Seismic Analysis of Structures
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Lecture – 18
Frequency Domain Spectral Analysis (Contd.)

In the previous lecture we discussed about the response analysis of structures subjected to multi points excitation.

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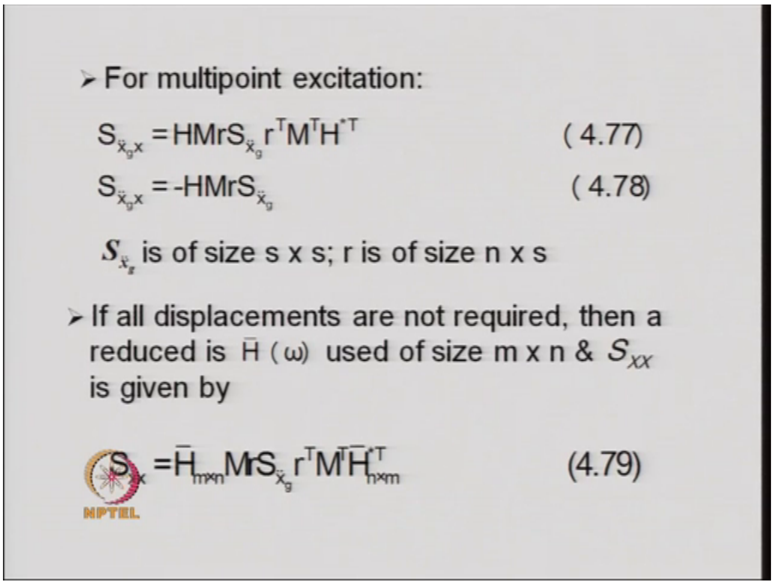

And how we obtain the power spectral density function matrix for the responses given the power spectral density function of the ground excitation and these power spectral density function of ground excitation was converted to a power spectral density function matrix of the excitations at different supports with the help of the inference coefficient matrix r and the various vectors or the vectors of excitation at different supports that lead to a matrix called S x double dot g matrix. Which denotes the power spectral density function matrix of the excitations at different supports in which the diagonal terms are the power spectral density of the excitations at different supports and the cross power spectral density terms represent the cross power spectral density between different excitations at different supports, and that is we are representing with the help of the matrix S x double dot g that is a matrix and it has cross power spectral density terms.

Once we have these matrix defined and the r the influence coefficient matrix, that is known then one can get the power spectral density function matrix of the displacements or responses S x x this will not be S x double dot g this will be S x. So, x x as H m r S x double dot g then the transpose of this quantities H m r on this side that is r T m T and H becomes now a complex conjugate and transpose of that.

So, that is a formulation that we have proved before and if we were wanting to obtain the cross power spectral density function between the excitation and the response displacement that is S x double dot g x that is equal to minus H m r x double dot g r x double dot g is a matrix of size S by S where is the number of supports and r of course, is a coefficient matrix of the size n into S. Also we discussed in the previous lecture that many a time we are not interested in obtaining all the responses, but we may be interested in only a selected few. In that case the a reduced frequency response function matrix is obtained from the total frequency response matrix of the system and that we denote by H bar and using that H bar one can obtain the power spectral density function of the selected or matrix of the selected responses.

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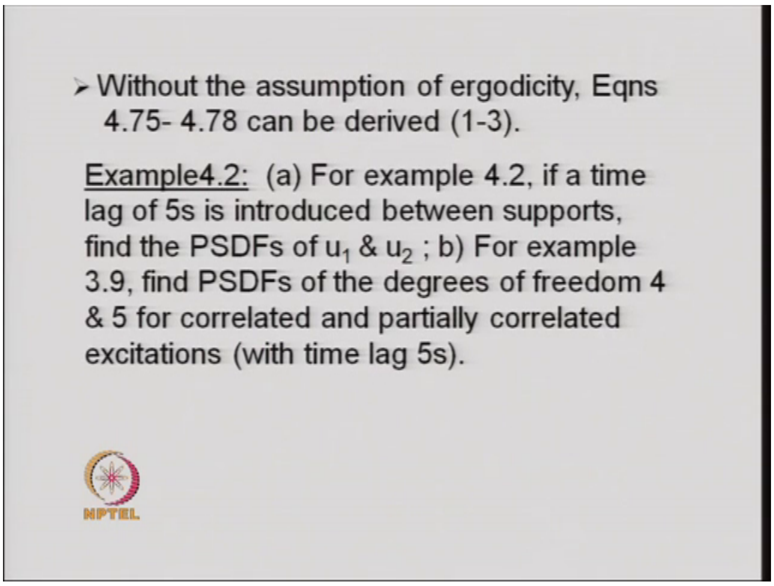

Then we saw as an example we solved a problem.

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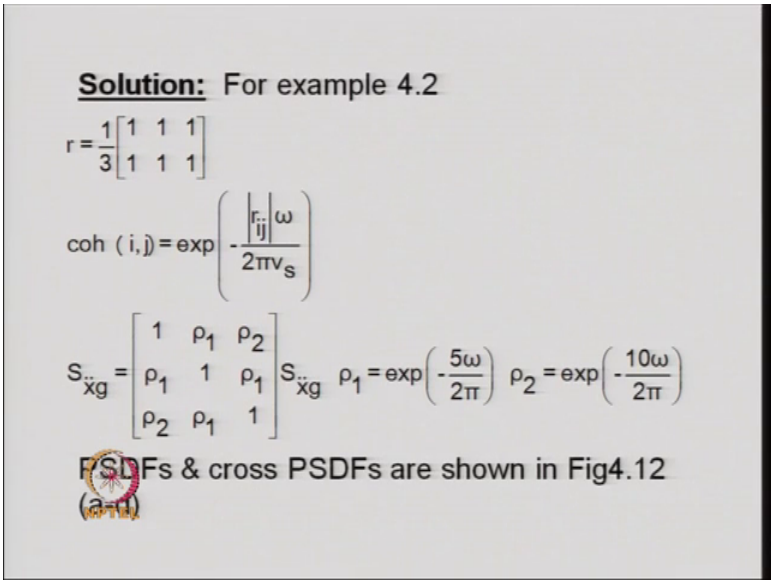

I(n which we had a 3 supports and two nonsupport degrees of freedom u 1 and u 2 and we discussed how we obtain the cross power spectral density function terms of the power spectral density function matrix of the excitation.

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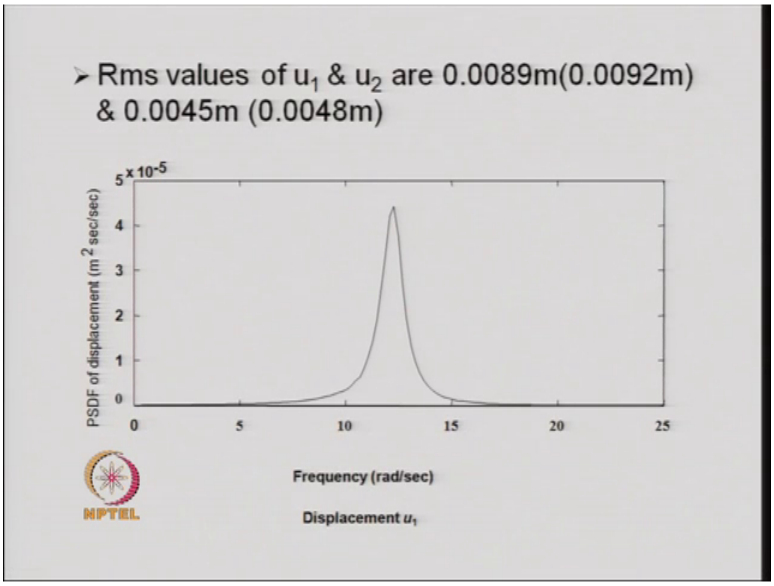

With the help of that we obtain the values of the power spectral density function or power spectral density function of the response u 1 and u 2 and from that the area under the curve provided us the root mean square value of the responses, and these root mean square values of the response were compared with the time domain analysis, note that the excitations where the elcentro earthquake record and the elcentro earthquake power spectral density function, they are the inputs respectively for the two kinds of analysis.

So, you can see that comparison was very good.

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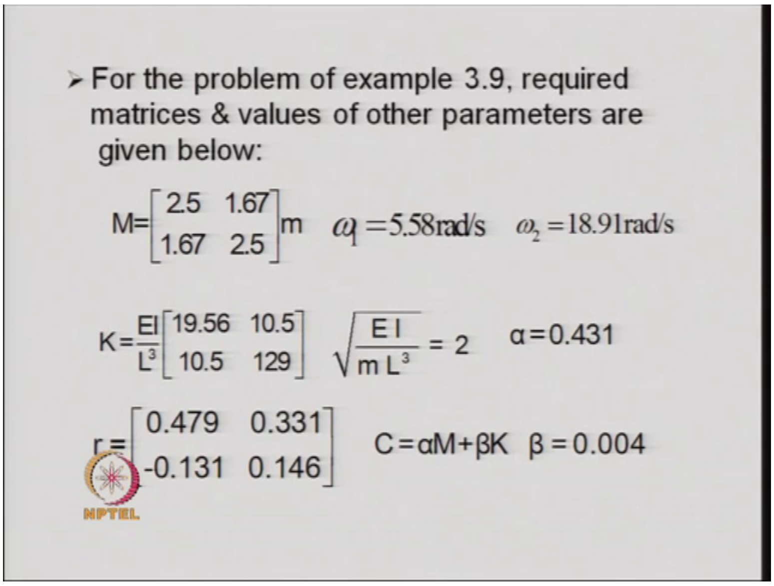

We now solve another problem in this class, the problem of the pitch reportal frame in which we had two degrees of freedom if we recall, one at the as the sway degree of freedom on the left hand side and the other one is a vertical degree of freedom at the crown. So, this is the problem that we had solved before for the ground excitations which had a time lag of 5 second, and for that we derived the r matrix and this was the r matrix and also we derived the mass matrix if we recall the mass matrix for such systems is not a diagonal mass matrix, but they are the couple mass matrix although we are using the lump masses at the different degrees of freedom.

So, the derivation of this also was worked out before this is the K matrix. So, with the help of the K matrix and the m matrix the frequencies two frequencies of the system where obtained they are 5.58 and 8.91 radiant per second and with the help of these two frequencies we obtain the values of alpha and beta to construct the C matrix and the C matrix thus was obtained then.

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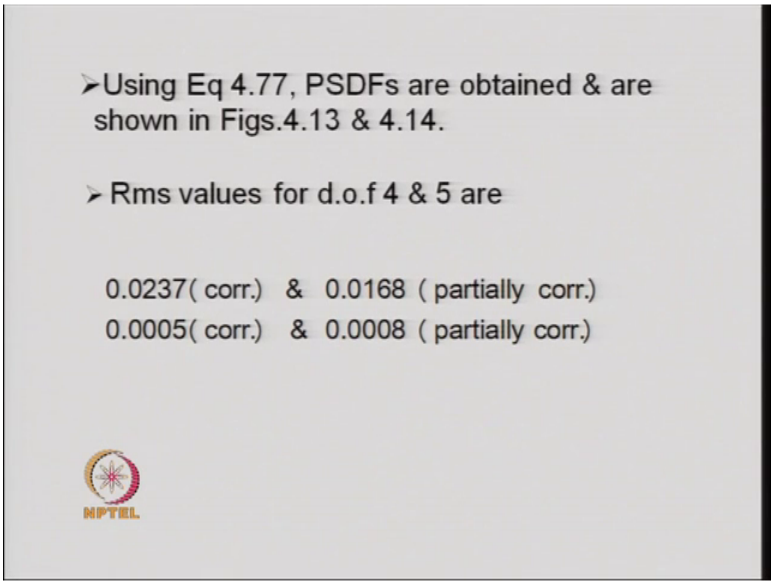

For obtaining the value of the C matrix, after we have obtained the value of the C matrix then we use the usual formulation that is the equation that I had shown before. So, we plugged into this equation and in constructing H matrix, we have k minus m omega square plus I C omega inverse of that that becomes the h matrix.

So, once we know this h matrix then we obtain the values of S x x and the S x double dot g matrix was obtained in this particular fashion and the cross terms where one cross term was exponential of minus 5 omega by 2 pi because between the 2 supports, the first support and second support there was a time like a 5 second and the first support and the third support the time lag was 10 second. So, therefore, they appear like this one row 1 and row 2 in to this matrix and the results where compared and for the two cases that is in one case we had a perfectly correlated excitation that is at the two supports the excitations where same and in another case we obtained the excitation, having a time lag of 5 second.

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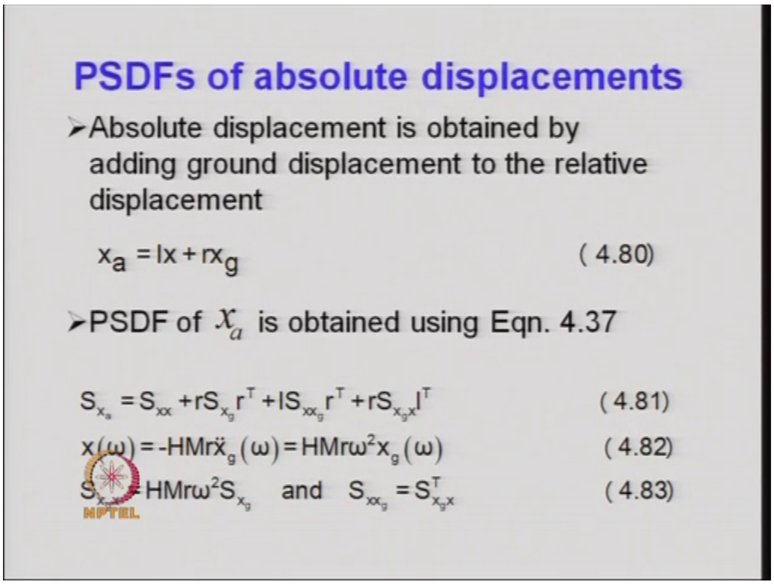

Next we discussed about the power spectral density function of absolute displacements and in order to obtain these power spectral density function of the absolute displacement what we have to do is that, we have to add to the relative displacement x to the displacement or quasi static displacement that is introduced at the degrees of freedom because of the ground displacements at different supports. So, in order to get that we multiply the x g vector that is the excitation vector at the different supports with the r matrix r matrix, and that gives us the contribution of the different displacements of the excitations to the nonsupport degrees of freedom and the relative displacement was x that is the with respect to the base that displacements of the different nonsupport degrees of freedom and that is multiplied by or pre multiplied by a identity matrix i.

So, this becomes the addition of these two becomes the absolute value of the total value of the displacement at different degrees of freedom and once this relationship is obtained then using this relationship one can write down the value of the power spectral density function matrix like this, that is S x x is for this term because I S x x I t will be S x x itself, then r S x g r T where S x g is the power spectral density function matrix of the ground displacement and then I is S x x g and S s x g and x.

So, they are the cross power spectral density function matrix between the displacement and the ground excitation or ground displacement and this is the other part of the cross power spectral density function matrix which generally is a complex conjugate of this and transpose that you have seen before. So, once we have this expression in this expression this is known this we have already obtained this is also given, because if the power spectral density function on matrix of the ground acceleration is given to us then from that one can construct the power spectral density function matrix of the ground displacement. And in order to get the value of S x g and x g x we use this equations that is first we obtain the value of the cross power spectral density function between the excitation and the displacement using this standard expression, in which it is S x double dot g and S x double dot g is replaced with the help of the S x g by multiplying it with the help of omega square.

So, this and the complex conjugate of this S x g is a complex conjugate of S x g x. If this matrix is a complex matrix then there will be a star over here star and transpose and if it is a real quantity then the star does not come into picture it is simply a transpose of that.

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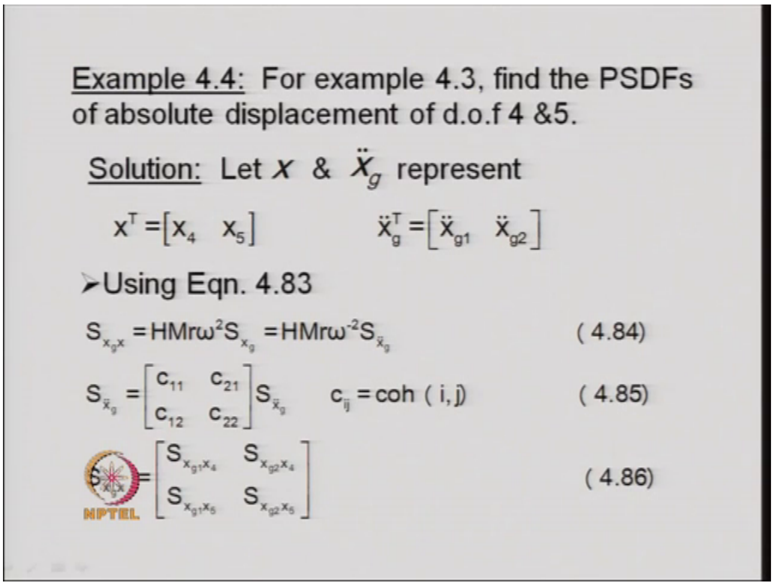

Now with the help of this we wanted to obtain the response of the pitch reportal frame and for that we are interested in finding out the response of the system not only for the not for the relative displacement, but for the total displacement.

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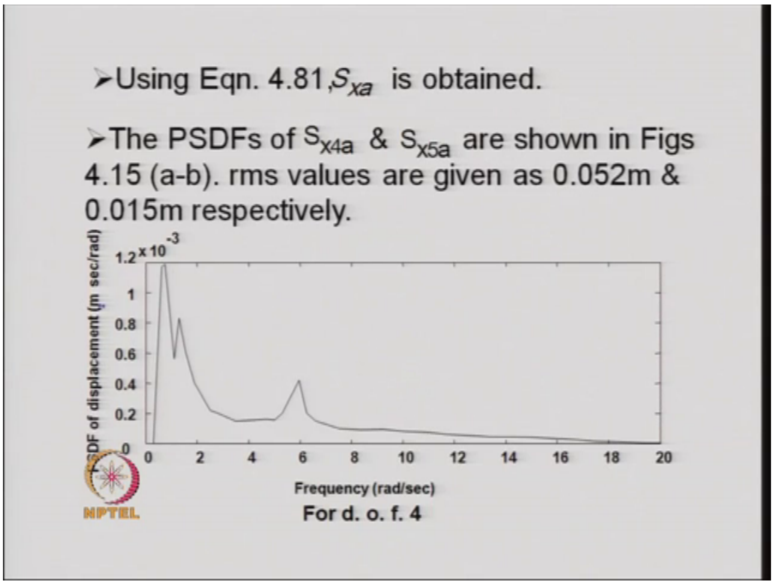

So, the for obtaining the total displacement we have to know the first S x g x, S x matrix. So, that is given in this particular form and the since S x g and S x double dot g they are related by a relationship that S x double dot g is equal to omega to the power 4 into S x double dot g that we have seen before. Then if you substitute that relationship over here then this become HM r omega minus to the power two into S x double dot g not S x g and this x x double dot g is known to us and S x double dot g can be constructed from the time lag that is given to us that is a 5 second time lag.

So, these S x double dot g matrix would be equal to this C 11 multiplied by S x double dot g mind you this S double dot g is a single quantity that is the power spectral density function of the earthquake that is the elcentro earthquake that we have consider and these when it gets multiplied by C 11 then it gives the power spectral density function matrix between at point 1. And at for point 2 the excitation becomes C t 2 and into x double dot g and; obviously, this C 11 and C 22 these values are equal to unity and C 12 and C 21 they are the cross power spectral density function on terms and we had seen before that for the same earthquake producing different kinds of excitations at different supports.

They are the cross power spectral density function between the excitations is equal to the power spectral density function of the earthquake that is a single quantity multiplied by the coherence function Cij and square n and its function is given or defined in this particular fashion. And we use some empirical relationship for obtaining the C 12 or C 21 etcetera. S x g x matrix will look like this that is the cross power spectral density function matrix between the ground displacement and the on nonsupport degrees of freedom displacement x will be like this, that is S x is g 1that is for the first support and this is x 4 is the displacement square displacement of the pitch reportal frame.

So, S x g 1 S x 4 represents the cross power spectral density function, between the excitation one support to the displacement x 4 similarly x g 1, x g 5 that indicates the cross power spectral density function between the excitation at support one with the displacement x 5 and in that fashion one can also describe the cross power spectral density function between the support 2 and the displacements 4 and 5.

So, that gives us the x g x matrix; however, this x g s matrix can be computed easily with the help of the above relationship.

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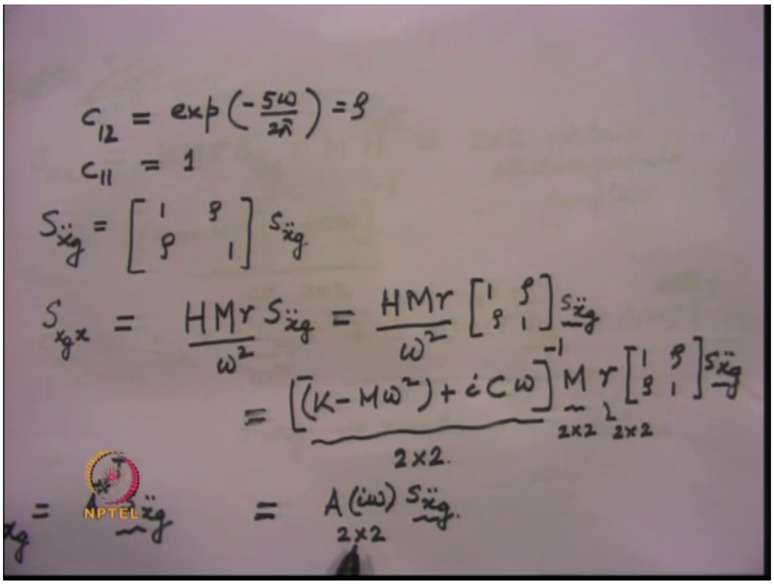

That is the relationship that we have used and once we take that relationship then what first we have to obtain is the cross power spectral density function matrix of excitation S x double dot g is the elcentro power spectral density function, and the C11 C 22 C 12 terms they will be this will be 1 1; obviously, and this will be the row a row will be equal to exponential of minus 5 omega by 2 pi is equal to row.

So, one can work out these value of C 12 for every value of omega and so, this matrix is completely defined and then we write down the cross power spectral density function matrix between the ground excitations and the displacement x and that is written as HM r x S x double dot g divided by omega square, that becomes the relationship for S x double dot g and this because S x double dot g over here is equal to omega to the power 4 into S x g.

So, using that relationship we get omega square below over here that is how the omega square at the denominator has come then HM r divided by omega square this we take out and then S x double dot g matrix that is written over here. The H matrix is nothing but K minus M omega square plus i C omega inverse of this. So, you substitute for H then this gets multiplied by M and then multiplied by r and then this matrix and this is a single quantity S x double dot g that is the power spectral density function of the earthquake that is the elcentro earthquake and all f them are 2 by 2 matrix therefore, this multiplication of the matrices for a particular value of omega leads to a 2 by 2 matrix of this form A i omega into S x double dot g.

Since there is a this term will be a complex term. So, this matrix is expected to be a complex matrix of size 2 by 2 and that multiplied by S x double dot g, and once we know the value of this matrix S x g x then one can obtain the S x x g as this is S x x g is equal to S star T, x double dot g see that here since it is a complex matrix we take a complex conjugate of that and then transpose in order to get S x x g from x g x x..

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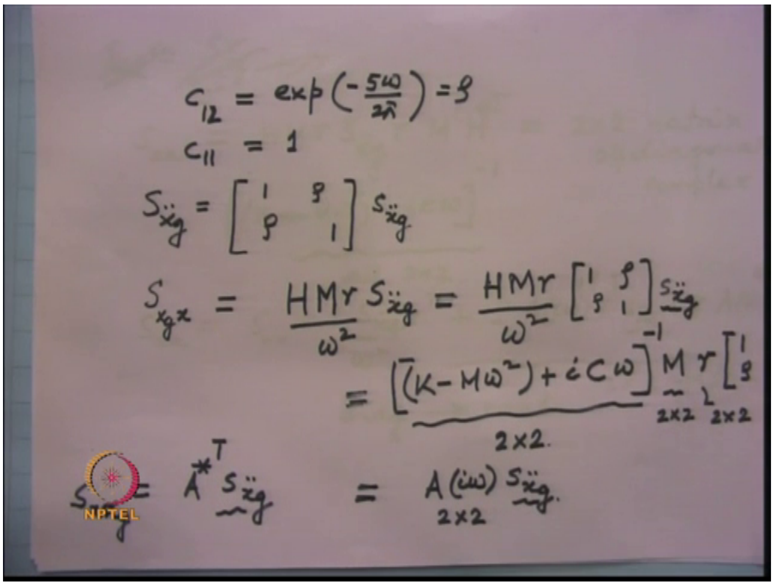

So, once these two quantities are known to us or this two matrices are known to us one can and substitute them into this particular equation, equation 4.81 and in equation 4.81 we know S x x then r x g that is also S x g is known. So, this r is also known. So, therefore, the this particular term is also known and we have obtained S x x g and x g x j and therefore, the all the terms in this equation are known.

So, one can get the power spectral density function matrix of the absolute displacement of the pitch reportal frame for degrees of freedom 4 and 5, 4 is the sway degree of freedom and 5 is the degree of vertical degree of freedom at the crown. So, that is how these power spectral density function earns of the degree of freedom 4 and degree of freedom 5 a represents the absolute value. So, the power spectral density functions of these two quantities were obtained and plotted against frequency that is for degree of freedom 4 and this is for.

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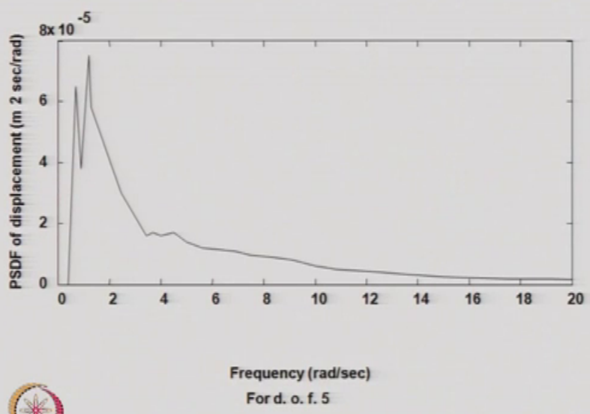

Fig 4.15

Degree of freedom 5 and one can see that the for degree of freedom 4 the value is more compare to the value of the power spectral density function of the degree of freedom 5, because this is of the order of ten to the power minus 5 the other one is of the order of 10 to the power minus 3. And the area under the curve of course, provides us the root mean square value of the absolute displacement of the degree of freedom 4 and degree of freedom 5.

So, that is how one can obtain the power spectral density functions of the absolute displacements if it is required.

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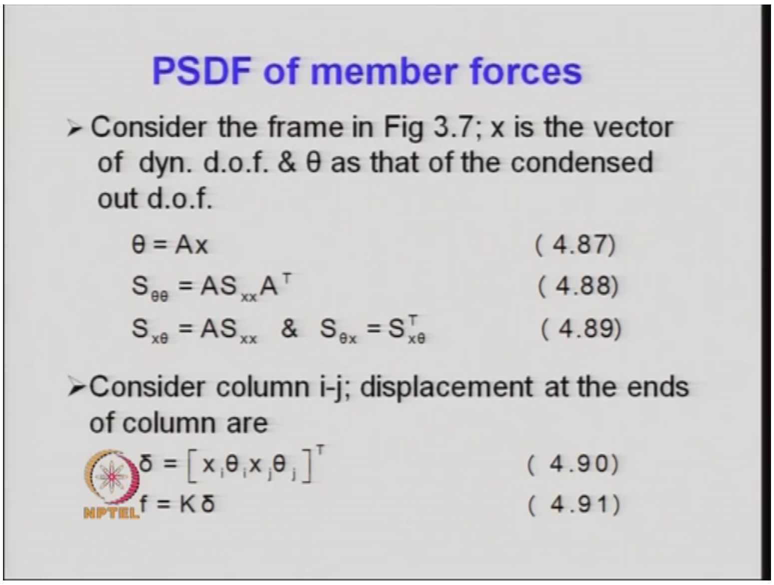

Next you come to the power spectral density function of the member forces, once we obtain the power spectral density function of the displacement that is the displacement of the various translational degrees of freedom, then from that one has to obtain the power spectral density functions of the member end forces and for that we take say this particular frame and here.

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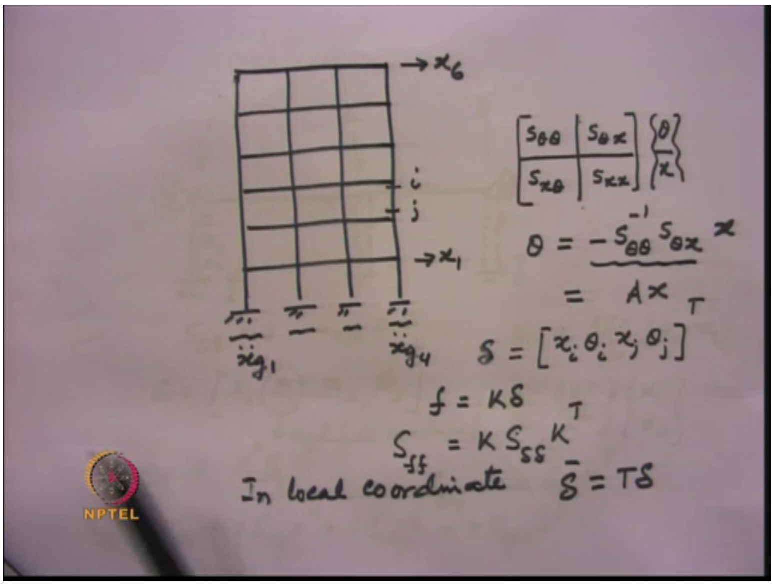

Say this is the frame in this frame we have say 1, 2, 3, 4, 4 supports and at these 4 supports we have got 4 different excitations because of the time lag between them and these are nonsupport degrees of freedom that is x 1 to x 6 they are nonsupport degrees of freedom and say we are interested in finding out the power spectral density function of the bending moment at i and j. So, that is of our interest.

So, then what we do in the beginning that we write down the matrix, the stiffness matrix of the entire structure with these rotations and once we know the rotations then from there one can get the relationship between the rotation and the displacement and this matrix is called A matrix A. Say once we know the relationship between the theta matrix theta vector and the x vector through matrix A then one can obtain the power spectral density function of the rotations in terms of the power spectral density function of the displacements using this equation and the cross power spectral density function between the displacement and the rotation is given by this A S x x and the S theta x will be simply the transpose of this.

Now, if x theta x is a complex matrix then of course, there will be a this will be a complex conjugate and then transpose. Now once we have obtained this S theta theta that is the power spectral density function matrix of rotation and the power spectral density function matrix of displacement and the cross power spectral density function matrix between rotation and displacement. So, once they are known then from that from those matrices one can select the specific terms of the power spectral density function to construct the power spectral density function matrix, which will be used in finding out the power spectral density function of member end forces.

Now, for example, if I take this column i j that I had shown in the figure then the values which are required to find out the bending moment at the ends i and j are x i theta i x j and theta j. So, these are the particular displacements and rotations which are required. So, we take out that and we write down the relationship between the member end forces and the member end displacement with the help of the usual equation f is equal to K into delta.

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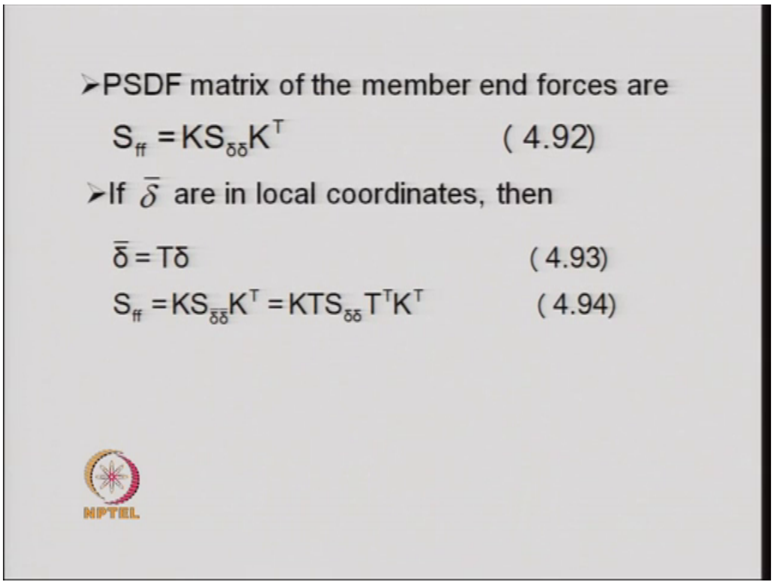

Once we are able to write that then S f f that is the power spectral density function matrix of the member end forces becomes equal to K into S delta delta K T, this K is known that is the member stiffness matrix and S delta delta is the or delta delta they are the member end displacement.

So, by knowing S delta delta once can easily find the power spectral density function of the member end forces. In many cases we want the member end forces in the local coordinate that is the member end coordinates. Then of course, there at a further transformation is required; that means, if the delta bar is the displacements in the local coordinate and if delta are the displacement in the global coordinate there you related through this relationship that is delta 1 is equal to T into delta where T is the transformation matrix and substituting for the delta bar that is one can obtain the power spectral density function matrix of the member end forces in local coordinate, to be equal to K into S delta bar delta bar and S delta bar delta bar is further converted to S delta delta by including the transformation matrix T. So, it is possible to obtain the power spectral density function of the member end forces both in the global and the local coordinates that is they are power spectral density function of individual quantities may be bending moment share force etcetera.

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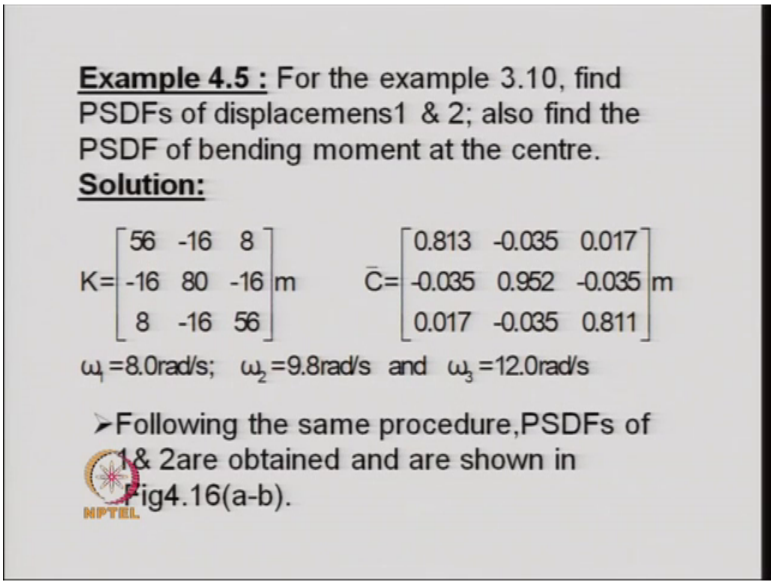

Now, these in this is example these that displacement.

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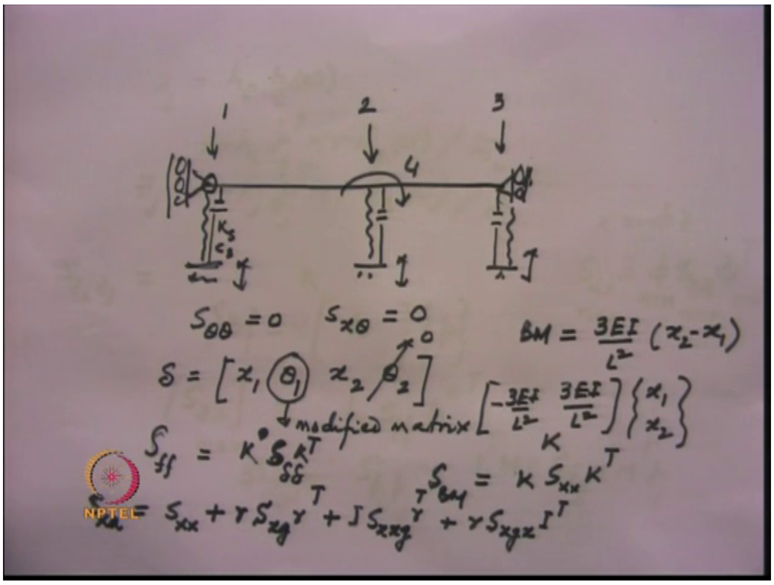

So, one and two that is the if you recall that is the example problem for the yes I think yeah, this is the problem if we recall we had solved for the for finding out the responses in time domain, where we specified the ground motions at this 3 supports and they are the same ground motion and this was a modal of a pipeline buried pipeline. And these supports there were 3 supports were taken, the soil damping and stiffness matrix or the damping and the stiffness are replaced by spring and dashpots and these values were given to us and the degrees of freedoms where 1 2 3 they are the translational degrees of freedom and a rotation here.

So, one can easily condense out this rotational degree of freedom and once you condense out this rotational degree of freedom, then with the help of the translational or matrix stiffness matrix corresponding to translation degrees of freedom 1 2 3 one can obtain the responses for the degree of freedom 123.

Now, the stiffness matrix for the system was this 3 by 3 stiffness matrix, we obtained a damping matrix using C bar is equal to alpha m plus beta k, and the 3 frequencies that that are obtained from this k matrix and m matrix so, that where 8 9.8 and 12 radiant per second.

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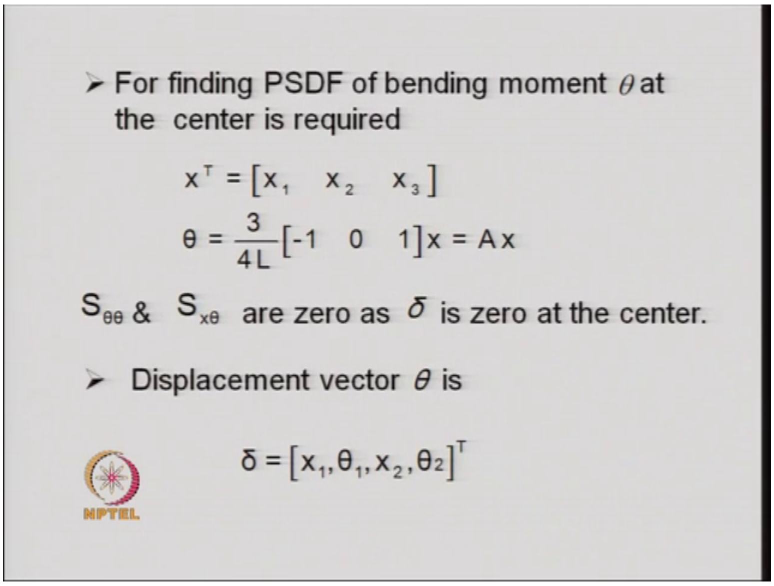

And then we obtained the displacement power spectral density function for the 3 displacement and using the same procedure that we have discussed before, but now we are interested in finding out the power spectral density function of the bending moment at the center.

Now, for that we recognize the fact that at the center if all the 3 degrees of or at all the 3 supports the ground excitations are the same then the we expect that there should be not be any rotation over here and in fact, it can be easily shown with the help of the condensational matrix. So, if there is no rotation then S theta theta will be 0 and x S x theta also will be equal to 0, that is the power spectral density function of the rotation and the cross power spectral density function of the rotation and displacement they will be equal to 0.

Now, if we wish to find out say bending moment here, then the displacement that we should take this will be the displacement 1 and then if there is a rotation over here say theta 1 this is the theta 1 and then the displacement here x 2 and then the rotation here theta 2. So, if you know them then with the help of that one can obtain the power spectral density function matrix of the bending moment at this point and at this point.

Now, since at this point the bending moment is equal to 0. So, we need not include this theta 1 into this vector and we can modify our stiffness coefficient for bending moment for this member accordingly. So, that these theta 1 need not be considered and since theta 2 is 0 then we get only x 1 and x 2 these are the two displacements if we know this two displacements only one can find out the bending moment at this point, and that is that bending moment can be written by simply by this equation 3 E I by L square into x 2 minus x 1 in the matrix form these becomes minus 3 E I by L square 3 E I by L square into x 1 and x 2 where x 1 and x 2 are the displacements at this two points and this can be further written in this particular form that is the S bending moment, that is the power spectral density function of the bending moment is equal to K S x x K T.

So, if we know S x x once can find out the bending moment at this particular point. Now the important part of this analysis is that if the ground motions are not the same at different points then the relative displacements at this point an in this point and from that one cannot obtain the bending moment at this particular point, that is at the center that the way that we have obtained the bending moment.

In that case what we have to do we have to find out the absolute displacement or the absolute displacement power spectral density function of point 2 and the power spectral density function of the absolute displacement at 1 that must be known and the cross power spectral density function must be known or in other words we must have a power spectral density function matrix of the absolute displacements at 123 degrees of freedom. And for this particular problem we need not bother about the rotation here because the rotation becomes 0 if the degrees of freedom are or the excitations are same at these 3 points, but if they are not same and we are wanting to find out the bending moment at this particular point then we wish to have the power spectral density function matrix of the absolute displacements at 1 2 3 and a power spectral density function of the rotation.

However the power spectral density function of the rotation at this point will not be effected by the ground displacements, that is taking place at this point at different points with a different displacements will not be effected by that this theta can be obtained S theta can be obtained from the relative displacement itself.

So, therefore, what we do is that in obtaining the cross power spectral density function matrix of these absolute displacements, we go back to the previous problem that we have solved in which we know we should know the power spectral density function on matrix of the relative displacement then power spectral density function matrix of the ground displacement then S x x g and S g x x that we had solved in the previous problem. And with the help of that we obtain the power spectral density function matrix of the absolute displacements and then we obtain the bending moment that is S bending moment here in place of S x S here, this S x x is the power spectral density function matrix of related displacement that would be replaced by the power spectral density function of the absolute displacements.

So, the problem in which we have got the same excitations at a different supports, the we need we do not have to find out this absolute displacement power spectral density function matrix, we can only obtain the power spectral density function matrix of the relative displacement. However, the problem becomes lightly complex, if we have a power spectral density of the ground excitations at different supports are different and for that remember that one has to find out the power spectral density function matrix of the absolute displacements or total displacements of different degrees of freedom. And also we have to find out the condensed out a power spectral density function matrix of the condense out degrees of freedom that that is like rotations theta theta theta theta and the cross power spectral density function between theta and x.

So, all those things must be known then only one can obtain the power spectral density function matrix of the member end forces. Thus we see that to find out the power spectral density of function of the member forces for excitations which are same as different supports the formulation is simpler, but if you are wanting to find out the power spectral density function of member end forces for a multi support excitation problem, then one has to obtain the power spectral density function on matrix of the total displacement and then from there we have to take out the relevant terms to form the power spectral density function matrix S delta delta and which can be multiplied with the stiffness matrix k and k t to obtain the power spectral density function of the member end forces.

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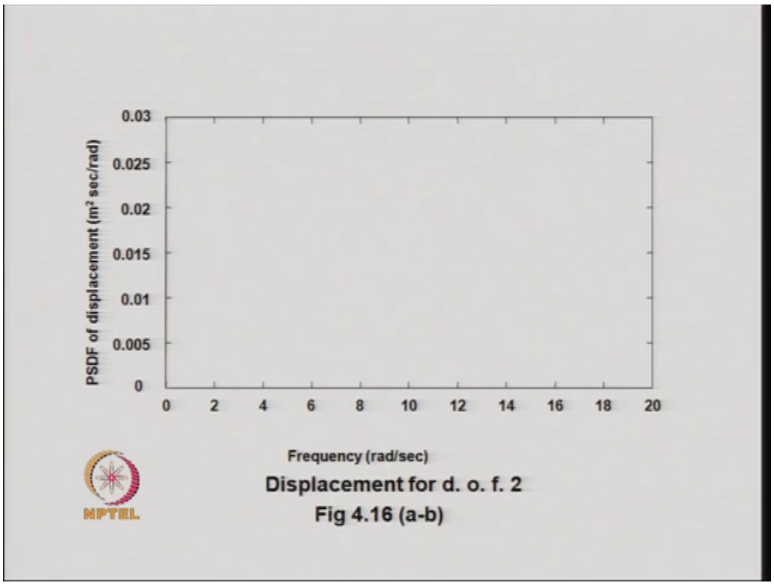

So, the these problems the power spectral density function of the absolute displacements where and the bending moment where plotted in this figures.

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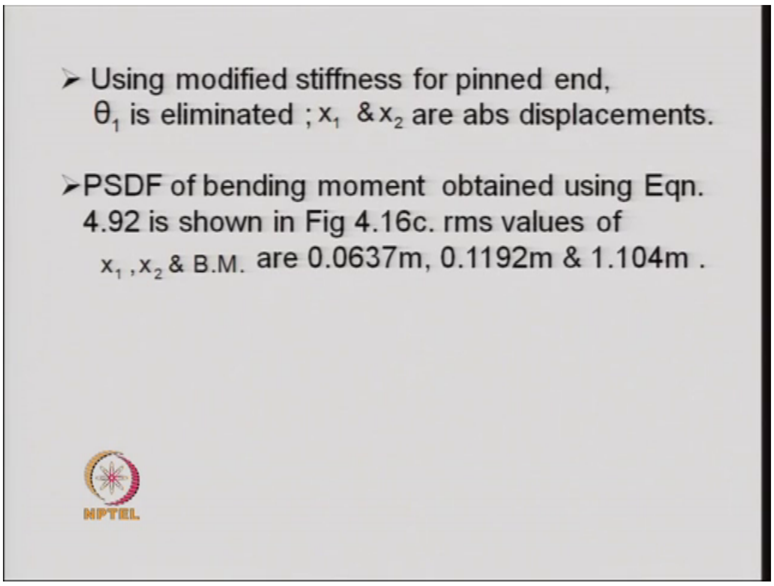

And the displacements and bend r m s values of the displacement and bending moments where obtained for different quantities.

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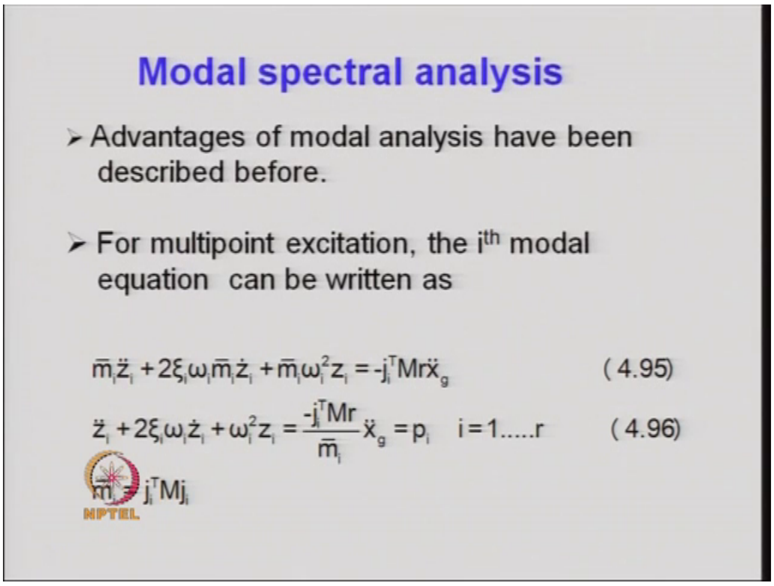

Next we come to what is called the modal spectral analysis, now since we use the mode super position technique you uncouple the different equations.

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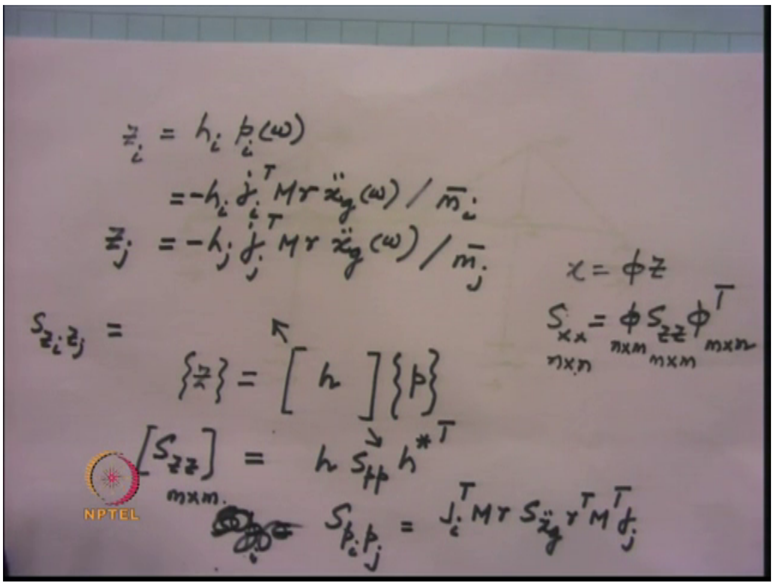

Of motion of a multi degree freedom system and can simplify the problem that is a multi degree freedom problem can be converted to a set of single degree freedom problems and for each single degree freedom problem one can obtain the responses in the generalized coordinate and from there one can obtain the displacement in the structural coordinate using the mode shapes.

So, ex extending that concept one can develop also a modal spectral analysis, the name modal spectral analysis comes because we are using the mode shapes of the structures for developing this method and the that is why it is known as the modal spectral analysis and it has the same advantages and that we observed for the case of usual time domain analysis and the frequency domain analysis.

So, for multi support excitation we have these standard expression that we have discussed in connection of with the mode super position technique. So, the ith generalized displacement velocity and accelerations are z z i z dot i and z double dot i and m bar iis the ith generalized mass and j i T is the transpose of the first mode shape r is the influence coefficient matrix associated with x double dot g considering x double dot g to be a multi support excitation, then one can convert this to this form by dividing by m bar i and these entire thing is called p i that is the generalized force at the for the ith note.

So, or in other words this the ith modal equation and in that m bar i is defined in this particular way that is already known to you now.

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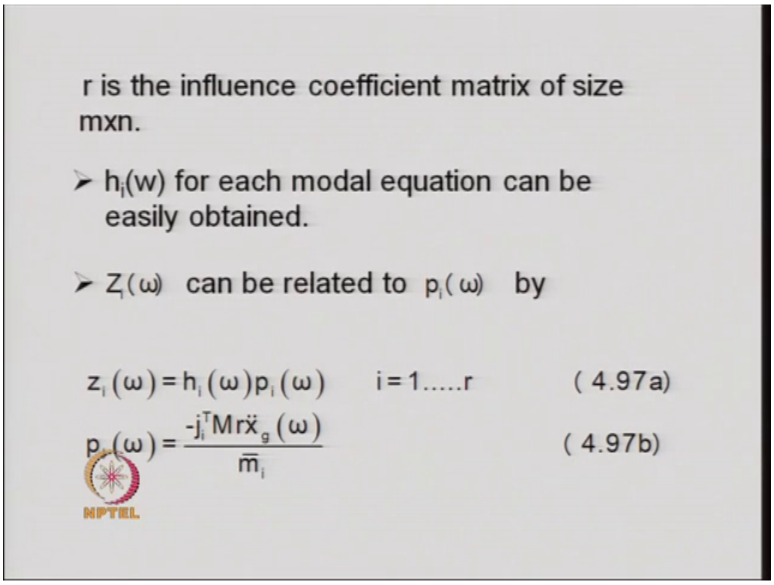

Once we have a single degree freedom equation for a mode i then one can write down z i omega to be is equal to h i omega into p i omega and once we have this then we can easily find out S z i z i very easily, that is the power spectral density function of the generalized displacement z that will be is equal to h i absolute square multiplied S p i omega and one has to simply find out what is the S p i omega ok.

Now, these S p i omega since it has a relationship like this, then one can obtain using the formulation that is a y vector or y is equal to a multiplied by x vector x x double dot g for that using that relationship we can find out S y that is the S p over here, that is the power spectral density function of the force S p that will be equal to a into S x double dot g into a t where a will be equal to these entire this j i T m r divided by m bar that will be the matrix a. So, that one can easily obtain the value of S p and once you obtain the value of S p then one can get the power spectral density function for z. Generally we rather than obtaining the power spectral density function matrix of a single z or that is for a particular value of mode we wish to find out the entire z matrix that is S z z that is the power spectral density function matrix of all the zs and say that matrix size will be equal to m by m where m is the number of modes that is considered.

So, in that case we can write down this basic relationship, that is z vector will be equal to a h matrix small h matrix which will be a diagonal matrix, because all the values of z they are uncoupled. So, therefore, with this h matrix will be a diagonal matrix, p is a modal load vector having the size of the same number of modes m and once we have that then we can write down S z z to be is equal to h into S p p into h star T and this is this is known these and these quantities are known only one has to find out what is the S p p matrix.

Now, S p p matrix any term of the S p p matrix can be obtained from this relationship that is S p i p j that is equal to j i T m r x double dot g into r T M T and j i oh sorry j j that is between i and j the cross power spectral density function of force that is the modal force p and modal force p j the cross power spectral density function between them. So, that can be given by this equation and only thing is that you note is that this is the i and this is the j.

So, one should only take note of these i and j over here that is this will be the ith mode shape and this will be jth mode shape and rest of m r M and r etcetera they are known.

So, one can obtain the value of S p i p j mind you S x double dot g here is again a matrix of the power spectral density function of ground acceleration and which requires again the requires a determination of the cross power spectral density function terms, which comes from the coherence function.

So, once we get S p i p j where terms or every term of the S p matrix then S z z is known and once S z z is known then we can go back to our modal equation that is x is equal to 5 into 0 or more super position equation and if this holds good then S x x will be equal to 5 into S z z 5 t and the say if we take only the f m mode m number of modes S z z will be m by m and there will be n by m and m by n respectively and we will get a power spectral density function of the displacements in the structural coordinate and we will get this matrix size will be n by n.

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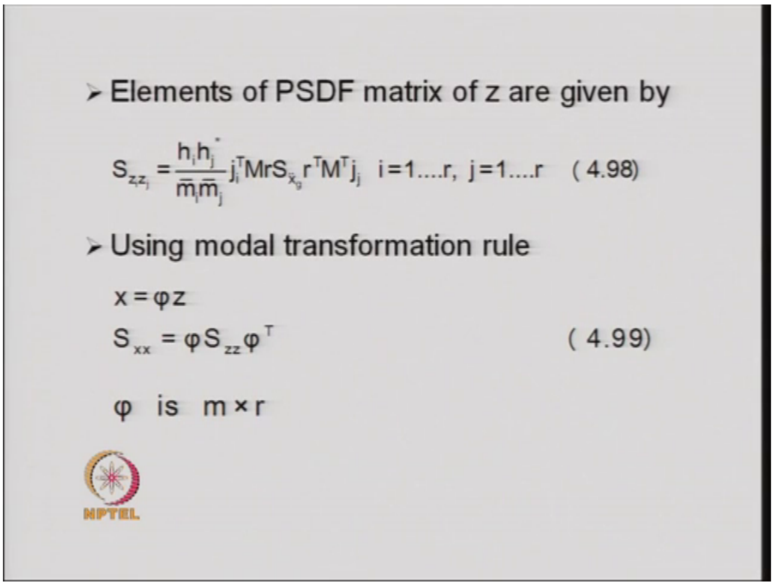

So, we can see that in the case of the modal spectral analysis, what we do is that we have the same formulation that we have done for the mode super position technique for solving the problem in the frequency domain and time domain, and extend it to obtain the power spectral density function of the generalized coordinate by using small h that is the frequency response function of a single degree freedom system. And we can obtain term by term that is we can obtain the S z for or each mode or one can obtain the power spectral density function matrix of S z z and the that would be given by this mind you this S z z is not a diagonal matrix it will have cross terms. So, this cross terms denote the correlation between the power spectral density functions of z 1 and z i and z j. So, I stop at this.

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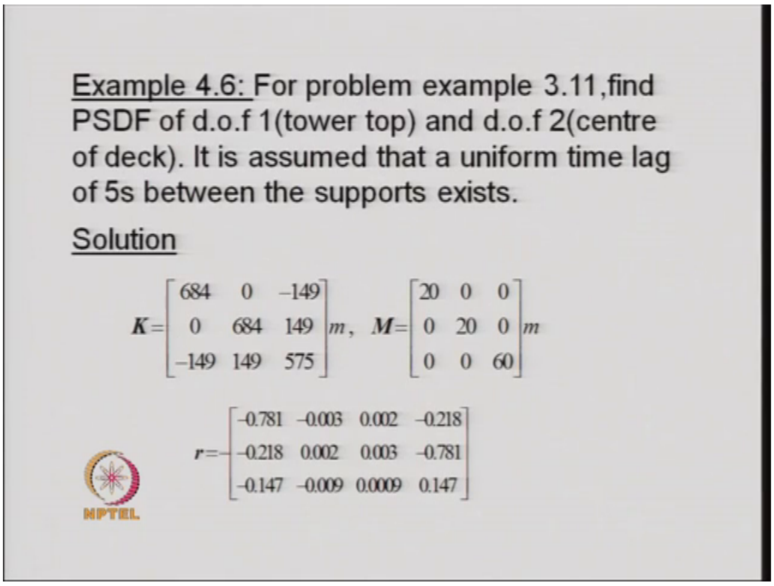

Today and we will solve a problem that is the problem that we solved for a cable stayed bridge and that we will take an example to illustrate the use of the modal spectral analysis.