the summation of the absolute value square of the absolute value squares at each frequency.

So, using this relationship one can write down the mean square value of the response to be is equal to h omega absolute square and p omega absolute square at each frequency because multiplication of these 2 is equal to this that we have shown before in the previous equation and if t tends to infinity then this entire summation becomes an integration and we have got this relationship that is 0 to infinity is equal to h omega square absolute square into. So, p d omega.

Now, this this thing can be shown with the help of these diagram say this is the power spectral density function or the p S power spectral density function of p, then if I take a small element over here over d omega.

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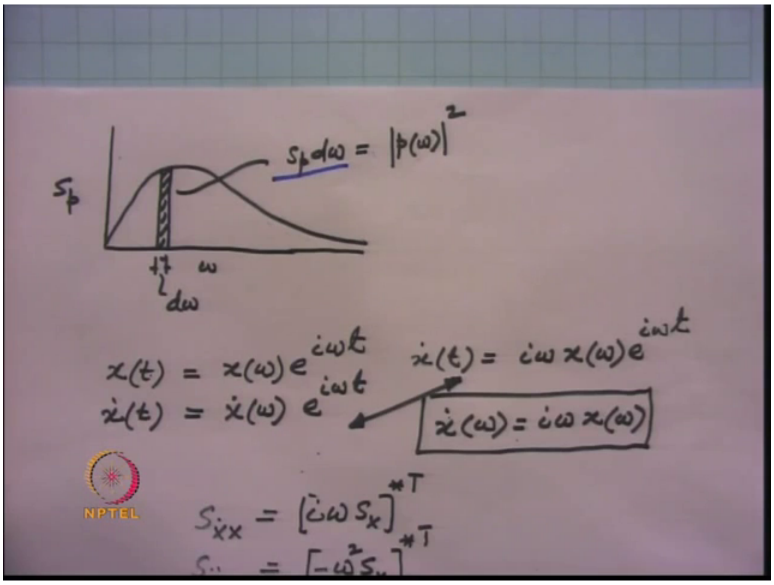


Then that d omega into S p; S p that becomes equal to absolute value square at this particular omega and if we sum up all the absolute value square in this diagram then they would be equal to that sum will be equal to the means square value of the process that we have seen.

So, S p d omega and p omega absolute square; they can be related and using that relationship what we have done we have replaced in this particular equation absolute value square that is the absolute value square of the excitation at omega that we have replaced by S p d omega. So, the mean square value of the response can be shown to be equal to product of this 2 now since the mean square value also of the response also can be written as 0 to omega S x omega d omega here what we have done instead of integrating up to infinity we integrate up to a finite frequency after which the value of S x becomes 0.

So, if we integrate up to that 0 to omega S x omega d omega, then we get the mean square value of the response that is one by t x t square d t that is replaced by this equation, then that becomes equal to the right hand side which you have shown before that is in the previous case; this particular equation in this we have replaced this integration by this integration S x omega d omega.

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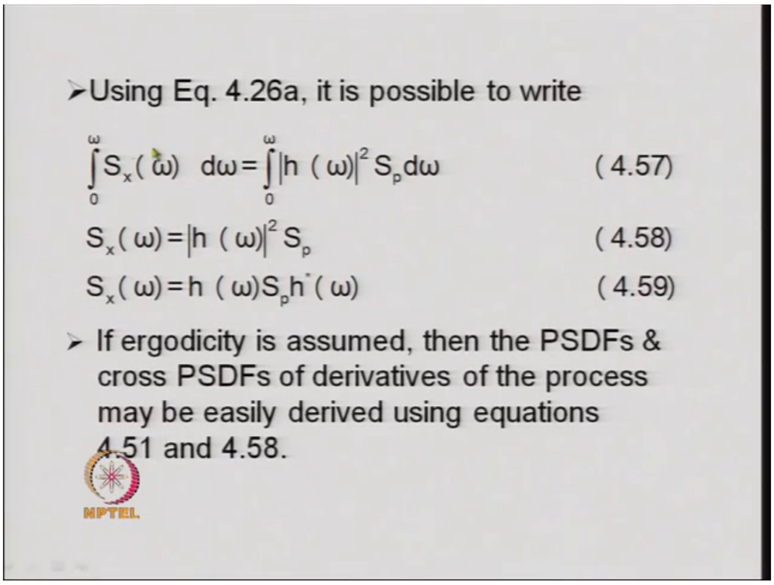


Now if we look into this 2 integrations then from that it immediately follows that S x omega becomes is equal to h omega absolute square into S p and S x omega also can be written as h omega into S p multiplied by the complex conjugate of h omega because this complex conjugate of h omega and this h omega they becomes equal to absolute value square.

Now, this derivation that we have obtained was apparent from the this equation itself in this equation we see that 2 random processes that is x omega is a random process p omega is a random process this is an input and this is an output and if x omega is related to p omega by a weighting function that is say in the previous case what we have done y is equal to a a S x, if we write then we have seen that S x becomes equal to a square into or S y is equal to a square into S x.

So, if we remember that then from this relationship itself we can say that S x will be equal to h omega absolute square multiplied by S p here it becomes absolute square because it is a complex quantity had it been a real number then simply it would have been and the square of that particular real number. So, if we assume and the ergodicity then one can derive the relationship between the power spectral density function of a process with the power spectral density function of the derivatives which we have done before through the differentiation of the autocorrelation function.

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$$\dot{x}(\omega) = i\omega x(\omega) \quad (4.60)$$

$$S_{\dot{x}} = \omega^2 S_x \quad (4.61)$$

$$S_{x\dot{x}} = i\omega S_x \qquad S_{\dot{x}x} = -i\omega S_x \quad (4.62)$$

$$\ddot{x}(\omega) = -\omega^2 x(\omega) \quad (4.63)$$

$$S_{\ddot{x}} = \omega^4 S_x \quad (4.64)$$

$$S_{x\ddot{x}} = -\omega^2 S_x \qquad S_{\ddot{x}x} = -\omega^2 S_x \quad (4.65)$$

$$S_{\dot{x}} = \omega^2 S_x \quad (4.66)$$

$$S_{\ddot{x}} = \omega^4 S_x \quad (4.67)$$

$$S_{x\dot{x}} = i\omega S_x; \qquad S_{\dot{x}x} = -i\omega S_x^T \quad (4.68)$$

$$S_{x\ddot{x}} = -\omega^2 S_x; \qquad S_{\ddot{x}x} = -\omega^2 S_x^T \quad (4.69a)$$

$$S_{x\dot{x}} + S_{\dot{x}x} = 0 \quad (4.69b)$$

Now, by using this relationship that we have; now using this relationships we will now try to prove the same thing again in this equation since x dot omega is equal to i omega in to x omega because of this equation that is x t is equal to x omega E to the power i omega t that is a frequency content of that. Now if I differentiate it with respect to T, then x dot T will be equal to x dot omega E to the power i omega T and also we know that x dot t is equal to i omega x omega E to the power i omega that is coming from a differentiation of this if this 2 are equal then one can write down x dot omega is equal to i omega x omega; that means, this is equal to this.

So, that is that becomes the starting point now if x dot omega is equal to i omega into x S x omega then the power spectral density function of S x dot that will be equal to the absolute value square of this quantity that is i omega absolute square into S x omega. So, that is how one can easily prove that S x dot is equal to omega square into S x that we have proved through the differentiation of the power autocorrelation function and before.

Now, the if we wish to find out the cross power spectral density function between x and x dot then S x dot simply will become equal to i omega into S x that we have you know seen before with the relationship that if y is equal to a into x then S x y that becomes equal to a into S x and the complex conjugate of this will be equal to the S x dot x now the x double dot omega that is the frequency content of acceleration that can be related to the frequency content of the displacement using this relationship that is minus omega square because if we differentiate this once more i omega multiplied i omega that will become minus omega square and if this relationship holds good then S x double dot that will be equal to simply square of that quantity into S x.

So, the 2 relationship that is the derivative of S x derivative of x and double derivative of x that is the velocity and acceleration they are power spectral density function are related to the power spectral density function of the parent process x with the help of all these 2 equations which we had derived earlier, but now we are deriving it in a different way that is the relationship we are using that exist between the velocity and the displacement for a harmonic excitation. Now we are writing the same thing over here, but now; the x is not a single random process, but is a vector of random process and then again the same relationship holds goods only the difference here would be that S x dot would represent a matrix and S x also would represent a matrix and S x double dot also represent a matrix of the power spectral density function of the acceleration.

Now, with this 2 relationships defined for the matrices of the power spectral density function of displacement and power spectral density function of acceleration and velocities one can find out the cross power spectral density function matrix between x and x dot and x dot and x and here this particular term it can be shown that this is basically i have wrongly written over here in this equation this should be is equal to this that is i omega S x the conjugate of this transpose that is here this minus term will not be there i omega and here there will be a a star; star means the complex conjugate of S x and in this relationship of course, it the it will have the complex conjugate term over here coming and this will be simply omega square.

Now, if we look into this 2 equations then one can see easily or one can prove easily that S x dot x and S x x dot that is the cross power spectral density function displaced between the displacement and velocity and the cross power spectral density function between the velocity and displacement, if we add them together they would become equal to 0 that is a very important relationship that we use in again solving many problems that is the cross power spectral density function between the displacement and velocity for 2 for stationary processes they turn out to be 0 because the sum of this 2 terms happens to be is equal you 0.

Similarly if we take the velocity that is the power spectral density function of velocity and the power spectral density function of acceleration again the sum of their cross power spectral density functions that turn out to be 0. So, in many problems of structural engineer we make use of these relationships that is the between the displacement and velocity and velocity and acceleration the sum of the cross power spectral density functions they turn out to be 0.

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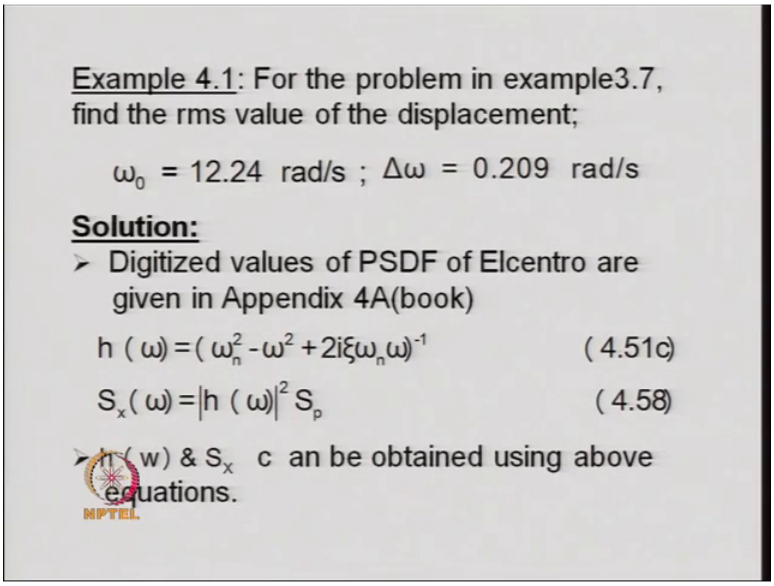


Now, let us try to solve a problem, say for example, for example, 3.7 that was the problem in which you have a inclined like portal frame and we had one sway displacement at the top the frequency of that was equal to 12.24 and it is excited by an excitation and whose frequency contents are obtained at delta omega is equal 0.209 and the excitation was that of a elcentro earthquake and the power spectral density function of the elcentro earthquake the is digital values are given in the book or the appendix of the book from that one can take the values of the power spectral density function or in it sampled at delta omega is equal to 0.209 using those values one can find out the value of h omega for each value of omega and the one can find out the S p S p here will be simply is equal to x double dot g that is S x x double dot g that is power spectral density function of the ground acceleration of the elcentro and h omega absolute square here you can see that it is that k and m they are replaced by omega n square and simply omega square over here because all through the equation is divided by m.

So, substituting here the values of different omegas we can find out absolute values square of h omega and we can also find out the values of the S x double dot g that is the power spectral density function of the ground motion given in the appendix.

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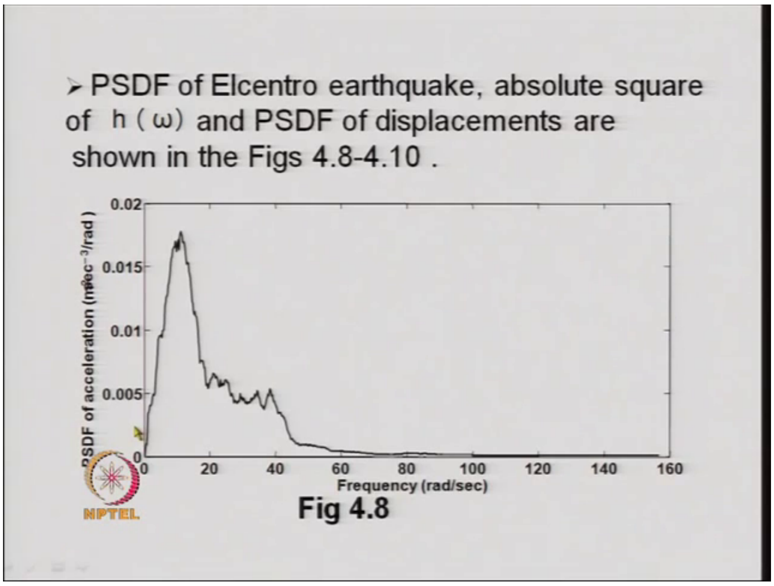


So, multiplying them together we get the value of the power spectral density function of the response here in this figure the power spectral density function of the elcentro ground motion taken from the appendix that is plotted.

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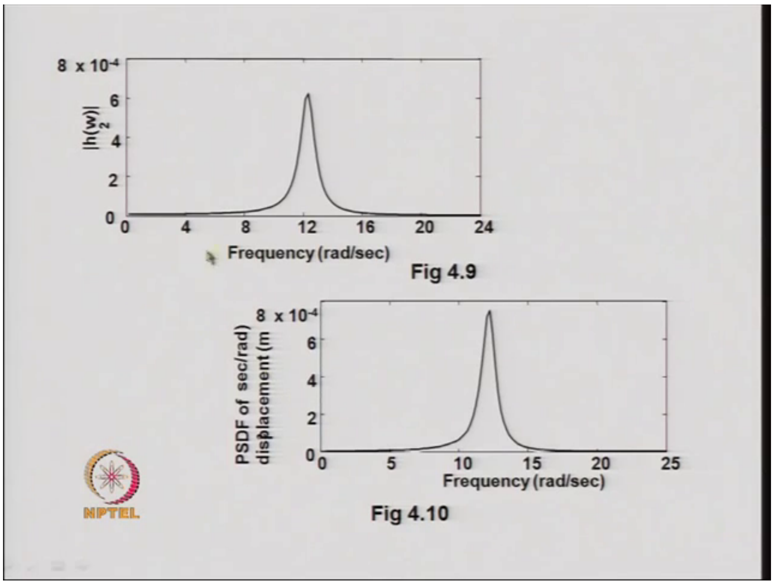

In this equation the frequency response square h omega absolute square that is plotted against frequency and the multiplication of these 2 gives you the power spectral density function of the response and that is again plotted over here against frequency.

So, this becomes the response of the system to the excitation. So, I think I stop at this what we have discussed over here is the single point and excitation to a single point output sorry the single point excitation and single point response known as SISO and for that the we obtain the relationship between the power spectral density function of the response to the power spectral density function of the excitation and they are related through absolute value square of the frequency response function and if we know the values of the power spectral density function of the excitation then immediately one can find out the power spectral density function of the response.

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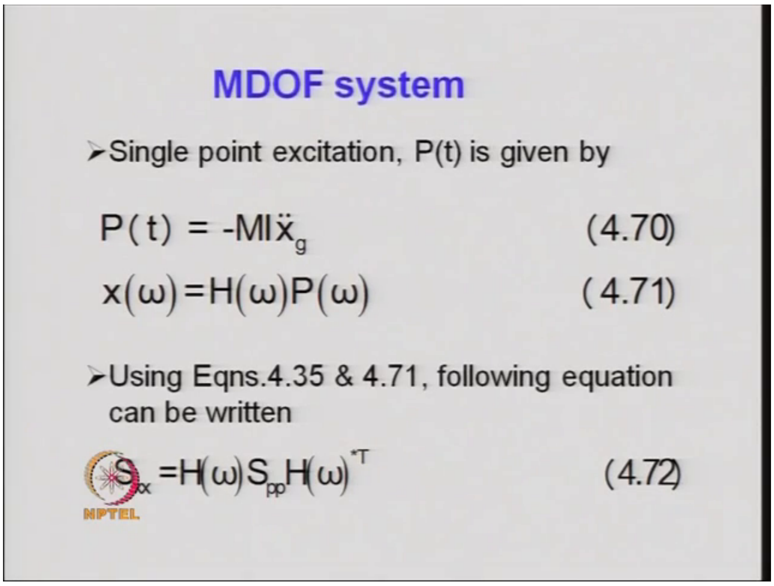


So, this can be extended and we will do it in the next lecture for a multi degree of freedom system in which we will not have a power spectral density function of a single input and the power spectral density function of a single output, but we will have the power spectral density function matrix of input vector and a power spectral density function on matrix of the output vector, those 2 will be related together with the help of the frequency response function matrix and we will see how we replace the absolute value square of the frequency response function that we have used for the single degree of freedom system that will be now replaced in a different way. So, and the proof of that would show in the next class.