

Seismic Analysis of Structures
Prof. T.K. Datta
Department of Civil Engineering
Indian Institute of Technology, Delhi

Lecture – 22
Response Spectrum Method of Analysis (Contd.)


In the previous 2 lectures, we discussed about the response factor method of analysis. And try to explain to you how the response spectrum method of analysis is developed from the principles of the normal mode theory. The equivalent static loads, which are calculated in each mode of vibration; how they are or developed, then we solved a problem for single point excitation system to illustrate the method of analysis for a single point excitation system.

At the end we started the extension of the method for obtaining the response spectrum method of, and all applying them response spectrum method of analysis for multi support excitation system.

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RSA for multi support excitation

- Response spectrum method is strictly valid for single point excitation.
- For extending the method for multi support excitation, some additional assumptions are required.




In that we started saying about the assumptions that are additionally used for extending the response spectrum method of analysis for multi support excitation.

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- Moreover, the extension requires a derivation through random vibration analysis. Therefore, it is not described here; but some features are given below for understanding the extension of the method to multi support excitation.
 - It is assumed that future earthquake is represented by an averaged smooth response spectrum & a PSDF obtained from an ensemble of time histories.




Those assumptions are the assumptions that the future earthquake is represented by an averaged smooth response spectrum, and a power spectral density function that is obtained from an ensemble of time histories.

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- Lack of correlation between ground motions at two points is represented by a coherence function.
- Peak factors in each mode of vibration and the peak factor for the total response are assumed to be the same.
- A relationship like Eqn. 5.16 is established between S_g and PSDF.

- Mean peak value of any response quantity r consists of two parts:



Then lack of correlation between ground motions at 2 points, that is at the 2 supports 2 different supports where the excitations are caused due to earthquake is represented by a coherence function.

Many expressions for the coherence functions were discussed in seismic input. One can choose any one of those coherence functions. The third assumption is the most important assumption that is a peak factor in each mode of vibration and the peak factor for the total responses are assumed to be the same. We have come across the term peak factor in relation to the seismic inputs for structures for the case of random ground excitation, the peak factor is a factor which is multiplied with the standard deviation or the Rms value of the ground motion, in order to obtain the peak value of the ground motion better it is called expected peak value of the ground motion.

Then expression for that was given in seismic input. So, in each mode of vibration similarly, if we wish to find out the peak value of the response, then one should multiply a peak factor with the standard deviation of the response for the mode for the response for that mode of vibration. This peak factor is obtained from the power spectral density function of the response in that mode. Once we get the power spectral density functions for the responses in different modes of vibration, from that one can calculate not only the standard deviation of the modal response, but also the peak factors associated with that.

Multiplying the peak factors with the standard deviation of the response, one can get the peak value of the response. Thus, the peak factors are involved in each mode of vibration that is also a peak factor for the total response in the structural coordinate system. This assumption means that all the peak factors in different modes of vibration, the peak factor which is associated with the total response of the system. All of them are assumed to be the same. Therefore, if we find out the peak factor for any mode of vibration, then that peak factor can be utilized throughout the analysis.

Next what is required is a relationship between the ground displacement spectrum, and a power spectral density function of response. These relationship is required, because we wish to find out also the power spectral density function of the ground motion, from the specified response spectrum. That is necessary for obtaining the coherence function the cross power spectral density function between the 2 responses, which are required in the derivation of this particular analysis. Mean peak value of any response quantity r consists of 2 portions.


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- Pseudo static response due to the displacements of the supports
- Dynamic response of the structure with respect to supports.

➤ Using normal mode theory, uncoupled dynamic equation of motion is written as:

$$\ddot{z}_i + 2\xi\omega_i\dot{z}_i + \omega_i^2 z_i = \sum_{k=1}^s \beta_{ki} \ddot{x}_k; \quad i = 1..m \quad (5.18)$$

$$\beta_{ki} = \frac{\phi_i^T M R_k}{\phi_i^T M \phi_i} \quad (5.19)$$


First part is the pseudo static response due to the displacements of the support. The second part is the dynamic response of the structure with respect to supports.

The first part is obtained by applying unit displacement at each support successively, finding out the responses to nonsupport degrees of freedom. So, that gives the pseudo static response due to the displacements of the support. They are some coefficients, and these coefficients are stored in the program so that this coefficient later on can be multiplied with the ground displacements that is taking place at different supports. The second part straight away comes from done response analysis of the structure or the dynamic response analysis of the structure with respect to the support.

Now, the using normal mode theory the uncoupled dynamic equation of motion for each mode of vibration, can be written by equations like equation 5.18, where we have the ith modal equation providing the ith generalized displacement z_i . There is a term β_{ki} which is the mode participation factor, these we discussed previously when you are solving the problem in time domain. There we have seen that if it is a case of multi support excitation, then there is not an unique value of the mode participation factor, like single point excitation system.

But the mode participation factor in a particular mode is associated also with the support. So, for the kth support excitation, there is a mode participation factor in the ith mode that is called β_{ki} and that is given by $\phi_i^T M R_k$ divided by $\phi_i^T M \phi_i$. R_k is the

column vector or the kth column vector of the R matrix. That is the influence coefficient matrix that we multiply with the mass matrix then multiplied with the \ddot{x} to find out the earthquake force. Or in other words for multi degree freedom system with multi support excitation on the right-hand side we write $m \ddot{r}$, then \ddot{x} . So, R_k is the kth column of that R matrix.

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➤ If the response of the SDOF oscillator to \ddot{x}_k is \bar{z}_{ki} then
$$z_i = \sum_{k=1}^s \beta_{ki} \bar{z}_{ki} \quad (5.20)$$

➤ Total response is given by
$$r(t) = \sum_{k=1}^s a_k u_k(t) + \sum_{i=1}^m \bar{\phi}_i z_i(t) \quad (5.21)$$

$$r(t) = \sum_{k=1}^s a_k u_k(t) + \sum_{i=1}^m \bar{\phi}_i \sum_{k=1}^s \beta_{ki} \bar{z}_{ki} \quad (5.22)$$

$$r(t) = \mathbf{a}^T \mathbf{u}(t) + \bar{\phi}^T \bar{\mathbf{z}}(t) \quad (5.23)$$

 and $\bar{\mathbf{z}}$ are vectors of size $m \times s$ (for $s=3$ $m=2$)

If the response of the single degree freedom oscillator to a specific excitation for a particular support is \ddot{x}_k , for that if the response is \bar{z}_{ki} , then one can write down the response or generally response in the ith mode is equal to the summation of β_{ki} multiplied by \bar{z}_{ki} . Because \bar{z}_{ki} is the response in the ith mode produce due to the kth support excitation. So, if we sum up for all the support excitation, then we get the value of the z_i in a particular mode of vibration.

So, this z_i is the dynamic response of the system with respect to the support. The total response is given by the equation 5.21, where the response $r(t)$ is given by this equation. That is, $\sum_{k=1}^s a_k u_k(t)$ summation over the supports and then this is the $\bar{\phi}_i z_i(t)$. That is the mode shape coefficient corresponding to the response quantity of interest. So, this response quantity of interest need not be displacement could be bending moment shear force or any other quantity of interest. So, the $\bar{\phi}_i k$ will be the mode shape coefficient for that response.

Then we multiply that coefficient with $z_i(t)$ and sum it over all the modes. So, that gives the response of that particular response of the system and that is the dynamic response with respect to the support. This part is the pseudo static response part and as we as I explained before this a k coefficient that is the responses, that is obtained at the non-support degrees of freedom due to displacement applied at its support that is stored and that is the coefficient a_k . So, that a_k is multiplied by the ground motion u_k to get the pseudo static component of the response at all nonsupport degrees of freedom.

These a_k , a coefficient could be also obtained for any response quantity of interest if this $r(t)$ is a response other than the displacement, then you use those values of a_k which we have obtained for the response quantity of interest. Substituting $z_i(t)$ from this equation we obtain finally, equation 5.22, and these 5.22 equation can be written in a compact form in matrix notation. That is $r(t)$ becomes equal to $a^T u(t)$ plus $\phi^b T \bar{z}(t)$ where ϕ^b \bar{z} are vectors of size m into s .

Say the number of support is 3 in the number of modes that you consider is 2.

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$$\phi_{\beta}^T = [\bar{\phi}_1 \beta_{11} \quad \bar{\phi}_1 \beta_{21} \quad \bar{\phi}_1 \beta_{31} \quad \bar{\phi}_2 \beta_{12} \quad \bar{\phi}_2 \beta_{22} \quad \bar{\phi}_2 \beta_{32}] \quad (5.24a)$$


$$\bar{z}^T = \{\bar{z}_{11} \quad \bar{z}_{21} \quad \bar{z}_{31} \quad \bar{z}_{12} \quad \bar{z}_{22} \quad \bar{z}_{32}\} \quad (5.24b)$$

➤ Assuming $r(t), u(t)$ and $\bar{z}(t)$ to be random processes, PSDF of $r(t)$ is given by:

$$S_{rr} = a^T S_{uu} a + \phi_{\beta}^T S_{zz} \phi_{\beta} + a^T S_{uz} \phi_{\beta} + \phi_{\beta}^T S_{zu} a \quad (5.25)$$

➤ Performing integration over the frequency range of interest & considering mean peak as peak factor multiplied by standard deviation,

Expected peak response may be written as:



In that case the ϕ^b vector would typically look like this, that is the $\bar{\phi}_1 \beta_{11}$ $\bar{\phi}_1 \beta_{21}$ $\bar{\phi}_1 \beta_{31}$. So, this β_{11} β_{21} and β_{31} are the mode participation factors, coming from the 3 modes for mode 1. Then we have $\bar{\phi}_2 \beta_{12}$ $\bar{\phi}_2 \beta_{22}$ and $\bar{\phi}_2 \beta_{32}$. They are the again the mode participation factors coming from supports 1 2 and 3 for the second mode.

Similarly, the \bar{z} vector consists of \bar{z}_{11} , \bar{z}_{21} , and \bar{z}_{31} . That is the responses coming from the support 1, 2 mode 1 the \bar{z}_{21} indicates the response that is coming from the second support 2 mode 1 \bar{z}_{31} . Similarly, is the response that is coming to the mode 1 from support 3. The last 3 terms, that is \bar{z}_{12} , \bar{z}_{22} and \bar{z}_{32} , they are a similarly the responses the model responses, that is coming for the second mode from the 3 supports. Then we assume that $r(t)$ and $\bar{z}(t)$ to be random processes, the PSDF of $r(t)$, then is given by equation 5.25, and this equation is shown over here.

The reason for assuming $r(t)$ and $\bar{z}(t)$ to be random process is that the excitation is an excitation which is assumed to be a coming to be an excitation coming from in ensemble of time history of records. That ensemble of time history of record is characterized by a stationary random process, as well as a smooth response spectrum is obtained for that ensemble time history of records. And therefore, we could relate the response spectrum of displacement with the power spectral density function of the ground acceleration, and a relationship of that type was shown previously, and that relationship is used to obtain the power spectral density function of the ground acceleration from a given response spectrum of displacement.

So, once we assume that $r(t)$ and $\bar{z}(t)$ are the are the quantities which are random quantities of random process, then one can write down state away the power spectral density function of the response by this equation. And the basis of this equation has been discussed before while discussing the spectral analysis of response for a specified ground for a specified power spectral density function of ground excitation. $\Phi^T a c u a$ that comes from the first part of the equation, that is $a^T u t$. The second part $\Phi^T S z \bar{z}$ $\Phi^T \beta$ that is coming from the second part of the equation. And these 2 parts are the cross terms that is the cross power spectral density function that exist between u and \bar{z} and u .

So, once we have this expression for S_{rr} , then one can perform an integration of this power spectral density functions. After we integrate this power spectral density function, then one can get the standard deviation of these quantities. Those standard deviations are multiplied by the peak factor, in order to get the mean peak values of the responses. And from that mean peak value of the responses one can get the mean peak value of the response quantity of interest.


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$$E[\max|r(t)|] = \left[\begin{array}{l} b^T 1_{uu} b + b^T 1_{i\bar{z}} \phi_{\beta D} \\ + \phi_{\beta D}^T 1_{z\bar{z}} \phi_{\beta D} + \phi_{\beta D}^T 1_{z\bar{u}} b \end{array} \right]^{1/2} \quad (5.26)$$

$$b^T = \left[a_1 u_{p_1} \quad a_2 u_{p_2} \quad a_3 u_{p_3} \quad \dots \quad a_S u_{p_S} \right] \quad (5.27a)$$

$$\phi_{\beta D}^T = \left[\begin{array}{cccc} \bar{\phi}_1 \beta_{11} D_{11} & \bar{\phi}_1 \beta_{21} D_{21} & \dots & \bar{\phi}_1 \beta_{s1} D_{s1} \\ \dots & \bar{\phi}_m \beta_{11} D_{1m} & & \end{array} \right] \quad (5.27b)$$

$$D_i = D_i(\omega_j, \xi_j) \quad i = 1, \dots, s; \quad j = 1, \dots, m \quad (5.27c)$$


So, once we do that then the expected maximum value of the response quantity of interest that is $E \max r(t)$. That can be written finally, in this form $b^T 1_{uu} b + b^T 1_{i\bar{z}} \phi_{\beta D} + \phi_{\beta D}^T 1_{z\bar{z}} \phi_{\beta D} + \phi_{\beta D}^T 1_{z\bar{u}} b$. Where b and $\phi_{\beta D}$ are the vectors like this; that is b vector is equal to $a_1 u_{p_1} a_2 u_{p_2}$ and so on. So, if there are S number S supports, then it will go up to $a_S u_{p_S}$ $\phi_{\beta D}$ is given by this expression $\bar{\phi}_1 \beta_{11} D_{11}$, then $\bar{\phi}_1 \beta_{21} D_{21}$, and that is how it goes up to $\bar{\phi}_1 \beta_{s1} D_{s1}$. Then for the last series of terms will be $\bar{\phi}_m \beta_{11} D_{1m}$ and the last term of the series will be $\bar{\phi}_m \beta_{s1} D_{sm}$.

Now, let me let us explain the terms of this b^T and $\phi_{\beta D}^T$. $a_1 a_2 a_3$ are the coefficients of the response or the pseudo static coefficient of the response quantity that we have discussed before. u_{p_1} is the p ground displacement for the support excitation 1. u_{p_2} is the p ground displacement corresponding to support excitation 2 and so on. In these expression $\bar{\phi}_1$ represents the first mode shape coefficient for the response quantity of interest that we can obtain, or that you have discussed before also. β_{11} is a mode participation factor, that is coming from the support 1 2 mode 1. D_{11} is a displacement response spectrum ordinate corresponding to the first-time period, and the first support excitation; that is, we are assuming that the response spectrums at support 1, support 2, support 3 and so on. These response spectrums as if are different. Then D_{11} represents the response spectrum ordinate for time period for the first mode, and for the

response spectrum that we have obtained or that is given for the support excitation 1. So, similarly β_{21} is the mode participation factor, that is coming to the first mode from the second support, and D_{21} is the displacement response spectrum for the first for the for the first mode or time period coming from the support excitation 2.

So, that way these terms basically can be easily explained and can be known. That is again for example, $\bar{\phi}_1$ here is the mode shape coefficient for the response quantity of interest in the first mode. And β_{s1} represents the mode participation factor which is coming to the mode 1 coming from support S in D_{s1} represents the displacement response spectrum ordinate for the time period for mode 1 coming from the support excitation at S. And that is how we are writing D_{ij} . And this D_{ij} means that the response spectrum ordinate coming to the or corresponding to the time period j, and it is contribution coming from the support i. If we assume that D_{ij} that is the support excitations for all the supports have the same power spectral density function, and they have the same response spectrum or displacement response spectrum, then D_{ij} simply becomes equal to $D_i \omega_j \psi_j$. Which means that for a given response spectrum which will be remain constant for all the supports for that the response spectrum ordinate corresponding to the time period, which will be the 2π by ω_j and for the damping ratio ξ_j .

So, for that we take the displacement response spectrum ordinate. And thus D_{11} D_{21} D_{s1} D_{1m} etcetera, all of them they basically becomes simplified. And we have for each mode we have a 1 particular value of the displacement response spectrum ordinate.


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➤ l_{uu} , l_{uz} and l_{zz} are the correlation matrices whose elements are given by:

$$l_{u_i u_j} = \frac{1}{\sigma_{u_i} \sigma_{u_j}} \int_{-\alpha}^{\alpha} S_{u_i u_j}(\omega) d\omega \quad (5.28)$$

$$l_{u_i \bar{z}_{kj}} = \frac{1}{\sigma_{u_i} \sigma_{\bar{z}_{kj}}} \int_{-\alpha}^{\alpha} h_j^* S_{u_i \ddot{u}_k}(\omega) d\omega \quad (5.29)$$

$$l_{\bar{z}_{ki} \bar{z}_{lj}} = \frac{1}{\sigma_{\bar{z}_{ki}} \sigma_{\bar{z}_{lj}}} \int_{-\alpha}^{\alpha} h_i h_j^* S_{\ddot{u}_k \ddot{u}_l}(\omega) d\omega \quad (5.30)$$


The elements of the correlation matrices l_{uu} , l_{uz} and l_{zz} are given in equations 5.28, 5.29 and 5.30. $l_{u_i u_j}$ are the elements of l_{uu} matrix and equal to $1/\sigma_{u_i} \sigma_{u_j}$. And integration of the a c y u j D omega, in which σ_{u_i} is the standard deviation of the displacement or ground displacement at support i. σ_{u_j} is the standard deviation of the ground displacement at support j. $S_{u_i u_j}$ is the cross power spectral density function between the ground displacement at support i and support j. $l_{u_i \bar{z}_{kj}}$ are the elements of l_{uz} matrix, and they are equal to $1/\sigma_{u_i} \sigma_{\bar{z}_{kj}}$ in which σ_{u_i} is the standard deviation of the ground displacement at the ith support. And the $\sigma_{\bar{z}_{kj}}$ is the generalized or standard deviation of the generalized displacement for mode j coming from support k, that is these \bar{z} is obtained for the jth mode using β_{kj} ; that is the mode participation factor β_{kj} that we had obtained before that is equation 5.19.

Then the within the integration we have h_j^* which is the complex conjugate of the frequency response function for the jth mode and $S_{u_i \ddot{u}_k}$; that is equal to the cross power spectral density function between the displacement at the ith support and the acceleration at the kth support. The $l_{\bar{z}_{ki} \bar{z}_{lj}}$ they are the elements of l_{zz} matrix, and are equal to $1/\sigma_{\bar{z}_{ki}} \sigma_{\bar{z}_{lj}}$. Where $\sigma_{\bar{z}_{ki}}$ is the standard deviation of the generalized displacement for the ith mode, and the kth support that is which is obtained using β_{ki} . Similarly, $\sigma_{\bar{z}_{lj}}$

$\sigma_{l j}$ is the standard deviation of the generalized displacement for the j th mode from the l th support.


Within the integration we have $h_i h_j^*$ is a complex frequency response function for the i th mode. H_j is the complex conjugate of the frequency response function for the j th mode. And $S_{u_k l}$, and \ddot{u}_k and \ddot{u}_l , they are the cross power spectral density function between the accelerations at the k th support and the l th support.

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$$S_{u_i \ddot{u}_k} = \frac{1}{\omega^2} S_{\ddot{u}_k}^2 S_{\ddot{u}_i}^2 \text{coh}(i, k) = \frac{\text{coh}(i, k)}{\omega^2} S_{\ddot{u}_g} \quad (5.31)$$

$$S_{u_i u_j} = \frac{1}{\omega^4} S_{\ddot{u}_k}^2 S_{\ddot{u}_j}^2 \text{coh}(i, j) = \frac{\text{coh}(i, j)}{\omega^4} S_{\ddot{u}_g} \quad (5.32)$$

$$S_{\ddot{u}_k \ddot{u}_l} = S_{\ddot{u}_k}^2 S_{\ddot{u}_l}^2 \text{coh}(k, l) = \text{coh}(k, l) S_{\ddot{u}_g} \quad (5.33)$$


Now, with this equations defined and after we integrate these equations, the elements of the matrices can be obtained. In those equations what we have to find out is $S_{u_i u_k}$ which is equal to $\frac{1}{\omega^2} S_{u_i}^2 S_{u_k}^2$ and coherence function i, k ; that is, it takes care of the partial correlation between the ground displacements at the i th support and the k th support. Since the power spectral density function of the ground displacements for all the supports are the same. Therefore, $S_{u_i}^2 S_{u_k}^2$ multiplication of these 2 terms out to be $S_{\ddot{u}_g}^2$, that is the power spectral density of the ground specified ground displacement. Then we have $S_{u_i u_j}$ that is equal to $\frac{1}{\omega^4} S_{u_i}^2 S_{u_j}^2$ multiplied by coherence function i, j . And which again turns out to be equal to $\frac{\text{coh}(i, j)}{\omega^4} S_{\ddot{u}_g}$. I am sorry, the in

the previous equation it was S_{uik} . So, that is again given as coherence $i k$ divided by ω^2 into S_{ug} .


And finally, we have S_{uk} and S_{ul} , that turns out to be simply coherence kl multiplied by S_{ug} . So, in terms of S_{ug} and a coherence function defined to take care of the partial correlation between the ground motions. One can obtain these terms S_{ui} , S_{uk} , S_{ul} and S_{kl} .

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- For a single train of seismic wave, $D_{ij} = D_i(\omega_j, \xi_j)$ that is displacement response spectrum for a specified ξ ; correlation matrices can be obtained if $\text{coh}(i, j)$ is additionally provided; S_{ij} can be determined from $D(\omega_j, \xi_j)$ (Eqn 5.6).
- If only relative peak displacement is required, third term of Eqn.5.26 is only retained.
- Steps for developing the program in MATLAB is given in the book.

Example 5.4 Example 3.8 is solved for El centro earthquake spectrum with time lag of 5s.



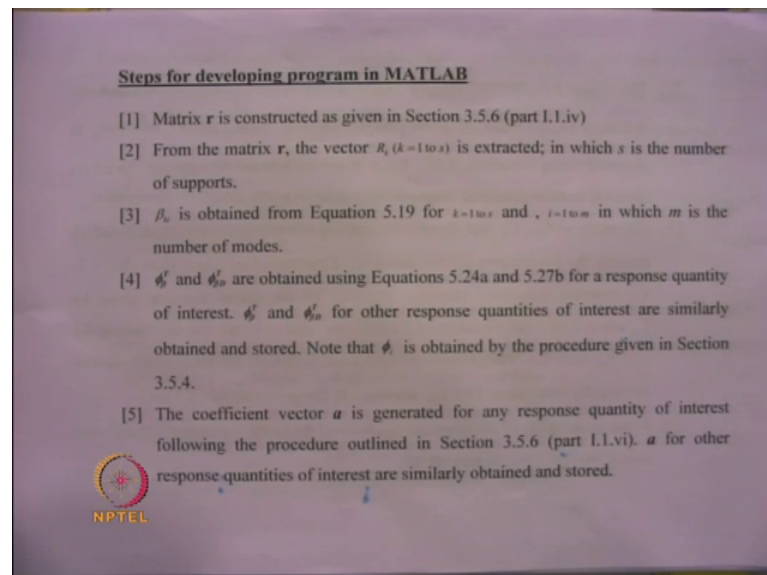
Now, for a single train of seismic wave D_{ij} becomes simply the value of the displacement response spectrum for the earthquake, which is given in this equation 5.27. They are D_{ij} is equal to becomes equal to $D_i(\omega_j, \xi_j)$ for the single train of earthquake; that is, it has a unique displacement spectrum. And S_{ug} can be obtained from the specified displacement spectrum from the relationship that we have discussed before.

So, for a given displacement spectrum of an earthquake, and then additionally specified coherence function to take care of the partial correlation between the ground motions, one can now obtain all the terms necessary to calculate the mean peak value of the response of the system. Does for a multi support excitation case, the one can find out the values of the expected peak value of the responses using the same response spectrum method of analysis. Only thing that is required over here is that in addition to defining

the displacement response function we have to define a coherence function, and a relationship that we equate the acceleration power spectral density function in terms of the displacement response spectrum of the earthquake.

Now, if only the relative peak displacement is required, then the thought term of equation 5.26, that is required. So, we the third term only gives the relative displacement, but if we take all the terms in equation 5.26, then this will give the absolute displacement. The steps for obtaining the response spectrum method of a response spectrum analysis for multi support excitation, using matlab is given in the book.

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In the matlab one can develop the method that is described, over here the first step would be to obtain a matrix R, which is constructed as we have discussed before. Then from the R matrix one can take out the R k vector; that is for the kth support and use it for finding out the mode participation factor for the mode for the kth support. Then beta k i is that is a beta k i that is obtained and this beta k i is obtained for all the supports from 1 to S, and for all the modes 1 to m.

Next phi T beta phi T b D they are obtained from the equations. And in obtaining phi T beta D, we require the ordinates of the displacement response spectrum at different modes corresponding to the time periods of those modes. We also require the mode shape coefficient for the response quantity of interest, as well as we require the mode participation factor for a particular mode coming from a particular support. So, all

these once these quantities are known. So, therefore, $\phi T b \beta D$ and $\phi T \beta$ can be constructed. Once we have these 2 vectors. Next vector that is required is the a vector; that is a influence coefficient vector which comes from the pseudo static analysis, and that can be stored for a particular response the pseudo static vector, and for the displacement the pseudo static vector they could be different.

Example 5.5: Find the values of mean peak displacement and bending moment at the centre of the pipeline, shown in Figure 3.15, for El Centro earthquake. Assume a time lag of 2.5s between the supports. Use the correlation function given by Equation 2.94 and use the digitised values of the response spectrum of El Centro earthquake given in the Appendix 5.


Solution: For the bending moment, the quantities required for the calculation of mean peak values are given below (taken from the example problem 3.10)

$$\phi = \begin{bmatrix} 0.5 & -1 & 1 \\ 1 & 0 & -0.5 \\ 0.5 & 1 & 1 \end{bmatrix}, \quad \bar{\phi} = \begin{bmatrix} 1 & 0 & -0.5 \\ 8mL & 16mL & -24mL \end{bmatrix}$$

$$r = \begin{bmatrix} 1 & 0.002 & -0.152 \\ -0.026 & 1 & -0.026 \\ -0.152 & 0.002 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.8139 & -0.0361 & 0.0182 \\ -0.0361 & 0.9562 & -0.0361 \\ 0.0182 & -0.0361 & 0.8139 \end{bmatrix} m$$

$\omega_1 = 8.1 \text{ rad/s}, \omega_2 = 9.8 \text{ rad/s}, \omega_3 = 12.2 \text{ rad/s}$

$\begin{bmatrix} -0.026 & 1 & -0.026 \\ 16.42mL & 0 & 16.42mL \end{bmatrix}$ m and l are shown in Figure 3.15.




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[6] The vector b is obtained using Equation 5.27a; u_n is obtained from Equations 2.19(a-c) for the PSDF of ground displacement at the support ($S_w = \frac{S_a}{\omega^2}$); S_w is obtained using Equation 5.16a for a given displacement response spectrum (derived from acceleration response spectrum) of earthquake.

[7] The matrices ℓ_{aa} , ℓ_{ac} and ℓ_{cc} are determined using Equations 5.28-5.30 which provide the expressions for the elements of the above matrices. Note that h_i is the complex frequency response function for the modal equation given by Equation 5.18, and Equations 5.31-5.33 are used to obtain the expressions for S_{u_i} , $S_{u_i u_j}$ and $S_{u_i \ddot{u}_j}$. Integrations of Equations 5.28-5.30 are numerically carried out and the cross terms like ℓ_{ca} are taken as the conjugate of ℓ_{ac} .

[8] Mean peak value of the response is obtained by using Equation 5.26 and step (xi) of Section 5.4.1, if appropriate.



Next is to construct a vector b , and these vector b we have is shown in expression 5.27. This vector b requires the peak ground displacement at different supports, but this peak ground displacement at different supports are assumed to be the same because we have the same train of earthquake moving through all the supports. This can be obtained from

the specified power spectral density function, on for the support excitation and then converting it to the power spectral density function for displacement. The area under that curve give the standard deviation. And that standard deviation can be multiplied by a peak factor in order to obtain the peak value of the or expected value of the ground displacement.

The matrices $\|u\|$ and $\|z\|$, or they require basically the integration of the cross power spectral density function multiplied by the frequency response function for each mode, which was shown in equation 5.28 to 5.30. And in that we had seen that we require also the standard deviation of the generalized displacement in a particular mode for the excitation law the k th excitation or the excitation at the k th support. So, that standard deviation can be obtained by assuming the power spectral density function, or not assuming by obtaining the spectral analysis for the specified power spectral density function of the ground motion. And from that one can get the power spectral density function of the generalized displacement.

The area under the curve will give the standard deviation. The cross power spectral density function that is required between the displacement at one support with the acceleration at the other support. The cross power spectral density function between the displacements at 2 supports and the cross power spectral density function between the acceleration at 2 supports which are required in those integrations. They are obtained again by a set of equation which can be finally, represented in the form of a coherence function. And the specified power spectral density function of the ground acceleration. Coherence function is additionally provided for this problem.

So, one can develop a program in easily in the matlab for finding out the response or the mean peak response for a systems having multi support excitation, and subjected to a set of ground motions which can be represented with the help of an averaged response spectrum; which can be again converted to a ground acceleration of spectrum through an empirical relationship. We illustrate the method with the help of a problem. This is a problem if we recall; is a problem for a 3-support frame.

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Solution : The quantities required for calculating the expected value are given below:

$$\phi = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix}; \bar{\phi} = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix};$$

$$r = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, w_1 = 12.24 \text{ rad / s};$$

$$w_2 = 24.48 \text{ rad / s}; a^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix};$$

$$\phi^T \beta \begin{bmatrix} \bar{\varphi}_{11}\beta_{11} & \bar{\varphi}_{11}\beta_{21} & \bar{\varphi}_{11}\beta_{31} & \bar{\varphi}_{12}\beta_{12} & \bar{\varphi}_{12}\beta_{22} & \bar{\varphi}_{12}\beta_{32} \\ \bar{\varphi}_{21}\beta_{11} & \bar{\varphi}_{21}\beta_{21} & \bar{\varphi}_{21}\beta_{31} & \bar{\varphi}_{22}\beta_{12} & \bar{\varphi}_{22}\beta_{22} & \bar{\varphi}_{22}\beta_{32} \end{bmatrix}$$

That is example 3.8 we had a a 2-study frame with 3 supports. The non-support degrees of freedom are 2, and therefore, the mode shapes are 2 by 2 mode shape matrix. And since we are wanting to find out the expected peak value of the displacements, then phi bar also becomes the same matrix. The r matrix for the multi support excitation case, was computed before for this problem, and r is given by this matrix. The 3 frequencies are 2 frequencies are omega one and omega 2. So, they are also given. A T that is the pseudo static coefficients at the nonsupport degrees of freedom, coming from all the 3 excitations they are same as r therefore, a T is same as r.

This is the phi T beta matrix, and the phi T beta matrix will have these sets of this is the first set phi bar 1 1 beta 1 1 phi bar 1 1 beta 2 1 phi bar 1 1 beta 3 1. All these things I have been explained before. And phi bar 1 2 phi bar 2 2 or beta 2 1 beta 1 2 etcetera they can be all obtained.

(Refer Slide Time: 45:28)


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$$\phi_{\beta D}^T = \begin{bmatrix} 0.0259 & 0.0259 & 0.0259 & -0.0015 & -0.0015 & -0.0015 \\ 0.0129 & 0.0129 & 0.0129 & 0.0015 & 0.0015 & 0.0015 \end{bmatrix}$$

$$D_{11} = D_{21} = D_{31} = D(\omega_1 = 12.24) = 0.056\text{m}$$

$$D_{12} = D_{22} = D_{32} = D(\omega_2 = 24.48) = 0.01\text{m}$$

$$\text{coh}(i, j) = \begin{bmatrix} 0 & \rho_1 & \rho_2 \\ \rho_1 & 0 & \rho_1 \\ \rho_2 & \rho_1 & 0 \end{bmatrix}; \quad \rho_1 = \exp\left(\frac{-5\omega}{2\pi}\right);$$


$$\rho_2 = \exp\left(\frac{-10\omega}{2\pi}\right)$$


The phi T beta D matrix that is computed over here, it requires as I told you before. The ordinates of the response displacement response spectrum, then phi terms and the terms for the mode participation factor for a particular mode. The displacement response spectrum which is assumed to be the same for all the 3 supports because it is the same train of earthquake that is passing through all the 3 supports; that is obtained for the time periods corresponding to this 2 frequencies and they are 0.056 and 0.011.

Coherence function or coherence matrix can be obtained easily. These will not be 0 this will be 1 1 1, and rho 1 and rho 2 will be minus 5 omega by 2 pi and minus 10 omega by 2 pi, because it is assumed that that is a 5 second time lag between the supports. So, between the first support and the second support this is the or this will be the value of rho 1 we have computed it also before also. And for first support and a third support or the rho 2 value will be equal to minus 10 omega 2 pi. So, plugging in the values of rho 1 rho 2 etcetera in this coherence matrix we get a complete coherence matrix.

(Refer Slide Time: 47:14)


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$$l_{uu} = \begin{bmatrix} 1 & 0.873 & 0.765 \\ 0.873 & 1 & 0.873 \\ 0.765 & 0.873 & 1 \end{bmatrix}$$
$$l_{u\bar{z}} = \begin{bmatrix} 0.0382 & 0.0061 & 0.0027 & 0.0443 & 0.0062 & 0.0029 \\ 0.0063 & 0.0387 & 0.0063 & 0.0068 & 0.0447 & 0.0068 \\ 0.0027 & 0.0063 & 0.0387 & 0.0029 & 0.0068 & 0.0447 \end{bmatrix}$$


L u u l u u z bar matrix they are again computed, that is the that integrations were performed and we finally, obtained each of these elements of those matrices and constructed this matrix l u u. Similarly, the matrix l u z bar that was constructed.

(Refer Slide Time: 47:38).

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$$l_{z\bar{z}} = \begin{bmatrix} 1 & 0.0008 & 0.0001 & 0.0142 & 0.0007 & 0.0001 \\ 0.0008 & 1 & 0.0008 & 0.0007 & 0.0142 & 0.0007 \\ 0.0001 & 0.0008 & 1 & 0.0001 & 0.0007 & 0.0142 \\ 0.0142 & 0.0007 & 0.0001 & 1 & 0.0007 & 0.0001 \\ 0.0007 & 0.0142 & 0.0007 & 0.0007 & 1 & 0.0007 \\ 0.0001 & 0.0007 & 0.0142 & 0.0001 & 0.0007 & 1 \end{bmatrix}$$


And l z z bar matrix also was constructed.


(Refer Slide Time: 47:45)

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➤ Mean peak values determined are:
 $(u_1)_{tot} = 0.106m$; $(u_2)_{tot} = 0.099m$
 $(u_1)_{rel} = 0.045m$; $(u_2)_{rel} = 0.022m$

➤ For perfectly correlated ground motion

$$I_{uu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_{\bar{z}\bar{z}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

 = null matrix

With the help of these matrices finally, we obtain the mean peak values of the displacements. That is for the displacement 1; the mean peak value was obtained as 0.106 and for the second non support degree of freedom. It was 0.099 meter. When we obtain the relative displacement for the response one, then the relative displacement mean peak value of the relative displacement came out to be 0.045, and for the second nonsupport degree of freedom the mean peak value of the displacement came as 0.022.

Note that we can obtain both the mean peak values of the total response, and the mean peak value of the relative displacement. The mean peak value of the total displacement means, the relative displacement plus the pseudo static component they are taken together and for that we obtain the mean peak values. And the relative displacements they come purely from the modal displacement \bar{z} multiplied by the mode shape factor. The general expression that we had shown in the development of the method are there if we retain only the these term then we get the if we retain only the third term and ignore all other terms, then we get the straight away the relative displacements or the mean peak values of the relative displacements.

So, that is what was done over here.

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➤ Mean peak values of relative displacement

<i>RSA</i>	<i>RHA</i>
$u_1 = 0.078m$;	$0.081m$
$u_2 = 0.039m$;	$0.041m$

➤ It is seen that's the results of RHA & RSA match well.

➤ Another example (example 3.10) is solved for a time lag of a 2.5 sec.

➤ Solution is obtained in the same way and results are given in the book. The calculation steps are self evident.



To get the value of the or mean peak value of the relative displacements. We also solve the problem assuming that the ground motions are perfectly correlated. In that case I_{uz} and $I_{z\bar{z}}$ they take this particular form. Using those we obtain the values of the mean peak relative displacement by response spectrum method of analysis for u_1 and u_2 , they turn out to be like this. For the time history analysis, for the same problem and for the same 1 sent to earthquake record for which the power spectral density function and the corresponding displacement response spectrum were used. That gave us a values which are 0.081 and 0.041, and we can see that these 2 values match quite well.

Thus, in spite of the different assumptions, that have been considered in the development of the response spectrum method of analysis for multi support excitation case. We see that the results that you obtained from the response spectrum method of analysis compare quite well with the time history analysis.

(Refer Slide Time: 51:45)

Cascaded analysis

➤ Cascaded analysis is popular for seismic analysis of secondary systems (Fig 5.5).

The diagram illustrates the cascaded analysis method. On the left, a building frame is shown with a 'Secondary System' mounted on a floor. The floor displacement is denoted as \ddot{x}_f and the absolute displacement of the secondary system as \ddot{x}_a . On the right, a single-degree-of-freedom (SDOF) model is shown, consisting of a mass m supported by a spring with constant k and a damper with coefficient c . The displacement of the mass is \ddot{x}_a . The relationship between the floor displacement and the absolute displacement is given by the equation $\ddot{x}_a = \ddot{x}_f + \ddot{x}_a$.

Secondary system mounted on a floor of a building frame

Fig 5.5

SDOF is to be analyzed for obtaining floor response spectrum

$\ddot{x}_a = \ddot{x}_f + \ddot{x}_a$

We explain the same method with the help of another example, and in that if you recall we solved a problem, which has a beam resting on 3 spring supports. That is the problem.

This problem, this was a beam which is resting on a soil.

(Refer Slide Time: 52:16)

The diagram shows a beam of length L supported by three soil springs. The beam has a total mass m , with $m/2$ at each end. The soil springs have stiffness k_s and damping c_s . The beam is subjected to three point loads: $m/2$ at the left end, m at the center, and $m/2$ at the right end. The damping constant for the soil is given as $c_s = 0.6m$ and the spring constant as $k_s = 98m$. The structural damping is given as $\tau = 2.55$.

$C = \alpha M + \beta K$ for structure. $k_s = 98m$

$\bar{C} = \begin{bmatrix} 0.813 & -0.035 & 0.017 \\ -0.035 & 0.952 & -0.035 \\ 0.017 & -0.035 & 0.813 \end{bmatrix}$ Non classical

Soil is replaced by the spring and dust pots, this as a spring constant case and case and the damping constants are c_s for the soil. The structural damping for the structure is taken as 5 percent; these are the masses which are lambded at the 3 degrees of freedom.

Translations are the 3 degrees of freedom. Rotation is condensed out. There is a time lag which is assumed between the supports the time lag is 2.5 second. C_s value is given as 0.6 meter, and k_{th} value is given as 48; this is not meter m , and this is k S is equal to 48 m . The structural damping matrix is obtained as c is equal to αm plus βk . The \bar{c} matrix that is the damping matrix considering the soil damping that turns out to be this.

Note that once we add the soil damping to the structural damping the damping becomes a non-classical damping. So, this problem is a problem of non-classical damping. Therefore, to use the response spectrum method of analysis for this becomes really difficult; however, by making an approximation, that is by diagonalizing the damping matrix that is the modal damping matrix or in other words $\phi^T \bar{c} \phi$. For that the off-diagonal terms are ignore. And with the help of that we solve the same or this problem and obtain the responses.