Engineering Hydrology

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Module: 1

Lecture 4: Derivation of Reynolds Transport Theorem Part - I

Hello all, welcome back to the NPTEL MOOC on engineering hydrology. In today's lecture, we are going to see the derivation of Reynolds transport theorem. So, before starting that, let us have a look into the previous lecture. The previous lecture we have seen for understanding the complex hydrologic processes, we need to model them.

For modeling these processes, we need to have a consistent mechanism which can be attained through well-known theorem known as Reynolds transport theorem. So, while proceeding towards Reynolds transport theorem, we have seen different approaches which are used for studying fluid flow problems, such as Lagrangian approach and Eulerian approach. We need to look into those approaches again in this lecture.

After that, we have seen different fluid properties which are dependent on mass and which are independent of mass. Those are termed as extensive properties and intensive properties. After that, we have seen the relationship between these extensive properties and intensive properties.

Now, let us move on to Reynolds transport theorem in this lecture. The derivation of Reynolds transport theorem is divided into 2 parts that will be covered in 2 lectures.

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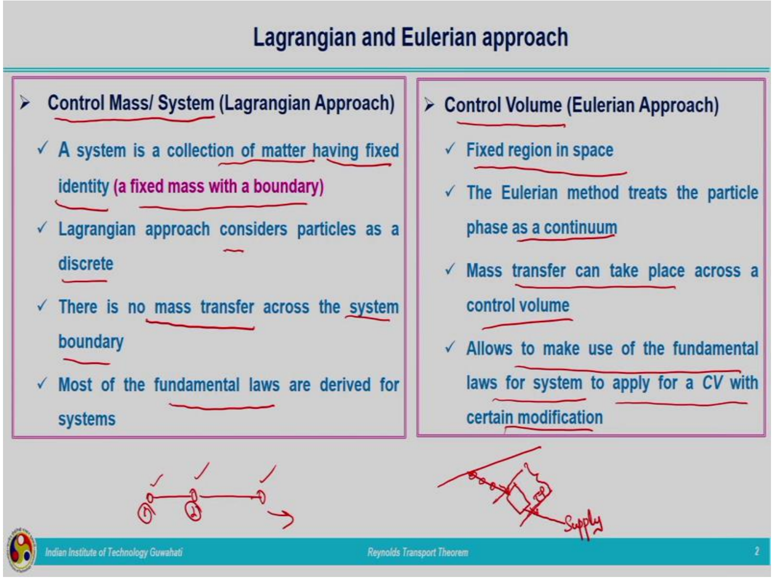

So, first let us see again in detail what is meant by Lagrangian approach and Eulerian approach? So, one thing is that control mass or system and second one is the control volume. So, if we are talking about the fluid flow, we can analyze the fluid flow by making use of Lagrangian approach and Eulerian approach.

So, if you are making use of Lagrangian approach, we are making use of control mass or system and if you are making use of Eulerian approach, we are making use of the control volume principles. Let us see in detail what is the difference between these two?

System is a collection of matter which is having fixed identity. So, that is, it has a fixed mass with a boundary. In the Lagrangian approach, we have seen earlier itself, we are tracing the fluid particle, it may be an individual particle or a number of particles together. So, we are tracing the flow path of these individual particles from the beginning to the destination.

But in the case of Eulerian approach, what we are doing? We are not interested in the entire flow path of the fluid particle. We are fixing a frame of reference and with that frame of reference, we will be analyzing the fluid flow properties, we will be studying the fluid flow characteristics.

And in the Eulerian perspective, control volume is the fixed region in space. In the case of Lagrangian approach, we consider particles as discrete, but in the case of Eulerian approach, we are considering the fluid phase as a continuum. That is in the case of Lagrangian, we are considering the particle as a discrete particle. It may be a number of particles together or a single particle. So, the mass of those fluid matter which is considered is not going to change in the case of Lagrangian approach. But in the Eulerian approach, we are fixing a frame of reference and whatever coming inside and some amount will be leaving outside that frame of reference and we are interested in the fluid matter which is within the frame of reference. So, the mass will be changing in that case. Here there is no mass transfer in the case of Lagrangian approach across the system boundary that you should understand. But in the case of control volume, mass transfer can take place across the control volume.

Basically, most of the fundamental laws are developed based on Lagrangian approach. But making use of those theorems which are derived based on Lagrangian approach, if you are going to apply for engineering problems, slightly difficult.

So, what control volume approach is suggesting, it allows to make use of the fundamental laws for the system, that is which is derived based on the Lagrangian concept to apply for a control volume with certain modification. So, I will tell you one thing, in the case of Lagrangian approach, we were considering an individual particle or a group of particles.

At time $t = t_0$, it has reached some other location and it is moving to another location as time passes. So, the observer is tracing the particle continuously from the beginning of the flow to the ending of the flow. But in the case of Eulerian approach, what we are doing? You imagine the case of a river, we are having a river flow, from that we are taking some water. That water will be treated in the treatment plant and from there, after treatment, water will be supplied for domestic requirements and all other requirements. So, water supply will be starting after the treatment. So, if we want to study something related to the flow within the treatment plant, what we will do, we will fix frame of reference which contains the treatment plant. We are not bothered about the fluid which is coming from the river. Okay, and wherever it traverses before it reaches the frame of reference, we are not giving importance to that. We are giving emphasis to the flow once it enters the frame of reference until it reaches the end of the frame of reference. After that, once it leaves, we are not tracing that fluid particle and we will be doing the analysis within this fixed space.

But in the case of Lagrangian approach, we are following the particle from first location to second location and finally, it reaches the destination. So, this is the difference, main difference between Lagrangian approach and Eulerian approach. Lagrangian approach, the observer is tracing the fluid particle. But in the Eulerian approach, we are not giving emphasis to a single particle, but we are giving emphasis within a fixed frame of reference, where we need to do the fluid flow analysis.

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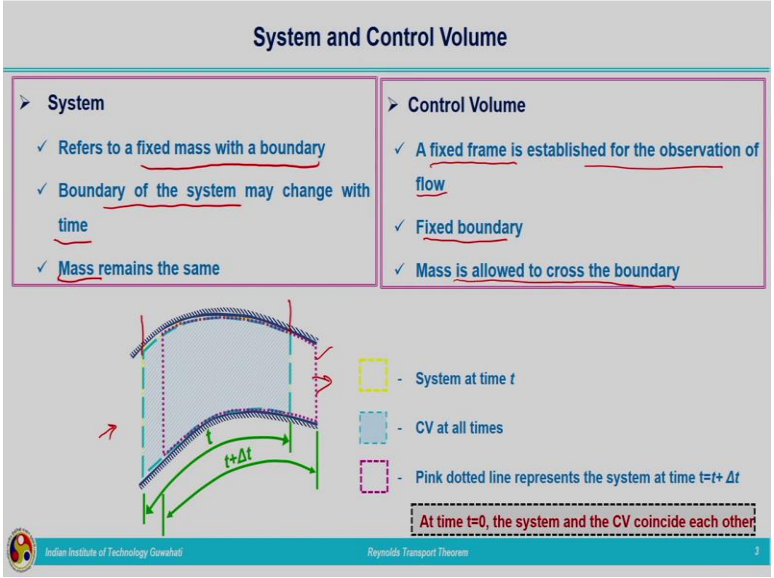

Again, I am coming to system and control volume concept. So, system refers to a fixed mass within a boundary. The boundary of the system may change with time, that is the system is moving from one location to another location as time changes. It is not stationary; it is moving from one location to another location. Thus, in the case of system mass remains the same.

But in the case of control volume, we are having a fixed frame that is established for the observation of flow. It has got a fixed boundary and the mass within the control volume will be changing because it is allowed to cross the boundary. These are the differences between system and control volume.

We can consider an example to understand the system and control volume. Let us take a domain in which fluid flow is taking place. So, within this region fluid flow is taking place in this way from left to right and we are considering the system in a time period of t that is represented by yellow dotted line.

The fluid within this flow field is contained within this yellow dotted line that is what is considered as the system at time t. Now, when we are talking about the control volume, we are considering the same system as the control volume. This is our frame of reference and within that we are considering the control volume at time *t*.

And when time changes to, this is our control volume for all times. When time changes to $t+\Delta t$, what happens, the system moves in the flow direction that is represented by this pink dotted line. So, whatever fluid was occupying the space marked by yellow dotted lines is now occupying the new space which is represented by this pink dotted line.

So, you should understand from this when time *t*, both the system and control volume were coinciding. But when time changed to $t+\Delta t$, the system has moved from the initial position, but the control volume is remaining at the same place, that is what is meant by this. That is at time $t=0$, the system and the control volume coincide each other.

So, control volume always remains at that particular location if we are considering a stationary control volume. There are certain problems which consider movable control volume, but as far as this course is considered, we will be making use of stationary control volume which is fixed in space.

So, as time changes, there will not be any change happening to the control volume. But as far as the system is concerned, it will be changing its position as time changes, but mass contained within the system will not be changing. But if we are considering the control volume, as the time changes, there will be changes taking place in the mass of the fluid which is contained within that control volume.

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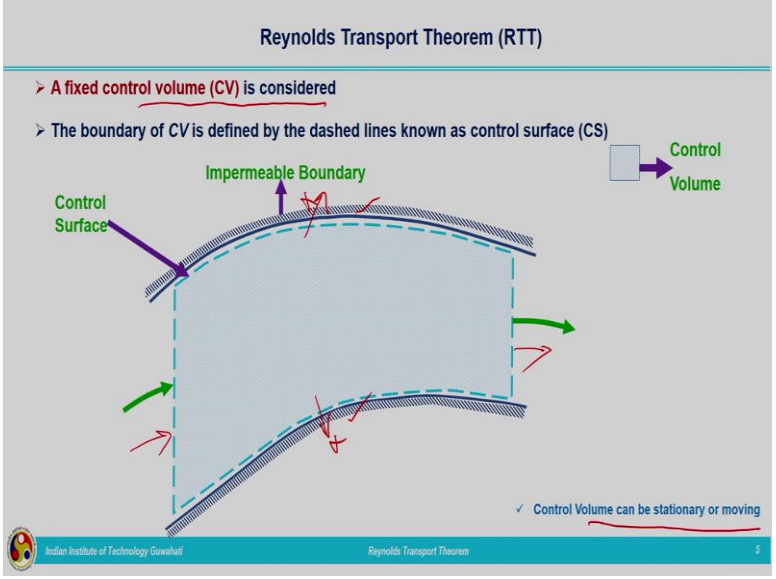

Now, let us move on to Reynolds transport theorem. Reynolds transport theorem is based on Eulerian approach. And here what we are doing? We are going to consider a fluid flow. We can have the pictorial representation of the fluid flow within this domain. So, you can see the flow is taking place within this domain and we are going to consider 2 sections, section 1 and section 2. The fluid flow is taking from left to right.

Now, within this domain, we are going to consider our control volume. Now, as far as the fluid property is concerned, the fluid is having a density of ρ. So, we know the relationship between density and mass of the fluid. What is density actually? Density is given by the expression mass divided by volume.

So, if we know, density and volume, we can calculate the mass of fluid which is contained within the particular domain. So, what we are going to see? We are going to see a fixed control volume within the flow domain, that is represented by this shaded region. Our control volume is this shaded region. This control volume is having a boundary that boundary is known as the control surface. These blue dotted lines represent the boundary of the control volume which is known as the control surface. Now, you look at the figure, this boundary and the bottom boundary both are impermeable. So, you can see water flow will not be taking place through these boundaries. There will not be any flow through this boundary and through this boundary, no flow.

But water flow will be taking place from the left-hand side to right hand side. Here again, I am repeating control volume can be stationary or moving. Here in our study we will be considering a stationary control volume.

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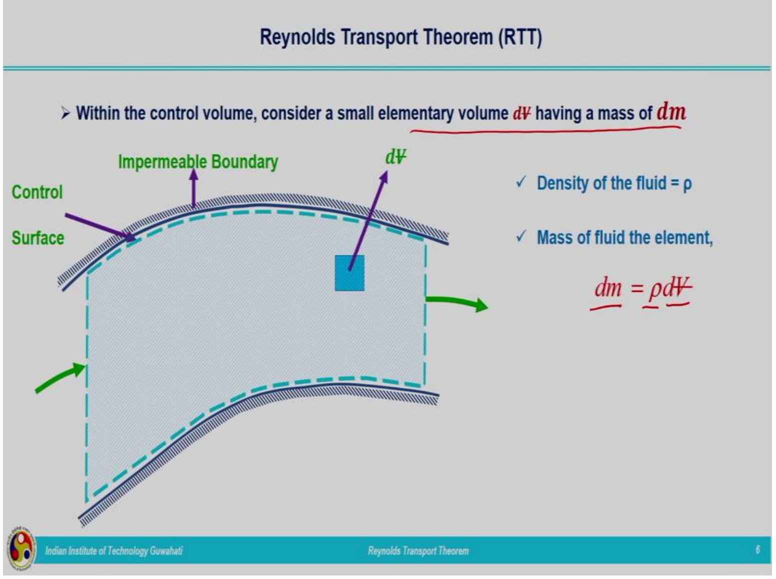

Now, what we have going to do? We are going to consider an elementary volume $d\forall$ having a mass of *dm* within the control volume. So, this is the elementary volume $d\forall$ which we are considering within the control volume and corresponding to this volume, there will be a mass because we know the density, density we have assumed to be ρ.

So, the mass of the fluid element contained within this volume $d\forall$ is *dm*, which is nothing but density multiplied by the volume, that is density ρ multiplied by volume $d\forall$ will give you the mass of the fluid contained within this elementary volume.

$$dm = \rho d\forall$$

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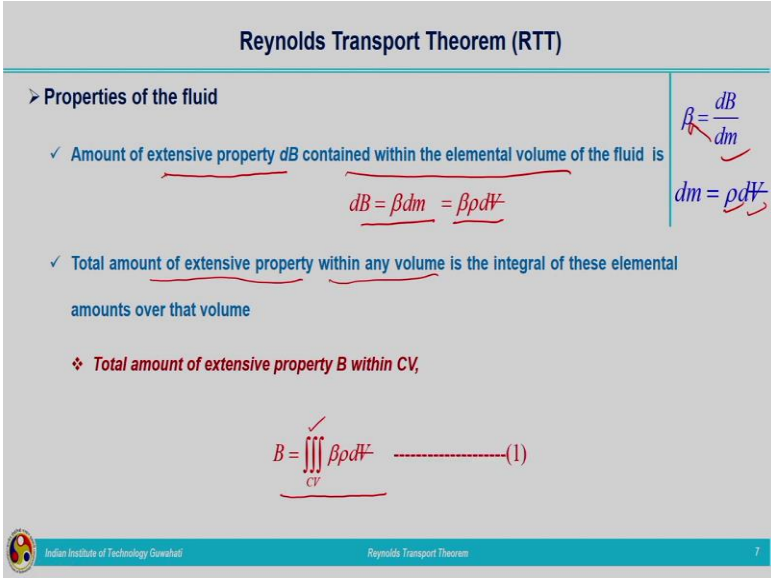

Now, coming to the fluid properties, that is the extensive and intensive properties. We know the relationship between them, that is β (extensive property to the unit mass), that is $\beta = \frac{dB}{dm}$. Now, what we are going to do? Amount of extensive property, dB contained within the elemental volume of fluid, we can write the expression for that.

So,
$$dB = \beta dm$$

dB is the amount of extensive property contained within the volume $d\forall$ that is the elemental volume which we have considered. So, dB is given by β multiplied by dm. Now, what is dm? dm we are having the expression because we know the density and we are considering a volume of $d\forall$,

$$dm = \rho d\forall$$

So, that we can substitute here

$$dB = \beta\rho d\forall$$

Now, this is the extensive property which is contained within the elemental volume. So, the total amount of extensive property within any volume, we can obtain by integrating the value corresponding to the elemental volume.

So, the extensive property within any volume will be equal to total amount of extensive property *B* within control volume.

$$B = \iiint_{CV} \beta \rho \, d\forall \text{ ----------- } (1)$$

This is triple integral or volume integral because we are talking about the extensive property within the control volume. This equation we can name it as equation 1.

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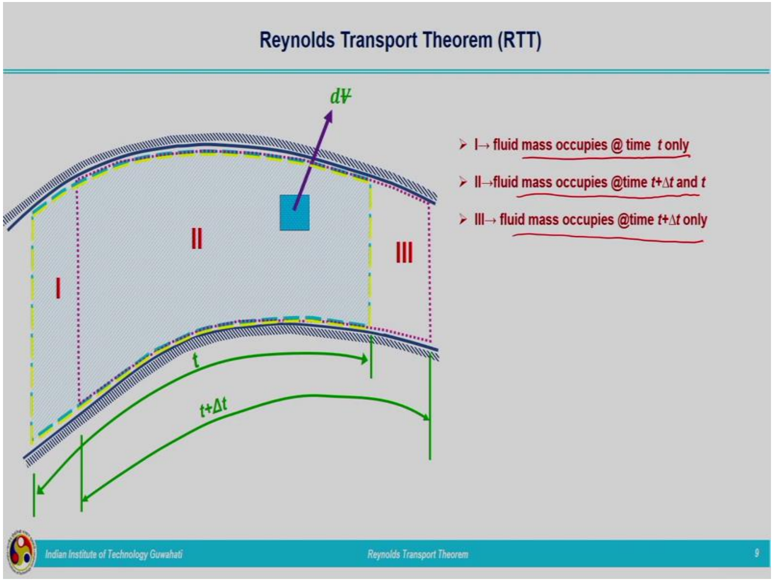

Now, what we are going to do? We are going to consider the control volume for the time *t* and as we know control volume is not going to change as time changes. So, for the time *t*, we are having the control volume and also, we are considering the system, that is the control mass. So, a system is marked by the yellow dotted lines which is coinciding with the control volume. System at time *t* is marked by this yellow dotted line.

Now, after a small interval of time *Δt*, what will happen to the system? System will be moving towards the right-hand side, that is represented by this pink dotted line. Initially at time *t*, the system was coinciding with the control volume, then the time changed to $t = t + \Delta t$, the system has moved to the right-hand side.

Now, the system is located at another location and again time changes, it will be moving again in the downstream direction, it will be moving in the right-hand direction in which the flow is taking place. So, the system at $t = t + \Delta t$ is marked by the dotted lines, pink dotted lines. Now, what we can do? While clearly looking into the figure, we are having 3 different regions, region I, region II and region III.

Initially control volume and system was occupying region I and II. When the time has changed to $t = t + \Delta t$, system has moved towards the right-hand side and it was occupying new region which is represented by region II and region III. That is region I is the region where the fluid mass is occupying at time *t* only. Region II is the region where the fluid mass is occupied at time $t = t + \Delta t$ and *t*. And region III is a region where the fluid mass occupies at time $t = t + \Delta t$ only. This should be very clear. We have divided the regions into 3. Region I is occupied by the flow which is entering into the control volume that is only there at time *t*. As time change to $t = t + \Delta t$, the system has moved to regions 2 and 3. But there is no change as far as the control volume is concerned.

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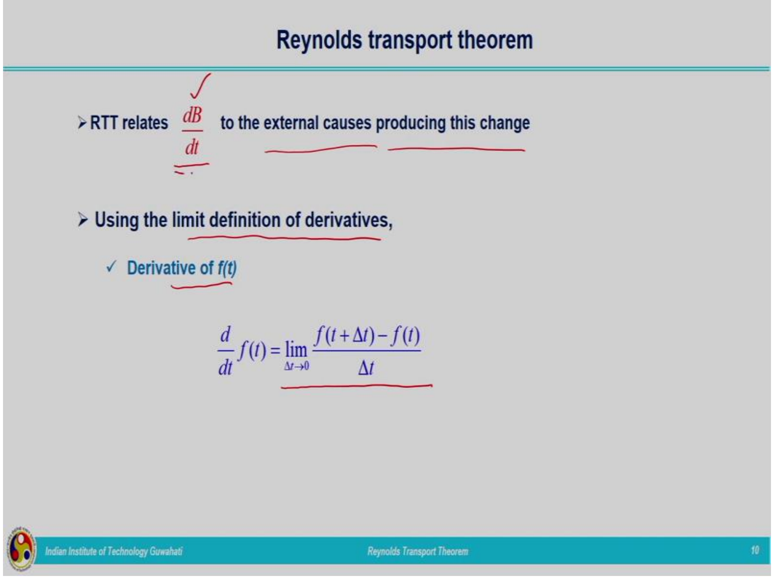

Now, coming to Reynolds transport theorem, what Reynolds transport theorem is doing? RTT is relating the time rate of change of extensive property that is given by $\frac{dB}{dt}$ to the external causes which is producing this change. For example, if you are having the fluid flow, if there is any change taking place within the fluid flow, some external factor will be causing that. So, RTT is relating the time rate of change of extensive property to the external cause, which is producing that change.

Now, for getting the time rate of change of extensive property, $\frac{dB}{dt}$ we can make use of the limit definition of derivatives that is from the basics of calculus.

$$\frac{d}{dt} f(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

We know what is the derivative of a function based on the limit definition. So, same thing we are going to apply, here in the case of extensive property also.

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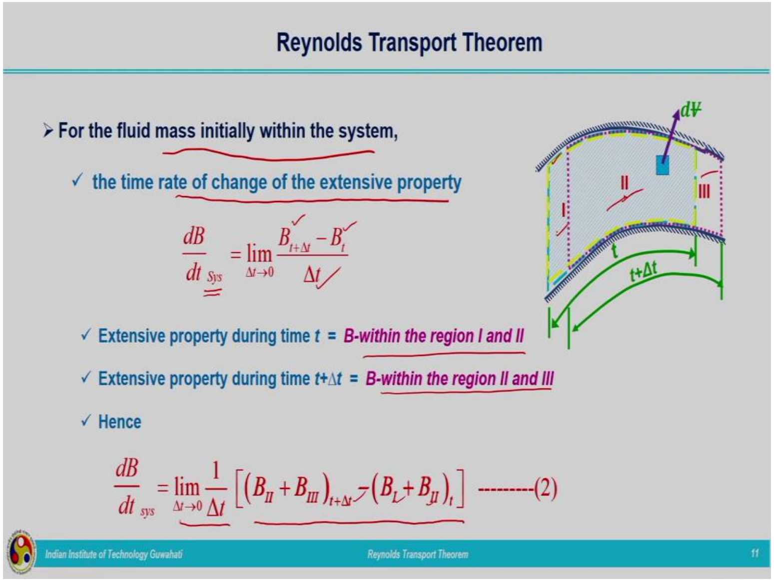

So, the same figure repeated here for getting clear idea. We are going to write the expression that is for the fluid mass initially within the system. We are going to write the expression $\frac{dB}{dt}$, that is the time rate of change of extensive property. $\frac{dB}{dt}$ is given by, that is it is something related to the system at time $t = 0$ or at the beginning of the time, system and control volume coincide each other, that is blue dotted line and the yellow dotted line coincide each other. So, here we are going to write $\frac{dB}{dt}$ for the system $\left(\frac{dB}{dt}_{Sys}\right)$, you can see at time *t*, it was occupying region I and II and when time changed from *t* to $t+\Delta t$, it was occupying region II and III. So, we can write the expression for time rate of change of extensive property corresponding to the system as

$$\frac{dB}{dt}_{Sys} = \lim_{\Delta t \to 0} \frac{B_{t+\Delta t} - B_t}{\Delta t}$$

Extensive property corresponding to the fluid within the system at time $t+\Delta t$ minus the same property within the system at time t divided by the time Δt. That is extensive property during time t is B within region I and II and at $t+\Delta t$, the corresponding quantity is within the region II and III. Now, we can modify this expression that is $B_{t+\Delta t}$ in terms of region, region II, III and B_t in terms of region I, II we can write.

$$\frac{dB}{dt}_{Sys} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(B_{II} + B_{III}\right)_{t+\Delta t} - \left(B_I + B_{II}\right)_t \text{-------}(2)$$

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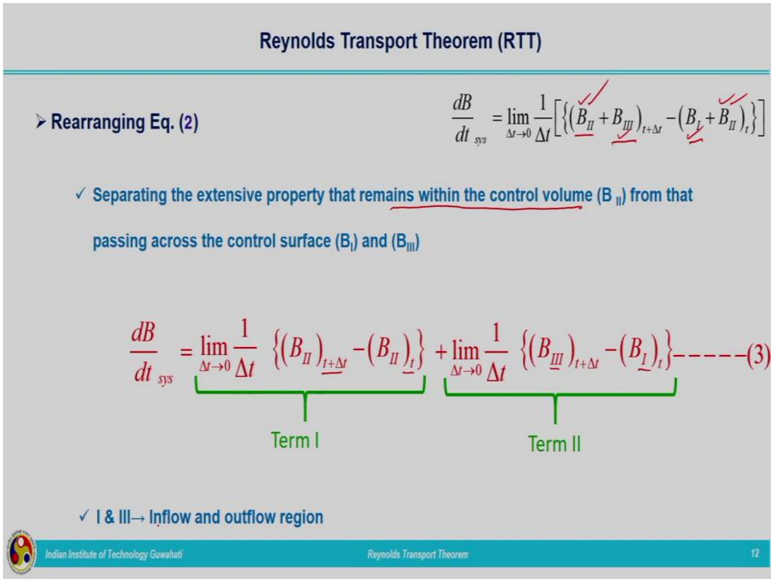

Now, what we are going to do? I have repeated the same expression here; this is equation 2 only. We are going to rearrange the terms. Look at the equation carefully. You can see B_{II} is there in both these terms. Other 2 terms are something related to region III and region I. So, what we will do? We will combine the terms related to B_{II} and rearrange the equation. That is, we are going to separate the extensive property contained within the control volume B_{II} from the other regions I and III.

$$\frac{dB}{dt}_{Sys} = \underbrace{\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\left(B_{II}\right)_{t+\Delta t} - \left(B_{II}\right)_t\right\}}_{\text{Term1}} + \underbrace{\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\left(B_{III}\right)_{t+\Delta t} - \left(B_I\right)_t\right\}}_{\text{Term2}} \text{------}(3)$$

Here you can see, 2 separate terms that is why we are calling the first term, term 1 and the second one term 2. Term 1 you clearly observe the expression terms, that is we are having the extensive property within the control volume at time $t+\Delta t$ and at time *t*.

Now, coming to term 2, we are having the extensive property corresponding to the fluid within the region III and within the region I. Now, what we are going to do? In order to make it simple, we will consider term 1 and term 2 separately and first let us see term 1. This equation can be named as equation 3. And II is corresponding to control volume, I and III inflow and outflow region.

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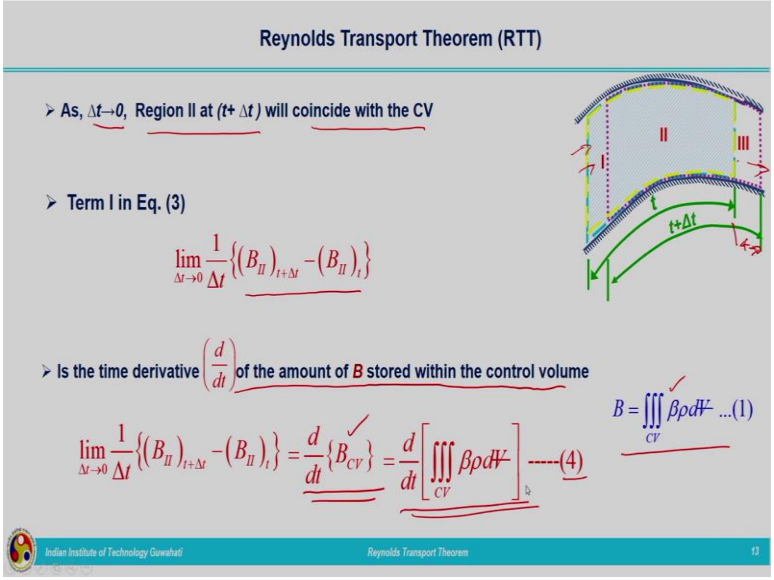

You look at the figure, as $\Delta t \to 0$, region II at $t+\Delta t$ will be coinciding with control volume. Say, region II that is initially we were having at region I and II, this is the region where the fluid is entering and this is the region where it is leaving. So, when Δt this much time, $\Delta t \to 0$, what will happen? It will be, this pink region will be coinciding with the yellow. When $\Delta t \to 0$ this will be coinciding with the control volume. Pink will be coinciding with yellow means, at time $t=0$ or at the beginning of the time, both the control volume and the system are coinciding each other that is what is written here.

As $\Delta t \to 0$, region II at $t+\Delta t$ will coincide with the control volume. So, the term 1 in equation 3, I am repeating here again, that is the term 1 is given by this expression,

$$\left.\frac{dB}{dt}\right|_{Sys} = \underbrace{\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left(B_{II}\right)_{t+\Delta t}-\left(B_{II}\right)_{t}\right\}}_{\text{Term1}} + \underbrace{\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left(B_{III}\right)_{t+\Delta t}-\left(B_{I}\right)_{t}\right\}}_{\text{Term2}}$$

So, both the terms are representing the extensive property within the control volume at time $t+\Delta t$ and t.

As $\Delta t \to 0$, you look at the expression this is nothing but time rate of extensive property within the control volume.

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left(B_{II}\right)_{t+\Delta t}-\left(B_{II}\right)_{t}\right\} = \frac{d}{dt}\left\{B_{CV}\right\}$$

So, regarding the first term which we got, we have started with time rate of change of extensive property within the system. When we have written the expression for time rate of change of extensive property within the system, we have made use of the principle of limits and then we got the right-hand side consisting of 2 terms.

We are dealing with both the terms separately. We have considered term 1. That was the property something related to control volume and we got the expression corresponding to term 1 as time rate of change of extensive property stored within the control volume.

Now, what is *B*? Expression for *B*, we have already represented by making use of the intensive property. Extensive property *B* is equal to

$$B = \iiint_{CV} \beta\rho\, d\mathcal{V}$$

So, within this expression, we can substitute for *B*, that is

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\left(B_{II}\right)_{t+\Delta t}-\left(B_{II}\right)_{t}\right\} = \frac{d}{dt}\left[\iiint_{CV} \beta\rho\, d\mathcal{V}\right] - - - - - - - (4)$$

So, when we are talking about the time rate of change of extensive property stored within the system, it has been separated into 2 terms. First term is found out to be time rate of change of extensive property within the control volume.

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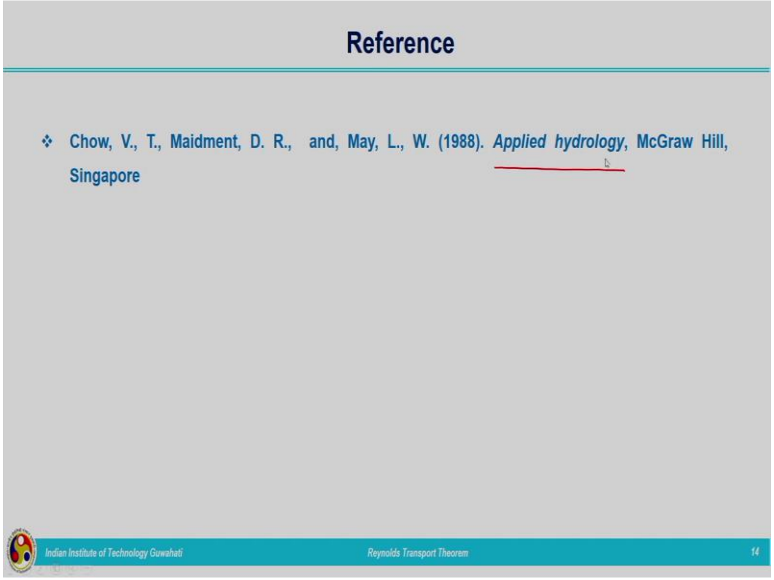

So, this part of the derivation, I have taken from the textbook by Ven Te Chow and others, applied hydrology.

So, I will summarize again. In this lecture, we have started with the Eulerian concept and the Lagrangian concept. Why we were discussing that? We wanted to understand the difference between them, because majority of the fundamental equations are derived based on Lagrangian concept or the system concept.

For the engineering problems with the help of that treatment plant, I told you, we need not have to analyze the flow which is starting from the river to the water supply. We just have to concentrate on the treatment plant for detailed analysis related to the flow within the treatment plant. So, we will be considering a fixed frame of reference, which is containing that treatment plant.

But, in the case of Lagrangian perspective, we will be considering each particle, we will be tracing the particle, each and every location of the particle will be traced by the observer. Reynolds transport theorem is developed based on the Eulerian principle. What it is doing? It is relating the time rate of change of extensive property within a system to that of an external factor which is causing the change. In that way, we have started deriving the equation and we have come up with 2 terms representing the time rate of change of extensive property within the system. And first term was found out to be time rate of change of extensive property within the control volume. Now, regarding the second part, we will see in the next lecture. Here I am stopping for today. Thank you. Have a nice day.