**Seismic Analysis of Structures**
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# Lecture - 20
# Response Spectrum Method of Analysis

In the last few lecture, we discussed about the Response of Structures for the Random excitation using the spectral analysis technique. And we had seen there that the input for the random vibration analysis of structures using spectral analysis technique is the power spectral density function of excitation and the cross power spectral density function between two excitations.

If you recall, we discussed in seismic input that there are many types of seismic inputs for analyzing structures. Time history records are one with which we can obtain the response for deterministic analysis of the structures for earthquake; that is we assume that the given earthquake is a deterministic quantity. And we analyze this structure, and this analysis is called the deterministic analysis of the structures for earthquake.

Then comes the analysis of the structures for the random vibration, where we use the power spectral density function as input. For the deterministic analysis not only the time history of the earthquake is an input, it could be also the Fourier spectrum of the earthquake record. That also can be used as an input for analyzing the structures.

Next type of input which we discussed at length is the response spectrum of earthquake. And we had seen that how a design response spectrum for earthquake is obtained, and the different kinds of response spectrums that are available in the literature. Out of these different kinds of response spectrum the response spectrum of acceleration, better it is called response spectrum of pseudo acceleration. That has been adopted as an input for the response spectrum method of analysis for structures.

In any earthquake code prescribing the code provisions for the analysis and design of structures for earthquake a response spectrum; that is the pseudo acceleration response spectrum is specified. And, generally most of structures are analyzed using that response spectrum which is specified in the code of practice. The response spectrum method of analysis has become a very very popular method of analysis for earthquake engineers for quite a few reasons.

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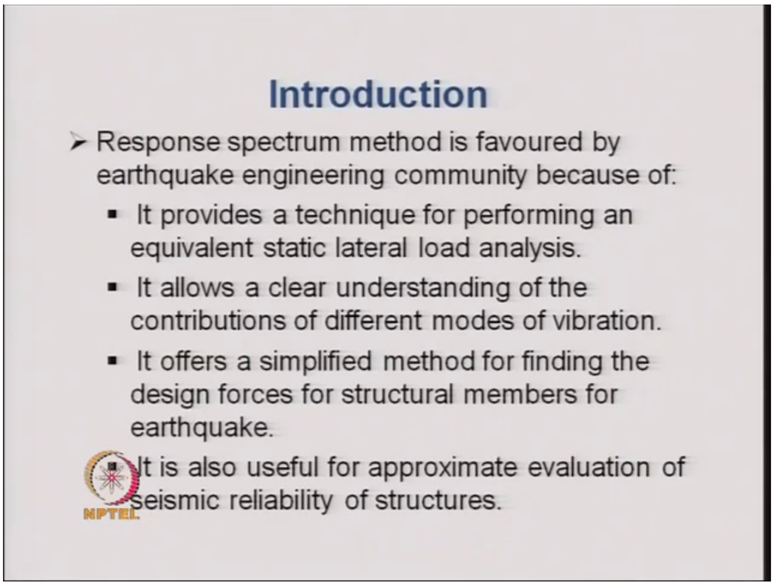


For example: the response spectrum method provides a technique for performing an equivalence static lateral load analysis. Now that is the most attractive part of this analysis, because it is very suitable for the designers. They do not want to perform a dynamic analysis at the design stage.

In the design stage it is better to have a equivalent static analysis so that the earthquake forces can be combined with other static loads or the response is produced due to other static loads. And a combined critical member force can be obtained for which the member can be designed. So, these equivalence static lateral load analysis which is performed for in the response spectrum method of analysis has become very attractive to earthquake engineers.

Secondly, it allows a clear understanding of the contributions of different modes of vibration. So that is another thing which a designer would like to understand, that is as the structure vibrates in different modes under an irregular time history of excitation. Then obviously, one would like to understand; what are the different contributions of the different modes of vibration in the structure. And once they understand that then one can consider how many modes they would take into account for finding out a good estimate of the response of the structure for which they would perform the design.

Further, this also gives a physical understanding of the behavior of the structure when it is undergoing a vibration under earthquake. Then it offers a simplified method for finding the design forces for structural members for earthquake. This is a so because the forces are computed using a static load analysis. Therefore, the method has is a simplified method and that a designer could consider in obtaining the design of this structures.

The dynamic analysis part of this structure obviously is more complicated compare to a static analysis. And therefore when this entire concept of response spectrum method of analysis was developed to find a simplified equivalent static load, lateral load analysis; then the method really become a very popular method. It is also useful for approximate evaluation of seismic reliability analysis. Many a time one has to obtain the seismic reliability of structures. And in place of doing a rather complex dynamic analysis one can perform a simplified equivalence static lateral load analysis for structures. And using the results of that one can also obtain a seismic reliability, analysis of structures.

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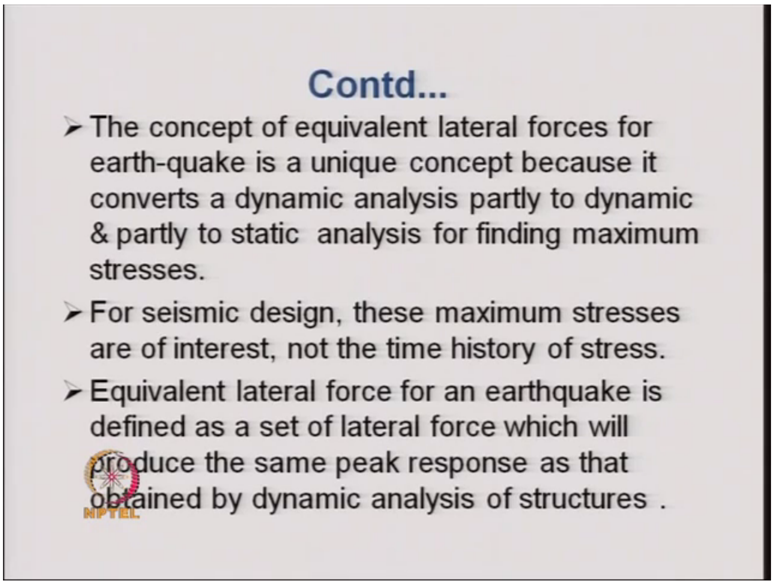


The concept of equivalent lateral force for earth quake is a unique concept, because it converts a dynamic analysis partly to dynamic and partly to static analysis for finding maximum stresses. It is partly dynamic and partly static because of the reason that: in the first part of the analysis one has to find out the mode shapes and frequencies of the structures. And in finding out the mode shapes and frequencies of the structure one has to perform a dynamic analysis of the system or rather you can say one has to solve a Eigen value problem.

Once the mode shapes and frequencies of the structures are known then rest of the analysis is a static analysis. And therefore it is called partly a dynamic analysis, partly a static analysis. The most important thing in the seismic design is the maximum stresses that is developed in the members so that the members can be designed for those forces. The designers really are not interested in the time history of the stresses, because the maximum stress that is of direct interest to them. And this method provides directly the maximum stresses which are induced in different members due to earth quake.

Equivalent lateral force for an earth quake is defined as a set of lateral force which will produce the same peak response as that obtained by dynamic analysis of structures. This is the most attractive part of the equivalent lateral load analysis of for earthquake, because we construct a set of equivalent static lateral load in each mode of vibration. And this set of lateral load when is applied to the structure it provides a response of this structure that response happens to be equal to the maximum dynamic response that the structure would have undergone if it where vibrating only in that mode.

So, if it is assumed that the structure is vibrating only in one mode then this equivalent lateral force can straight away provide us the maximum response the structure would undergo in that particular mode of vibration.

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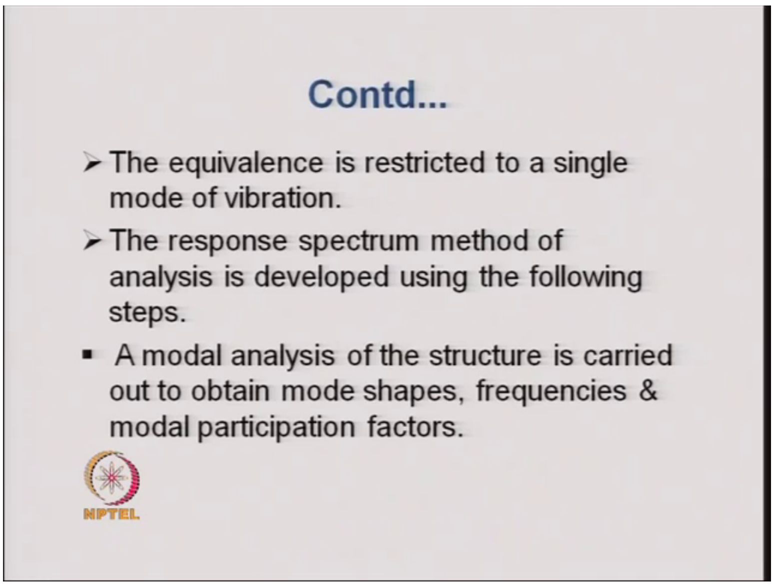

Obviously, the equivalence is restricted to a single mode of vibration. The response spectrum method of analysis is developed using the following steps. First a modal analysis of the structure is carried to obtain mode shapes, frequencies and modal participation factor. All of you have done this exercise when you have undergone a course in dynamic analysis of structures.

The mode shapes and frequencies they are very routinely obtained for any given structures provided the mass matrix and the stiffness matrix corresponding to the dynamic degrees of freedom are known. And once we have the mode shapes and frequencies then one can find out the mode participation factor for each mode that we will see little later.

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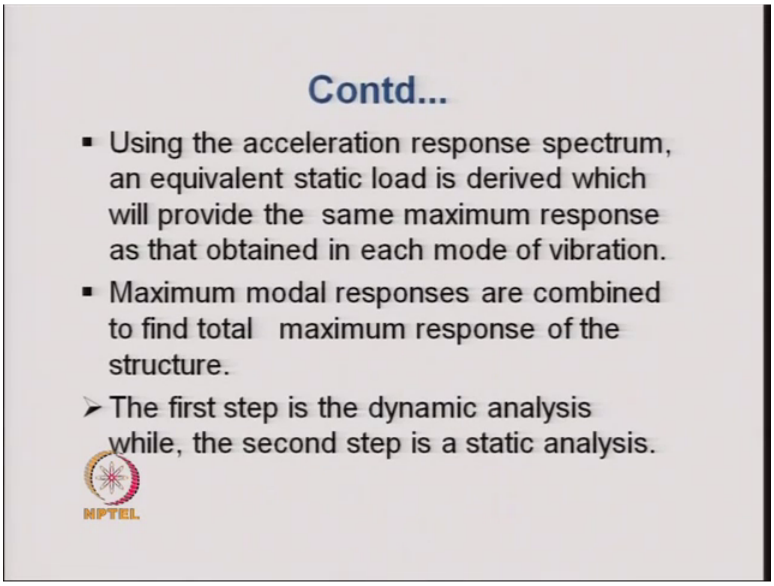


Then using the acceleration response spectrum which is given in a code, an equivalent static load is derived which will provide the same maximum response as that obtained in each mode of vibration; that I have explained earlier. The maximum modal responses are combined then to find total maximum response of the structure. The first step, that is the finding of the mode shapes and frequencies of the structure that is a dynamic analysis while, the second step is a static analysis because we obtain an equivalent static load and carry out a static analysis.

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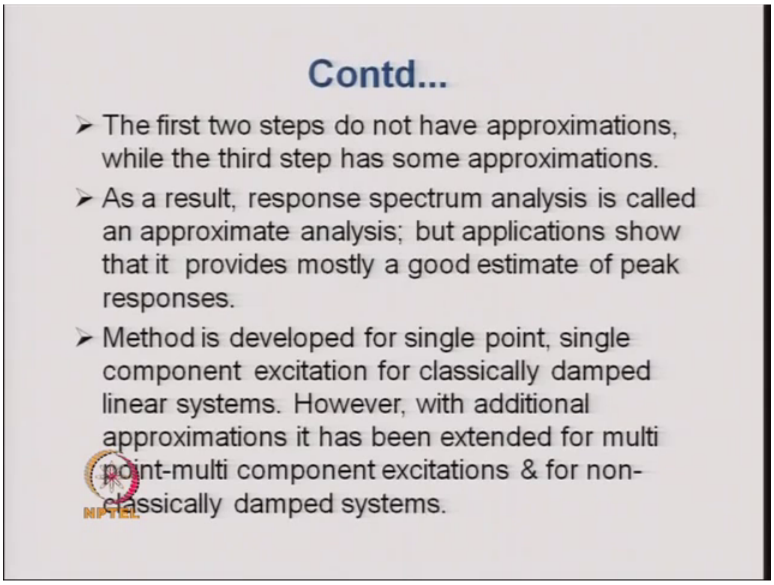


The first two steps do not have any approximation. That is we do not compromise any accuracy in this first two steps and they can be derived from the first principles. While the third step that is when we are combining the maximum responses in different modes of vibration, then we make some kind of approximations. As a result of that the response spectrum analysis is called an approximate analysis, but applications show that it provides mostly a good estimate of peak responses.

So, the response spectrum method of analysis all though it is called an approximate analysis because of the combination of the contributions of the different modes, yet the method is found to be a very useful and it can provide or estimate the peak responses quite satisfactorily.

The method as such is developed for single point single component excitation for classically damped linear systems. However, with additional approximations it has been extended for multi point-multi component excitations, and for non-classically damped systems.

So, first what we will do is that we look in to how we come into the response spectrum method of analysis or develop the required equations for obtaining the response using the response spectrums of earthquake for single point, single component, excitations and for classically damped linear system that is where the damping of this structure is constructed out of the modal damping ratio. And it is defined for one type of material;

that means, if the structure is constructed out of one type of material for which the damping ratio is specified then we can obtain or derive the required equations for the response spectrum method of analysis.

When we encounter a structure which is not classically damped systems, like a structure in which two kinds of materials are used for example steel and concrete in that case it no more remains a classically damped system. Similarly, if we consider the soil structure introduction in a problem then also the system does not remain a classically damped system. Also for the multi support excitations and multi component excitations this response spectrum method of analysis cannot be directly derived without making some kind of additional approximations. And we will look in to this later.

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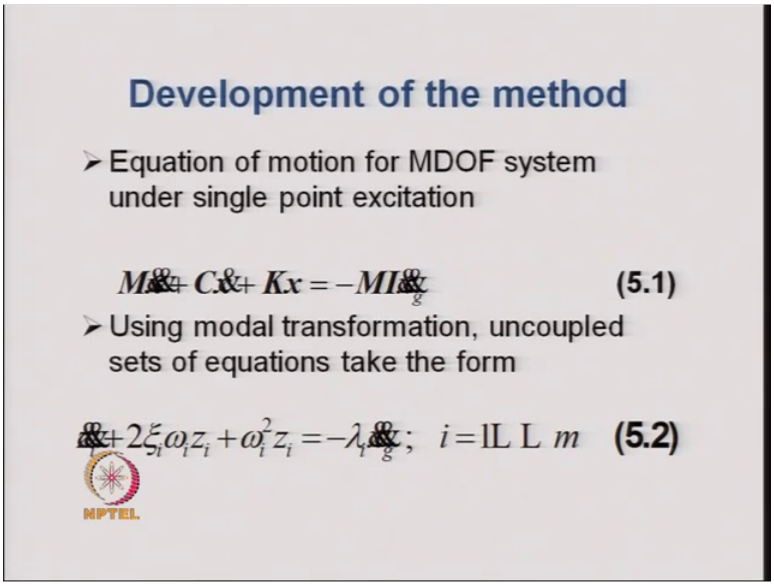


The let us take the equation of motion for a multi degree of freedom system under single point excitation. The equation which is shown in 5.1 it is not very clear, it will be M x double dot plus C x dot plus K x is equal to minus MI x double dot g. And if we use the normal mode theory which all of you are acquainted with, an uncoupled sets of equation can be obtained which take the form of Z i double dot plus twice zeta i into omega i into Z i dot plus omega i square Z i is equal to minus lambda i into x double dot g. That is the i-th modal equation and the number of modes that is considered is equal to m. Therefore i varies from 1 to m.

Now each one of these modal equations how it is derived using normal mode theory is known to all of you. The Z i is the i-th generalized coordinate for the i-th mode, lambda i is the mode participation factor for the i-th mode, and omega i is the natural frequency for the i-th mode.

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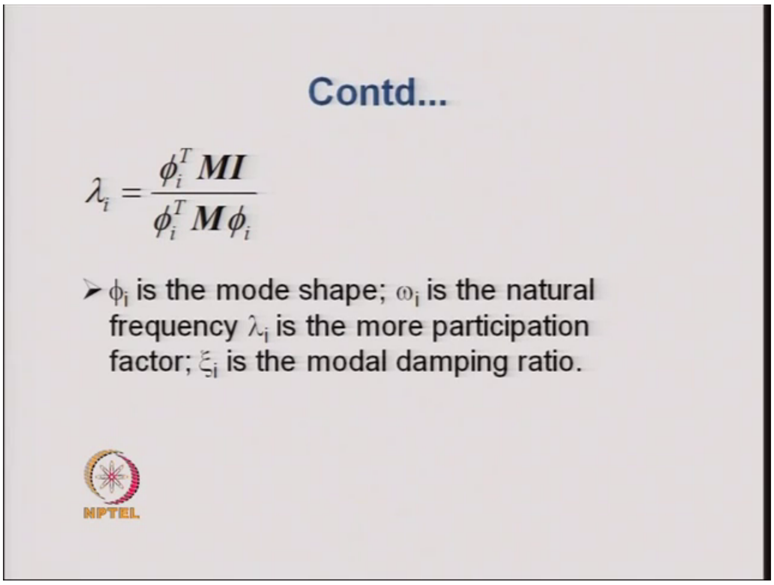


The lambda i that is the mode participation factor is obtained using this formula that is phi i T MI divided by phi i T M phi i; where phi i T is the transpose of the i-th mode and i is an influence coefficient vector generally consisting of 1 1 1 1 or 1 0 1 0 1 0 which we discussed in connection with the response analysis of structures to the deterministic ground motions.

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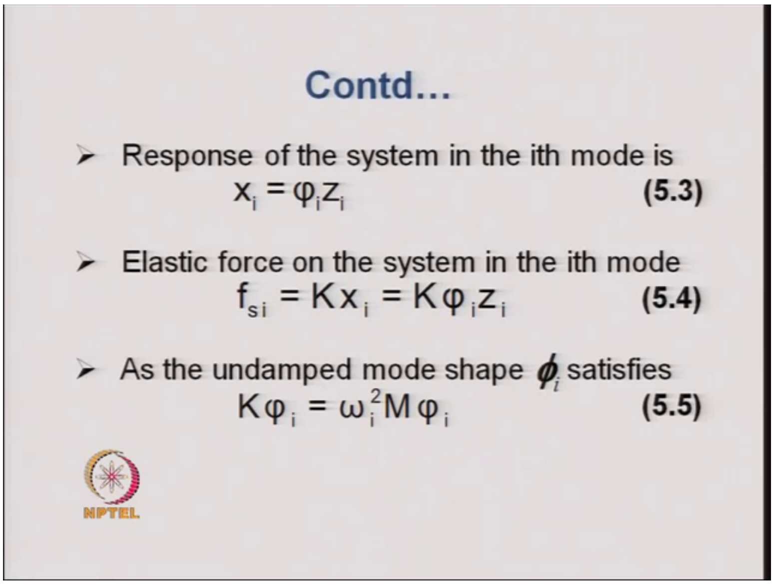


So, using these formula one can obtain the value of lambda i, and once we have the lambda i then one can solve each one of these modal equations in time. And then the responses for the original structural coordinate system that is x i can be obtained by using equation 5.3; that is we multiply the phi i with Z i, where phi i is the i-th mode shape and Z i is the time history of the generalized coordinate for the i-th mode. And these X i is also a time history of displacement in structural coordinate.

And the value of x i that I get is the response of the structure if it vibrates only in the i-th mode. So, in this fashion one can find out the responses in any mode of vibration. And once we obtain the vibrations in different modes of vibration, then we sum up the time histories of these responses to obtain the final response of the structure. And the summation is carried out for M number of modes that we consider for the analysis.

Now say if we wish to find out the maximum value of the force that is generated in the structure during its vibration in the i-th mode, and we call it as f s i, then f s i can be obtained by simply multiplying the stiffness matrix of the structure with X i, that is the displacement that is taking place in the i-th mode of vibration. And this can be written as K into phi i into Z i, which is given in or rather using equation 5.3.

Now, as the undamped mode shape phi i satisfies the following equation; that is K phi i is equal to omega i square M phi i and giving rise to the what is known as the Eigen value problem, then one can substitute for K phi i.

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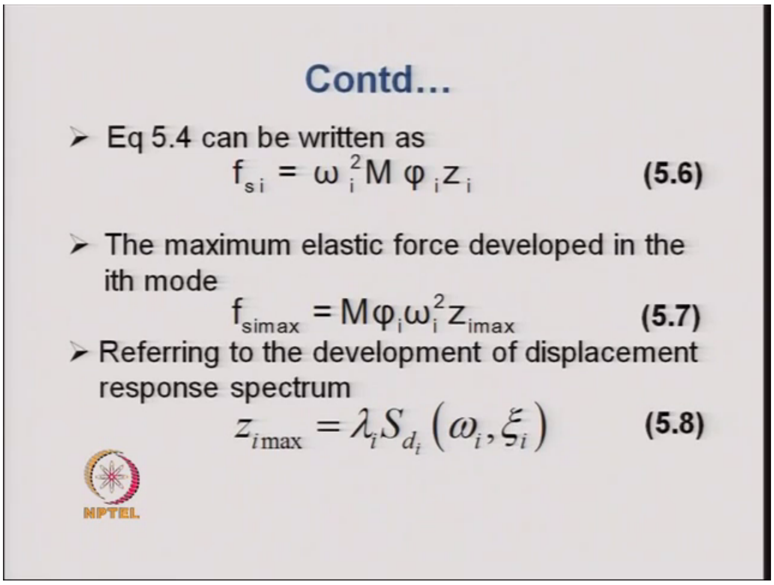


And can write down the f s i that is the force which is attracted in the i-th mode of vibration equal to omega i square M phi i into Z i. Since, Z i is a single quantity or the time history of a single quantity, one can find out a maximum value of Z i. And then the f s i would be the maximum force that is developed on to this structure in the i-th mode, and this is called as f s i max equal to M phi i into omega i square into Z i max, These Z i max is the maximum displacement in the generalized coordinate in the i-th mode.

Now let us go back to the development of the displacement response spectrum. If you recall the displacement response spectrum for an earthquake is obtained by analyzing the single degree of freedom system with a specified damping and a frequency, for a given record of earth quake and finding out the maximum displacement. And this maximum displacement that we get for the single degree of freedom system is plotted against t i which is equal to 2 pi by omega i. And this way one can have a plot of the maximum displacement of the oscillator against time period t, which is called the displacement response spectrum. And these displacement response spectrum is obtained for a given damping ratio psi i.

Therefore if we use that definition of the displacement response spectrum, then Z i max that is obtained is nothing but is equal to lambda i into S d i where, S d i is the displacement response spectrum ordinate at time t i or period t i is equal to 2 pi by omega i, and is valid for a damping given damping ratio psi i. It is to be multiplied by lambda i, because originally the equation that we had is this equation that is on the right hand side of this single degree of freedom equation it is not x double dot g, it is multiplied by lambda i. Therefore, lambda i comes into picture here, so Z i max can be write be as lambda i into S d i.

So therefore, for a given earthquake if we know the displacement response spectrum, then corresponding to the time period omega 2 pi by omega i and the damping ratio psi i, one can obtain the value of S d i from the displacement response spectrum curve and that gets multiplied by lambda i and that gives you the value of Z i max.

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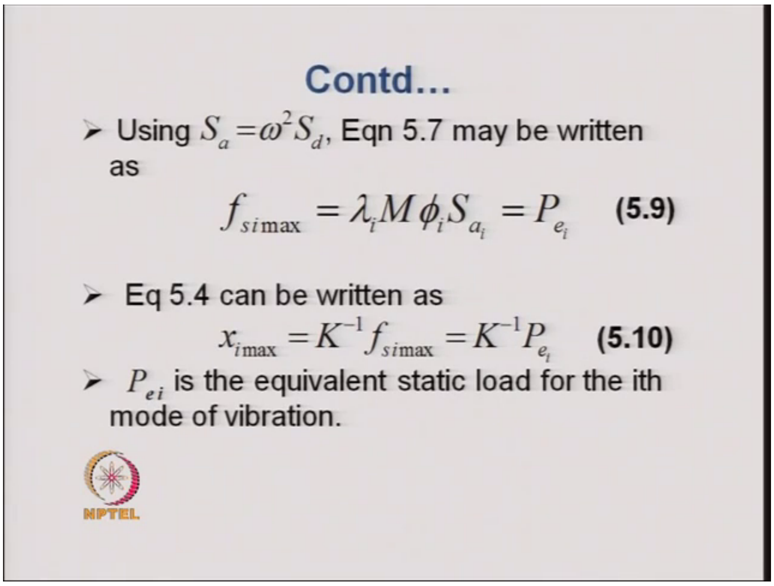


Then we know the relationship between the displacement response spectrum, and the pseudo acceleration response spectrum loosely called the acceleration response spectrum. And that is S a is equal to omega square into S d. Then that equation now can be written as f s i max is equal to lambda i into M into phi i into S i; that is we substitute in place of omega i square into S d the value of S a i. And these quantity is called P e i that is the equivalent static load. And these equivalent static load produces a maximum displacement of the structure if it where vibrating only in the i-th mode.

Therefore, these load P e i is an equivalent static load, and if this equivalent static load is applied on to the structure statically it will produce a displacement in the structure which will be equal to the maximum displacement that the structure would have if it where vibrating only in the i-th mode. Then once we obtain the value of P e i using the formula

5.9. Then the things that are required for us are S a i that is the spectral acceleration that must be defined, and that is generally given in all course of practice, phi i that is the mode shapes in the i-th mode and the mode participation factor lambda i that can be obtained using the formula that I had shown before.

So, with the help of this now one can obtain P e i, and once you obtain P e i then one can obtain the maximum displacement that the structure would have if it were vibrating only in the i-th mode. And the analysis that we perform in equation 5.10 is a purely a static analysis. So therefore, we see that in finding out the equation 5.9 we step by step follow the normal mode theory come to a single degree of freedom equation, and for that single degree of freedom equation one can find out the maximum displacement of the structure if it were vibrating only in that particular mode.

So, this is the main derivation that goes behind the development of the response spectrum method of analysis. And once we obtain the value of P e i, then the entire problem is converted to a static problem that is P e i is applied statically on to the structure. And we can not only obtain the response or the displacement of the structure, but also can find out bending moment sheer force or any other quantity of interest for a particular mode. And for this reason we called P e i as the equivalent static load for the i-th mode of vibration.

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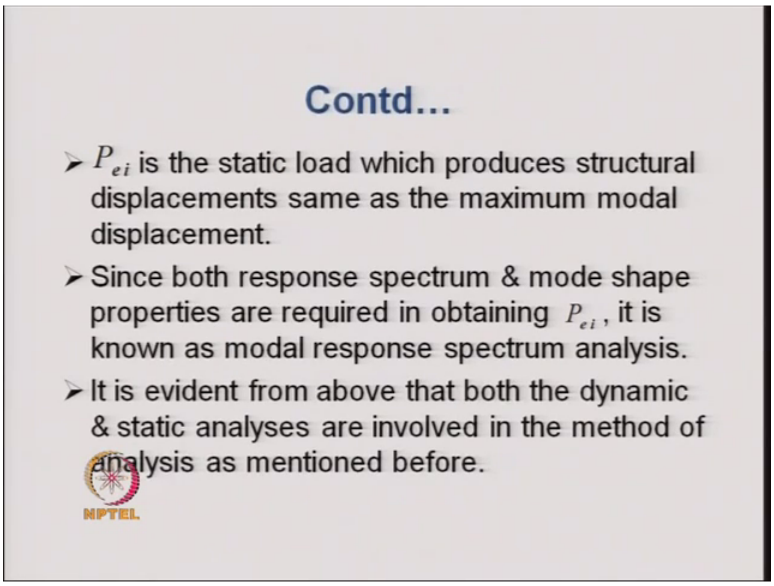

Since both response spectrum and mode shape properties are required in obtaining P e i it is also called modal response spectrum analysis. And it is evident from the above discussion that both dynamic and static analyses are involved in the method of analysis using response spectrum.

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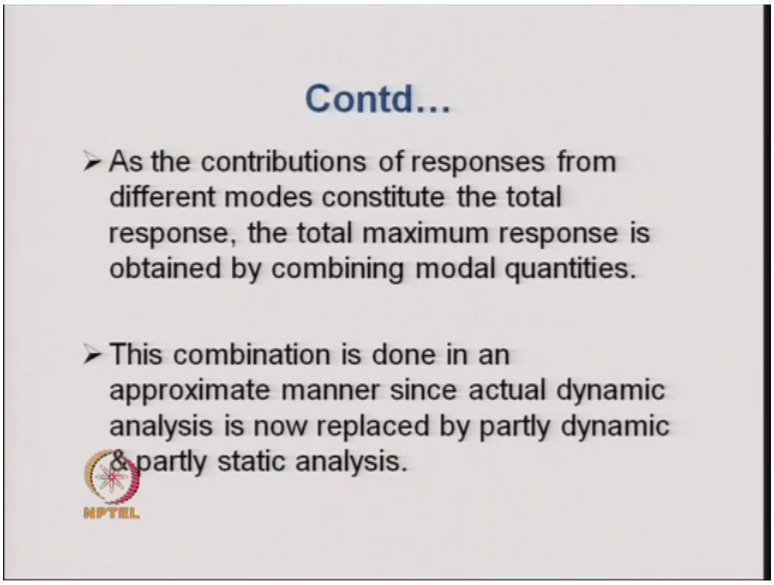


So, up to this part here is no approximation it can be straight away derived using the normal mode theory, and one can find out an equivalent lateral load for each mode of vibration. Then one has to combine these modes or the modal responses obtained in each mode of vibration. And these modal responses are the maximum responses and that would have occurred if the structure were vibrating in that particular mode. And then we have to combine these responses in order to find out the total response of the structure.

Now, in this part we make an approximation and there are different combination rules, that is prescribed and we adopt one or the other combination rule. And this combination rules are somewhat approximate. The reason for this will be explained shortly.

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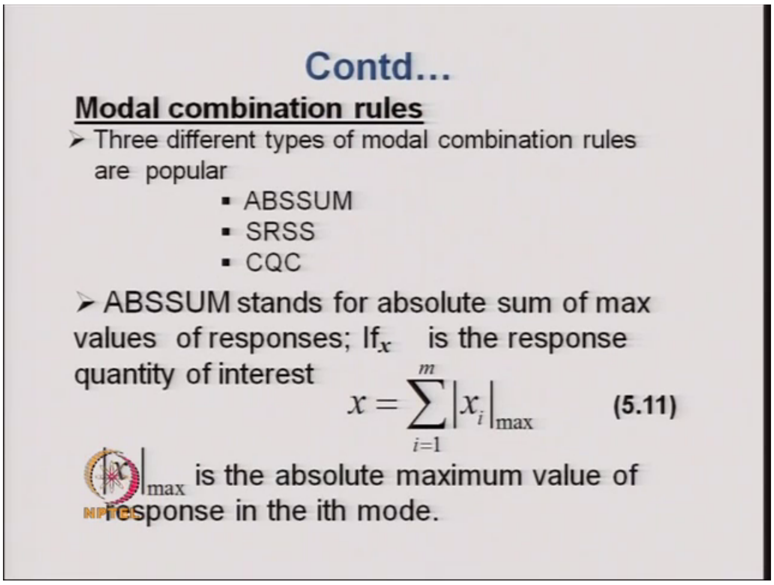


The modal combination rules that we have are ABSSUM, SRSS and CQC. ABSSUM stands for absolute sum of maximum values of responses. Say if x is the response quantity of interest say displacement or bending movement or whatever be the response quantity of interest, then we sum up the absolute maximum value of the response quantity in each mode of vibration, and the resulting quantity is equal to the desired value of the or desired maximum value of the response.

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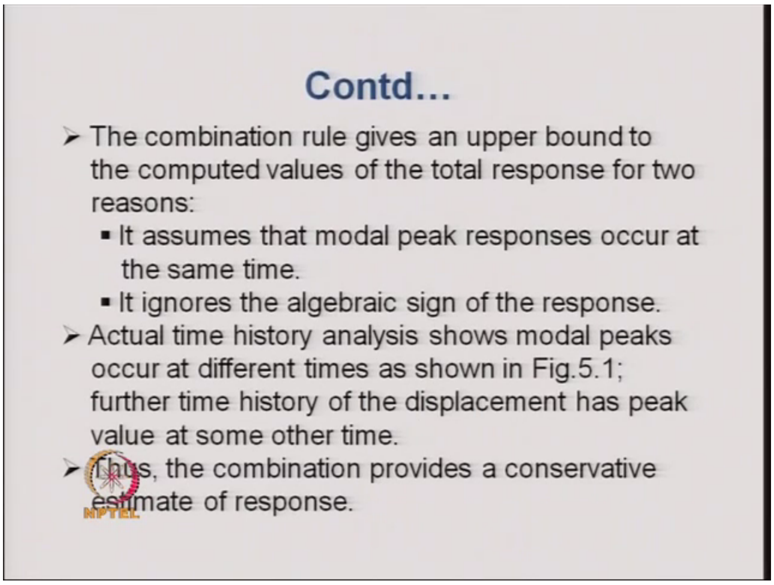

Now, this is an approximate one because of two reasons. Number one, it assumes that modal peak responses occur at the same time; that means, if we have the plot of the Z i that is the generalized displacement at the i-th mode, and i is equal to 1 to m that way one can have n number of such plots.

If we look at these plots we will find that the maximum value of Z that is the generalized coordinate in a particular mode these occurs say at a time t i. And this t i is not same for all the modes that is for i is equal to 1 to m we will get n numbers of time at which the peak occurs, and these times will not be the same.

So, if the peak of the generalized responses in each mode of vibration do not occur at the same time, then the addition of the absolute value of the maximum responses that clearly do not give the actual peak value of the response.

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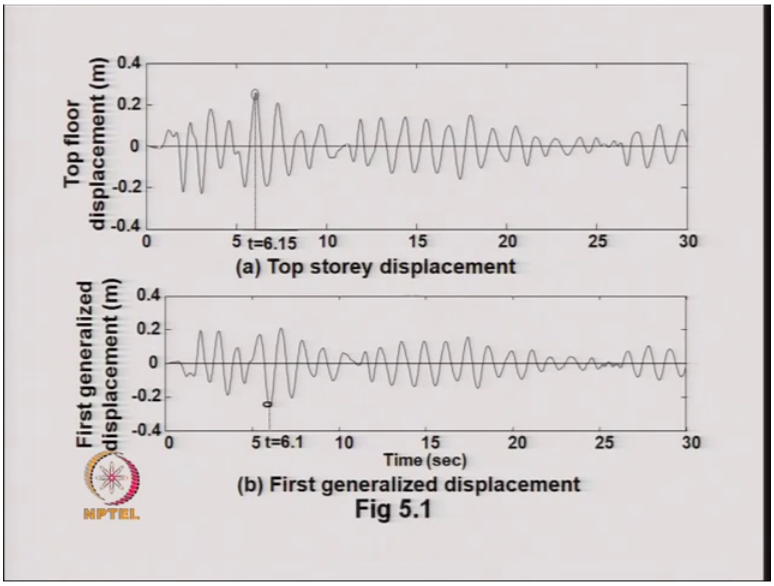


Secondly, it ignores the algebraic sign of the response that would be made clear with the help of figure 5.1. And say this is the figure said we are interested in a particular top story of displacement in a multi storey building. And when we combine the effect of all the modes and find out the time history of displacement using normal mode theory, then the maximum displacement that take place say is at t is equal to 6.15.

Now, if we consider the first generalized displacement, then the first generalized displacement we find that the maximum value is occurring at t is equal to 6.1, and at that time the value is negative not positive. Therefore we can see that if we sum up the absolute values then it is ignoring the sign. Similarly the second generalized displacement, here we see that the time at which the maximum displacement that is occurring is equal to 2.5.

So, there is a difference between the times at which the maximum generalized displacement in each mode of vibration that takes place, but also the signs could be different that is it could be sometimes plus or sometimes minus. And the total response or displacement time history has a different kind of plot where the maximum or peak displacement occurs at a different time. Thus, the summation that we have considered in equation 5.11 is an approximate 1.

Now, this value of the response that we get from ABSSUM generally is conservative, because it is firstly ignoring the sign, and then also it is assuming that the responses in each generalized coordinate occur at the same time.

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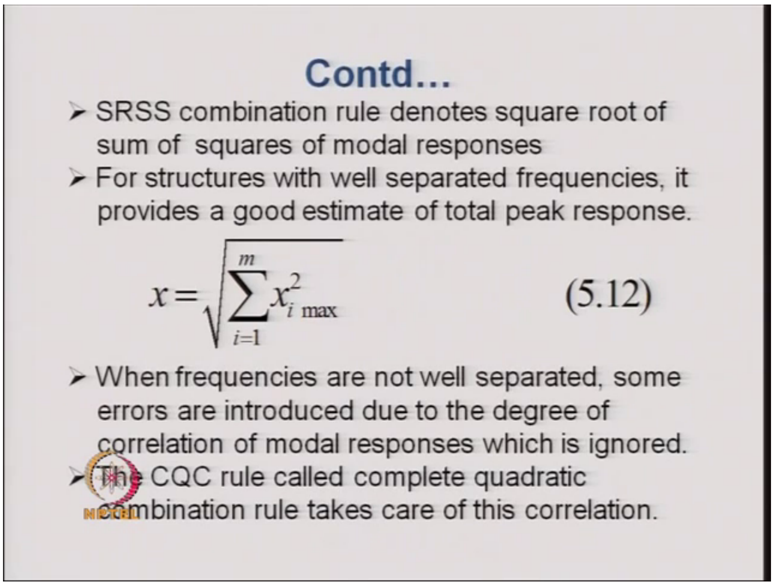


Next rule is the SRSS combination rule which denotes square root of sum squares of modal responses. Now the idea behind this is that the maximum responses that occur in each mode they are occurring at different times and these quantities that is the maximum response in each mode of vibration there is no relationship between them. And each one of them is assumed to be a independent random variable.

Now, if you are wanting to combine n number of independent random variables, then the best way to combine them is by taking the square of those values, and then add them together and take a square root of that. So that is what is known as SRSS combination rule and 5.12 explain that that is the maximum displacement is equal to the summation of x i max square and the summation extends from i is equal to 1 to m and then we take a square root of this summation.

Now, this particular formula is valid when frequencies are well separated; that is they are not close enough, because if they are close that means 2 frequencies in the 2 modes are very close to each other. Then it is expected that there is there will be a correlation between the responses is of the 2 modes. And if the frequencies are well separated for the 2 modes, then we expect that the maximum values that you obtained in the 2 modes will not be that much correlated.

So, therefore these equations 5.12 is developed with an assumption that the maximum values in each mode of vibration are independent random variables. And they are summed up with the help of SRSS rule. Now the effect of the correlation between the 2 responses in 2 modes of vibration, that can be taken care of by just modifying equation 5.1 and the rule that we use for that is called CQC rule or complete quadratic combination rule.

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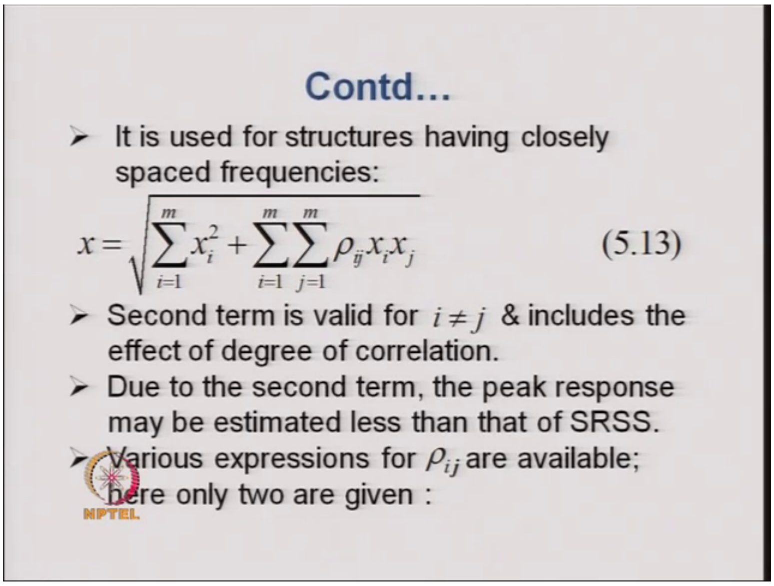

The complete quadratic combination rule is given in equation 5.13. And we can see that it consist of two parts, first part is simply ah the SRSS rule that we are used before, and the second part of the equation contains rho i j; where rho i j is the correlation between the modes of vibration, and x i and x j are the maximum values of the responses in the i-th and j-th mode.

Now second mode or the second term is valid for i not equal j, when i is equal to j, then it is perfectly correlated and the equation only then becomes only simply x i square. The second term in fact includes the degree of correlation that exist between the 2 modes. And since the second term has the value rho i j in to x into x j, then these particular quantity could be plus or minus that is depending upon the value of x i and x j.

So the SRSS rule which ignores the second term can provide a higher value compared to the CQC rule. Or in other words the CQC rule quite often estimate less value, then that proved by SRSS rule. Now, the various expressions for the correlation between the 2 modes are available in the literature.

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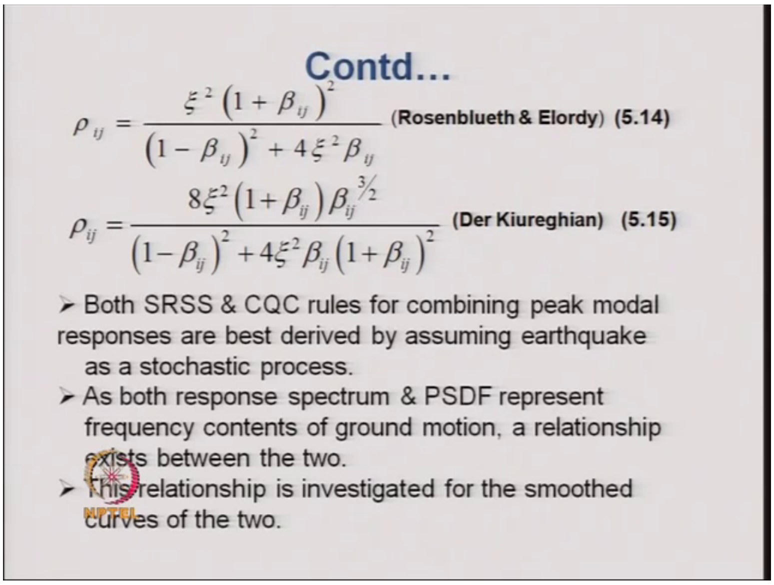


We have taken here two such correlation coefficient expressions: one is given by Rosenblueth and Elordy that is given in equation 5.14, where psi is the damping ratio and beta i j is the ratio between the natural frequency in the i-th mode and the j-th mode. So, you can see here these the two equations are slightly different in their form. And using these equations one can obtain the value of rho i j that is required for finding the response using CQC rule. And to find out this rho i j what we should know is the ratio between the frequencies of the 2 modes which are known to us and the damping ratio psi which is specified.

So, the CQC rule can be easily implemented in place of SRSS rule, and that is done for cases where we find the 2 frequencies are very close to each other in a particular structure. And one can compare as such the values of the responses obtained using SRSS rule and CQC rule and can compare them for any given problem.

The SRSS and CQC rules for combining peak modal responses are best derived by assuming earth quake as a stochastic process, and there are exclusive papers for that. And using those principles one can show that or one can justify the combination rule that we classically use for combining the modal responses in the response spectrum method of analysis. Now, as both response spectrum and the power spectral density function, represent frequency contents of ground motion a relationship exist between the two.

So that is a thing that one should ah take note off, because the response spectrum of any earth quake is nothing but the maximum value of the responses or the displacement response, plotted against time period or it can be plotted against the frequency. And from that one can obtain the response spectrum for acceleration simply by multiplying the response spectrum for displacement with omega square.

So, this gives basically a distribution of the energy of the earth quake with frequency or indicates a distribution of the energy of earth quake, and that is what is the major frequency component that exist in the earth quake. Similarly power spectral density function; since it is a plot of the variance or the mean square value of the process with frequency, then this also provides an idea about the major frequency content of the earth quake process.

So, therefore there is a relationship or there is expected to have a relationship between the PSDF and the response spectrum of an earthquake, and this has been established by many researchers, and the relationship between the two for the smooth response spectrum, and smooth power spectral density function and have been given.

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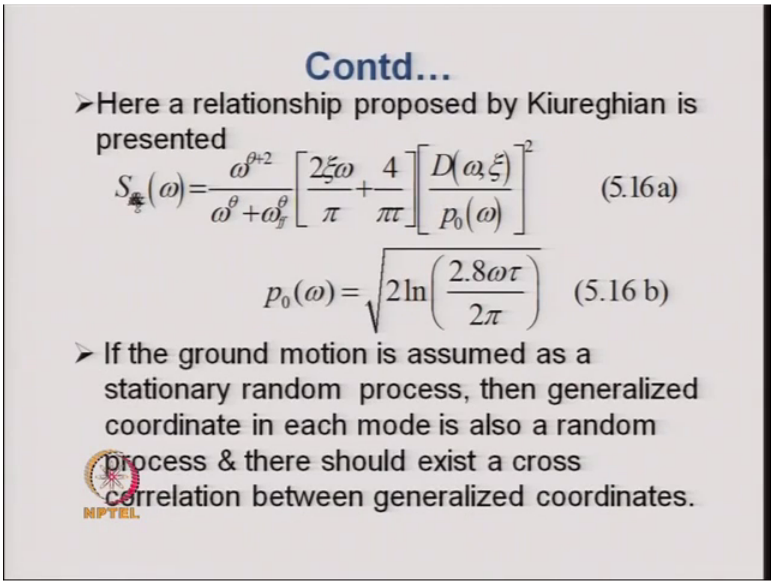


And one such relationship is shown over here, and here in this we can see that the power spectral density function is related to the displacement response spectrum D and P 0 is called the peak factor and tau is the duration of the earth quake.

So, if the duration of the earth quake is known and the frequency at they are different frequencies omega one can obtain the relationship between the displacement response spectrum for a given value of psi with the power spectral density function of the acceleration which is S x double dot g.

Now this derivation is obtained using what is known as or is derived using the concept, that the earthquake is modeled as a stationary random process, and using that one can find out or this particular expression. Apart from that we will require the cross power spectral density function between the two processes in an earthquake. And these relationships would be required when we will extend the response spectrum method of analysis for multi support excitation.