Numerical Methods in Civil Engineering

Prof. Arghya Deb
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 10
Iterative Methods - III

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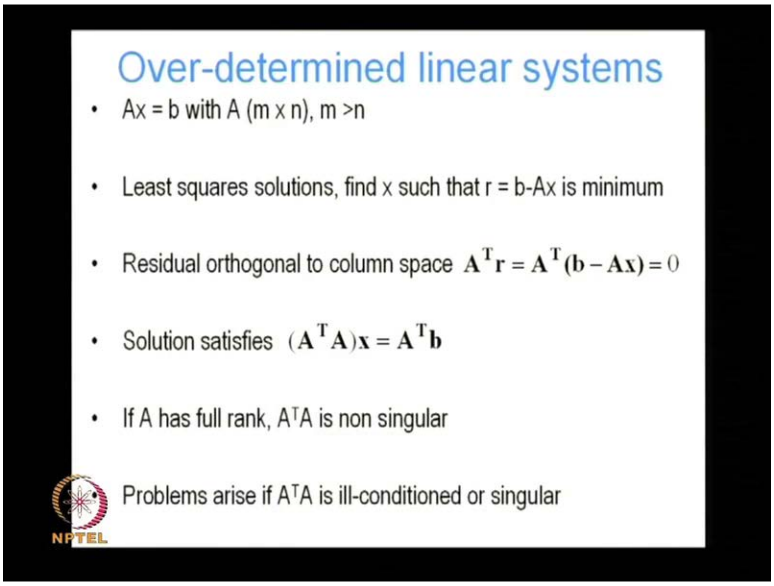

Once again, our series on numerical methods in engineering, recall last time, we talked about iterative method for over determined linear systems. What are over determined linear system? Where there are where there is a co-efficient matrix A, which has size m by n where m is greater than n, basically we have to note. So, those are over determined linear systems. So, for instant for example, for over determine linear system is when we have least when we want to find the least square solution to a problem A x is equal to b. So, we want to find the solution x, which is the best solution in some norm. So, instance for least square solution we want to find the solution which give the minimum residual in the l to norm right.

We found to find x such that r equal to b minus x is minimum and the minimum in the l to norm and we also recall that the residual if we minimize the function in the and find the minimize residual the l to norm of the residual, we find an x and we also find the residual which is equal to b minus A x and the residual is orthogonal to the space span by column vectors of A. The vector space span by the column of A the residual is orthogonal to that space or A transpose r is equal to zero right. So, the residual basically projects out the component it is basically the component left of the projecting b on to the space which is span by the column vectors of A and the solutions also satisfy.

We also recall this condition this relationship A transpose A x is equal to A transpose b and if a has full rank, we found last time that A transpose A is always going to have full rank two that is a transpose always going to have the positive determine it is going to be non singular. So, by solving this equation by inverting A transpose A we can find out a least square solution x. However, problems arise when A transpose A is in ill condition or singular if it is ill condition we cannot invert it with appropriate level of accuracy and if it is singular we cannot invert it at all.

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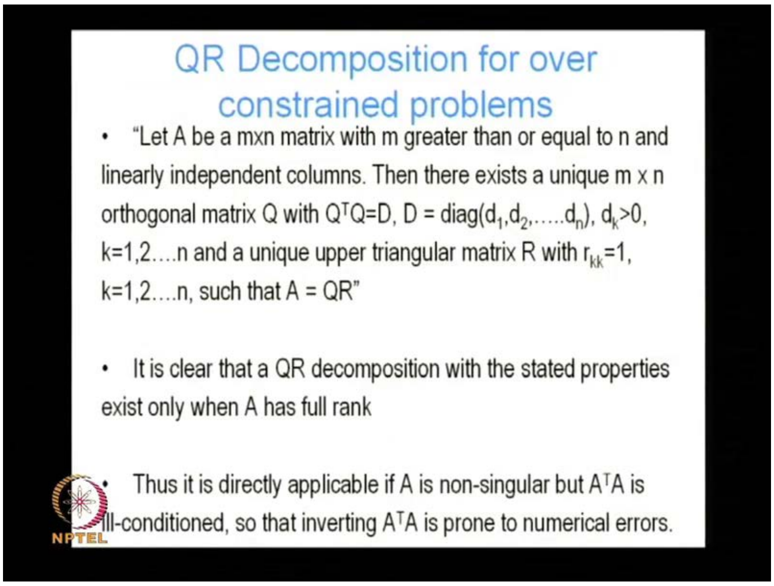

In that case, we said that the proper way to go about doing this problem is probably to do QR decomposition QR a decomposition in a QR D decomposition, which says there is the QR theorem with which say, let A be a m by n matrix with m greater to or equal to n and linearly independent column. Then there exist a unique m by n orthogonal matrix Q such that Q transpose Q is another diagonal matrix D with all positive entries and an unique upper triangular matrix R with one on all the diagonals right. You need upper triangle matrix R. So, A is equal to QR were Q is orthogonal matrix and R is unit upper triangle matrix.

It is clear that QR decomposition with the stated property exists only when A has full rank why is that because we have said that A has n linearly independent column right.

So, if it has n linearly independent column it is got to have full rank. So, the QR decomposition, assume that A has full rank thus it is applicable if A is non singular, but A transpose is ill condition. So, it is directly applicable to the case where A transpose A is ill condition. So, that a transpose A is prone to numerical errors.

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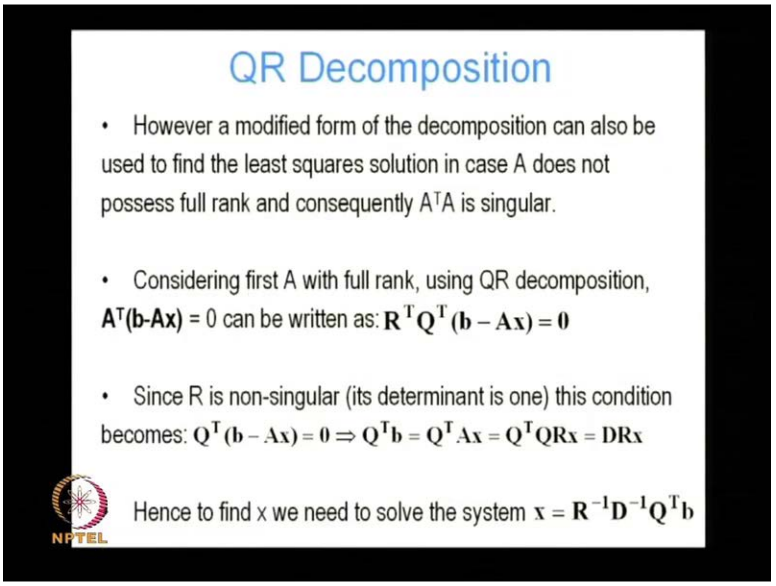

However even if A is singular that is a does not have all linear independent column we can use the modified form of QR decomposition to find the least square solution. Let us first consider the case were A has full rank. So, using QR decomposition we can replace A transpose by R transpose Q transpose R transpose Q transpose b minus A x must be equal to zero. Since A is non singular we call R is a unit upper triangular matrix and also recall that upper triangular matrix the determinant is given by the product of its diagonal term and since all it diagonal term is one it is evident that R must be non singular.

So, since R is the non singular is a non singular upper triangular matrix we know that R is invertible. So, this inflation is effectively becomes Q transpose b minus A x right because R is always invertible. So, Q transpose b minus A x is equal to zero implies Q transpose b is equal to Q transpose A x or Q transpose A x is again is equal to QR using the QR decomposition Q transpose QR x Q transpose Q is equal to the diagonal matrix D. So, Q transpose b is equal to D R x. Hence to find x we have to solve this system we have to invert this D R may product D R and we get x is equal to R inverse D inverse Q transpose b.

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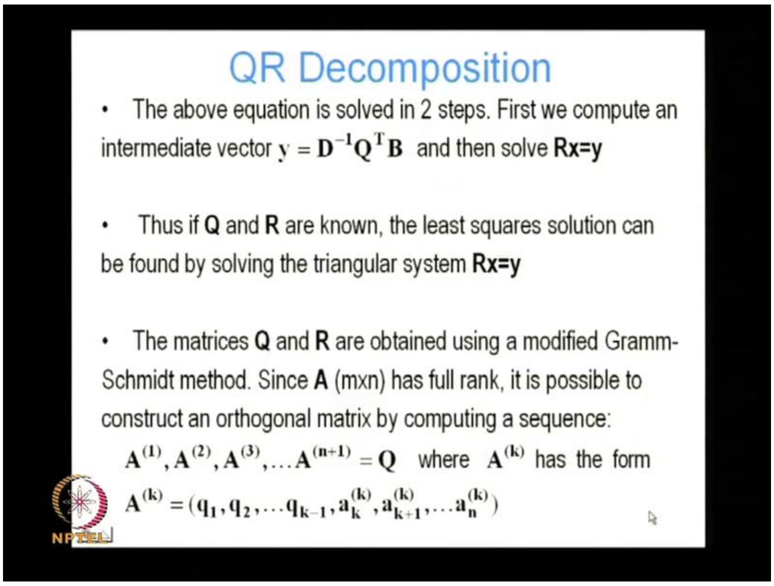

The above equation is solved in two steps, recall whenever we do operations with triangular matrixes are the both upper triangular matrixes then, which we found when we did our decomposition in Gaussian elimination. So, all this sometimes make sense to do this problem in to steps, but in order to do this problem in two steps we have to define the inter mediate vector y is equal to D inverse Q transpose B. So, basically we define intermediate vector comprising these terms right D inverse Q transpose B and then we can calculate and then we solve R x is equal to y.

So, if Q and R is known then least square solution can be found by solving the triangular system R x is equal to y. Off course assuming that, we have already found the intermediate vector y from this operation D inverse Q transpose B. So, again this is a type this is not the V this is because this is that is not the matrix it is the vector right. So, this is D y equal to t inverse t transpose the vector p which is right hand side of my system equations. How do we find the matrix u and r.

Basically, this we can this is the good time to introduce what is known as the method of graham smith, graham smith orthogonalization. So, Q and R are found using a modified graham smith method. Since A has full rank it is possible to construct the orthogonal matrix by computing A sequence A one, A two, A three and so on up till A n. Until we get orthogonal matrix Q were a k has the form a k is equal to q one q two q k minus one and so on and so forth.

So, basically let me try to explain the idea in words first basically we start with initial matrix A and we repeat we change we change one column at a time until we get an orthogonal matrix to start with. So, from A we transform A into Q. So, how do we one column at time? So, first we form the first column of the Q. What is the first column of Q? What is the first column of A? Right, but and then at the same time we make sure that the remaining column of A are orthogonal to the first column of Q. How do we do that well we do that by the orthogonal the remaining column of A with my chosen first column which is Q and we go on and we continue in this fashion. So, the first k if you look at here we are assuming that we have done up to k minus 1 right. So, we have to form the q k minus 1 k minus columns of q and then we are trying to form the k column of q.

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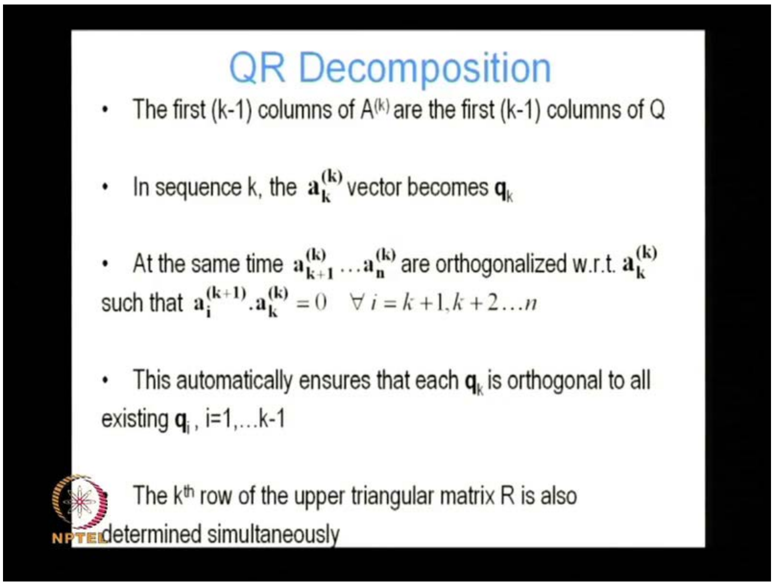

So, how are we going to do that well what we are going to do is that we will make the a k vector q k. So, this a k k vector right this vector this vector in the k th steps this vector is going to automatically become q k. But in order to become q k we have to make sure that the remaining a k plus 1 k through a n k, they are orthogonal to the vector a k k. How are we going to do that well we are going to do that by taking out the projection the projection of a k with respect of q k a k k, which become q k from all this remaining vector.

So, the same time we make the vector a k k q k we orthoganalize a k plus 1 through a n with respect to a k, such that a i k plus one dotted with a k k is equal to zero. So, that all these vector also now after orthoganalize a k plus 1 k become a k plus 1 k plus 1 because that is next iterative right how does a k plus 1 k change to a k plus one by orthogonalizing it with respect to my new q k which is acutely a k k.

So, this automatically ensure that each q k is orthogonal to all the existing q i. So, basically let me go over it again. So, we start with the first vector of a the first column vector of a and make it first column of q, but in order to do that I have to make sure that all the remaining column vectors of a are orthogonal to that q, how are we going to do that from each of those column vector remaining column vectors, I am going to project out the component of q right.

So, if my second column vector is a 2 right I am going to compute my new a 2 by a 2 minus a two dotted with the unit vector in the q k direction. So, I am just projecting out from a 2 the part which is lies along q. So, simultaneously as I compute my orthogonal vector simultaneously as I compute my q i also compute k th row of my upper triangular matrix r.

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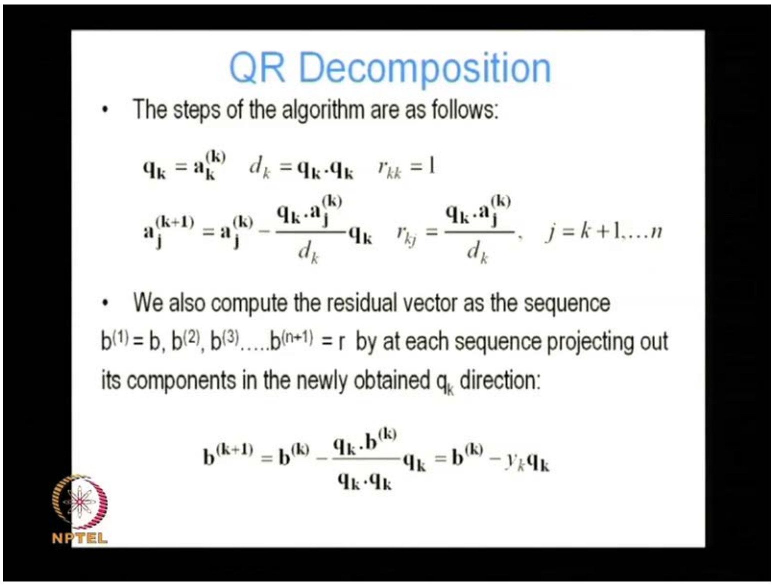

The step of the algorithm is as follows suppose in the q at the k th step first thing that we do is we make the a k th column a to k of q k vector right then from all the remaining columns of a from j is equal to k plus 1 to n for all the remaining column. I project out the component of that vector along the q k vector and of course, I normalized it by the magnitude of the q k vector right. So, this make sure that all my remaining columns vector after my k th vector right they all orthogonal to all the previous q q k vector that as been found out till now. So, in the next iteration automatically take the first vector and make it my next q k vector because i am assured that all those vector are orthogonal to my previously formed q k vectors. So, at the same time we compute r which is nothing, but r k j is equal to q k dotted with a j k divided by d k.

So, this is how we form our q r decomposition. So, we start with the matrix a which is m by n and then we step by step we orthogolize that matrix may each column orthogonal each of the remaining other columns we do it systematically using gram smith orthogonalization we are going to end up with an orthogonal matrix q and a upper triangular matrix r. At the time we all compute the residual vector as the sequence b one b one we said b one equal to b b two b three b n as we do the as we compute each additional column of q we also change our right hand side b. So, that at.

The end we are going be left with the residual vector r because at the each iteration there are also projecting out from b we start with b zero b one equal to b and each step we project out from b the component its components in the current q k vector direction. So, we start with my original right hand side at each iteration at each step my q r q r algorithm i project out from my right hand side the component the projection of the right hand side in the current q k vector direction. So, at the end i am left with whatever is left is just going to be orthogonal to all the q k vectors basically since all the q k vector and my space a; that means, my residual is going to end up orthogonal to the space span by the column vectors of a. So, this is how i compute the b k plus one which is b k minus y k q k were i have defined y k is equal to q k dotted with b k divided by q k dotted with q.

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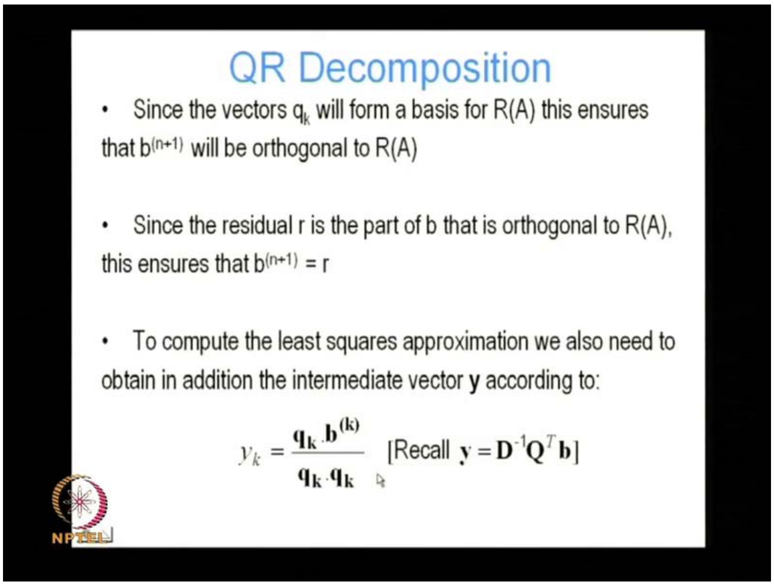

Since the vector q k will form the basis for r a this ensure that b n plus one will be orthogonal to r a recall what is r a it is the vector space spanned by all the column vectors of a. So, we since we have to ensure that the b n are orthogonal to all the q ks we will ensure that the end my b n plus one is orthogonal to all the column.

Vectors of a and therefore, what is that is that is the residual to compute the least square approximation we also need to obtain in addition the inter mediate vector y because we need to compute the intermediate vector y which we defined to be y equal to d inverse q transpose b and during the q r process we also compute y like this y is equal to q dotted with b k divided by q k dotted with q k we call it y is equal to d inverse q transpose b. So, this is basically that part q transpose b and then we are dividing by q k dotted with q k this is d inverse that operation right.

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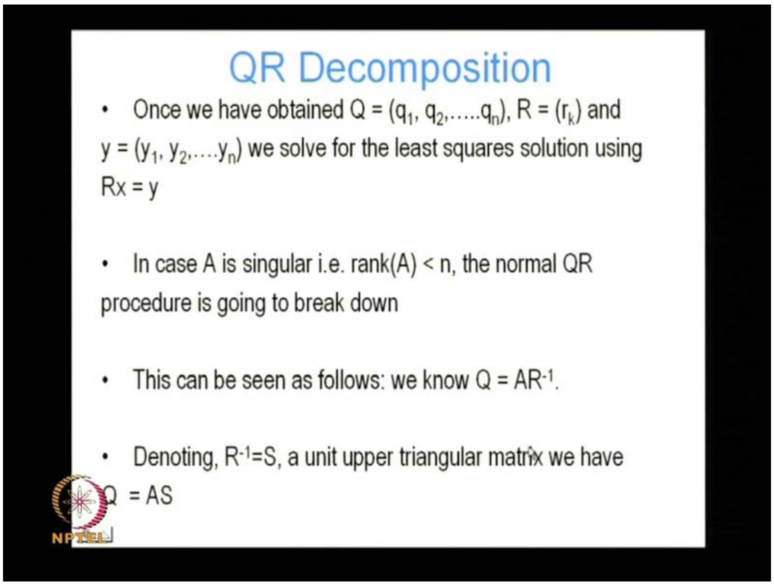

So, once we have obtained q q one q two q n. So, all these column which are orthogonal to each other right and we have obtained our right angular matrix r and we are obtained the vector intermediate vectors y we solve for the least square solution using r x is equal to y. So, this is what we do when a transpose a is ill condition we can of course, use it a transpose is well condition also dot, but it is particularly situated for when it is ill condition because then with the normal method is going to break down what happens when instead of being ill condition it is acutely singular that is a transpose a is actually singular because a does not have full value what happens then in that case the normal q r procedure is going to break down how is this going to happen we can see it as follows we know from the q r decomposition q is equal to a r inverse right q is equal to a is equal to q r. So, q is equal to a r inverse let us denote r inverse is equal to s an upper angular matrix we are assuming that since r is the upper triangle its true it is not assuming it is you have to take, but it is true that since r is the upper triangular matrix r inverse is also going be a upper triangular matrix and this also is going to be unit upper triangular matrix right. So, we can write q is equal to a times s a product s matrix product s.

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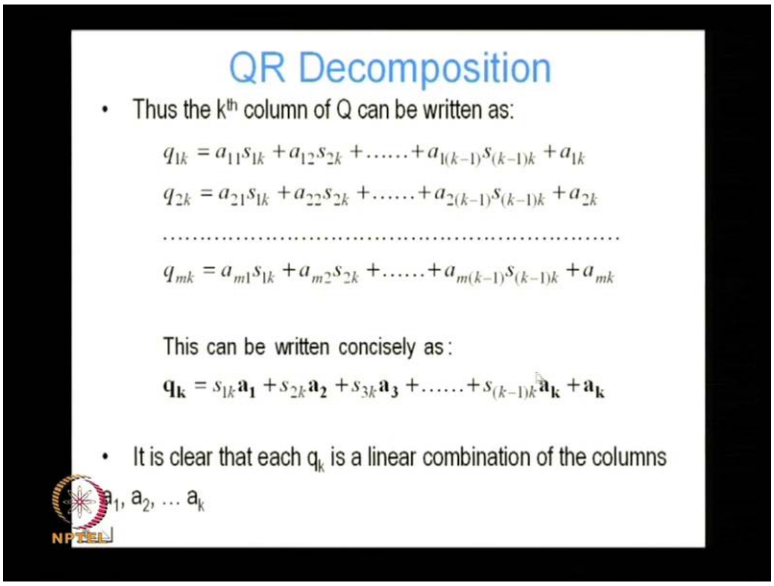

Then the k th column of q can be return as q is equal to a s. So, that k th column of q is nothing, but q. One k the k th column all the entries in the k th column q and k q two k through two q m k which is going to be nothing, but a the first row of a multiplied with the k th row of s right. So, this is the first row times the k th row of s second row k th row of s n th row of k a times the k th row of s. So, this gives me the k th column of q. So, we can write this concisely as q k is equal to some scalar times the first column of a right plus another scalar s two k times the second column of a plus another scalar s k minus one k this time times the m th column of oh sorry is this this is k minus one column of a and finally, we are going to have a one k right why there is nothing, but because there is unit triangular matrix right. So, s k k is going to be one right. So, s k minus one k s k k is going to be one because it is a unit triangular matrix. So, we can write we can write each q k as a linear combination of the a vector of the columns of a matrix a one a three up to A x again this is the typo a k minus one plus a k right. So, it is clear that each k is the linear combination of the columns of a one a two through a k.

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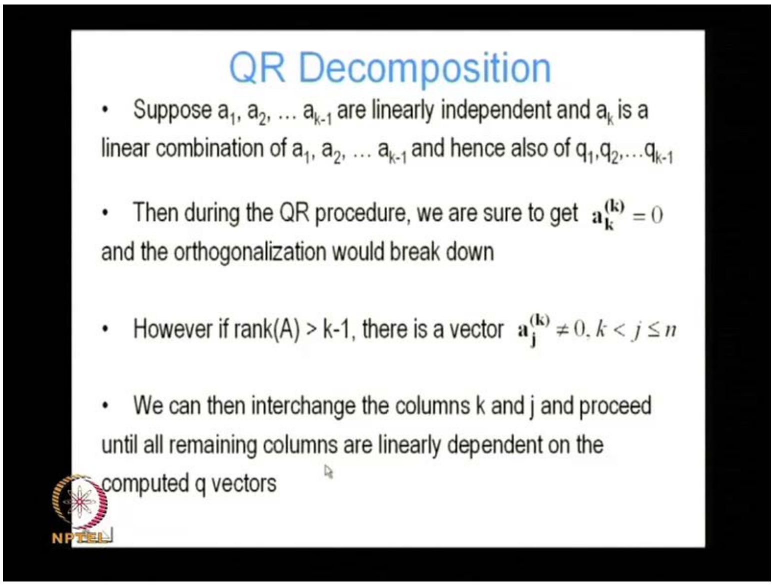

Since suppose let us suppose that a one a two a through a k minus one a linear recall now we are considering the case were a does not have full rank what does; that means, that means that there are column in k in a.

Which are linear combinations of the other columns right not all the columns of a are linear independent right a does not have full rank. So, let us assume that the a k a th column that the a th column of a is a linear combination of first q k minus one columns right since it is a linear combination of first k minus one column and we are just shown that each q k is the linear combination of the all the previous k column so; that means, that a k is also a linear combination of all the q one through q k minus one thus during the q r procedure we are sure to get a k k is equal to zero and the orthogonalization is going to break down why is that because re call at each step at each time we fall we form a new q vector v orthogonize all the remaining vector with respect to that vector. So, you project out from each remaining vector the q k vector that have been formed previously now since a k is the linear combination of all of k minus one k minus one columns of a when we project out from a k the previous a minus one column we are going to get zero because a k is basically a linear combination of k minus one column right eventually we are going to land up with an a k which is zero and then when we do that our orthogonalize procedure breaks down because the q r procedure depends on making the first column the first of the remaining column the next q k vector, but the first of the remaining column is now a k, but a k is identically equal to zero. So, our algorithm is going to break down, but; however, if the rank of k is greater than k up till now we.

Have found k minus one independent column, but suppose rank of k rank of a is more than k minus one if the rank of k is just minus well that is it we cannot go any further all the remaining columns are assure to be zero right because they have been all orthogonal with respect to the previous k minus one column right, but if the rank is greater than k minus one there are still some column which are non zero right there are still some of columns of a which are non zero. So, what we do we do the pivoting we do the column interchange we replace the k suppose the j th column is non zero and the j is greater than k, but lesser than or equal n we can then interchange the column k and j and proceed until all the remaining column are linearly independent on the computed k vector. So, we keep on doing this we keep one interchanging until we reach a situation where all the remaining columns are orthogonal to the computed q vectors which means that all the reaming columns are zero right.

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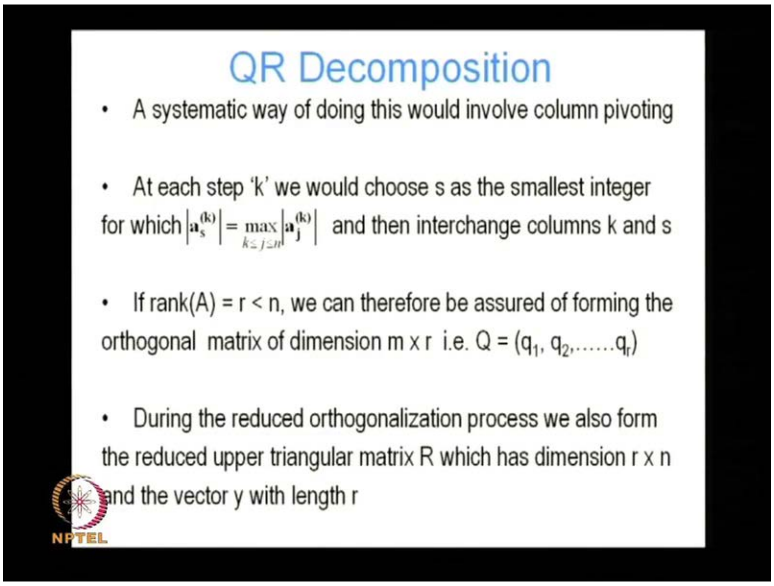

So, systematic way of doing this would involved column pivoting at each step we would chose s as a smallest integer for with k s k is equal to max of a j k j varying from k to n and then interchange the.

Column k n s basically we at each step we instead of directly taking the next column and making the q vector we look at all the reaming columns and choose the column which as the maximum norm maximum norm in the infinite since right infinite norm right is the maximum infinite norm make that the q vector and. So, on and. So, forth right. So, if rank of a is suppose r which is less than n we can therefore, be assure of forming orthogonal matrix of dimension m by r that is q is equal to q one q two to q r. So, that to form r since it has rank r we can form r orthogonal vectors right we can form r orthogonal vectors q is of size m cross r during the reduce orthoganilization process we also form the reduce upper triangle matrix are which has dimension r by n and the vector by the length r using the same procedure that we talked about earlier we also forming r and the y vectors.

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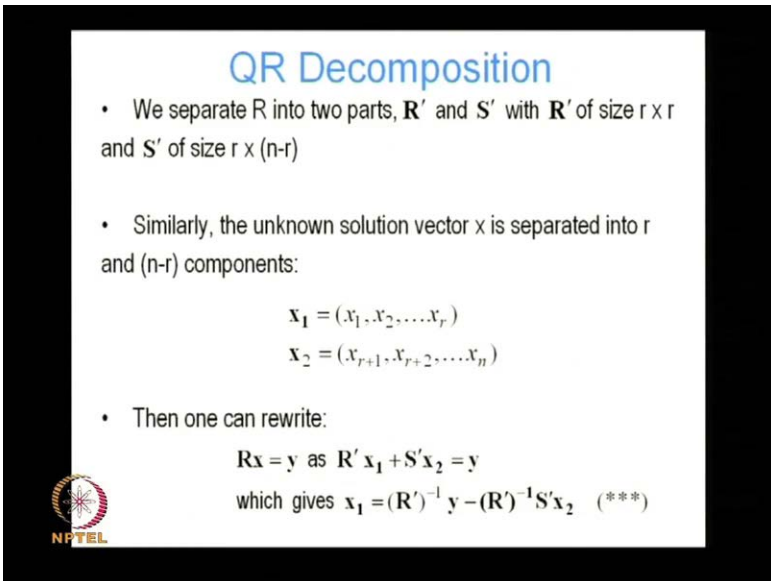

We separate r into two parts r prime and s prime with r prime of size r by remember that q is of size m by r. So, e must be of size r by n because a is of size m by n right. So, m by n is equal to m by r product with r by n. So, now, . So, basically we are saying that we are going to split r into two parts r prime and s prime p prime of size r by r and s prime of size r by n minus r similarly the unknown solution vector x is separated into r n minus r components we separate out the first r component of x like this as x one and r plus one to the nth component as x two then we can write r x is equal to y such that.

Now, see we have split out r into two parts r prime which is of size r by r and s prime this is size r n minus r similarly we have split up this vector into this little part x one which is of size r and this part x two which is of size n minus r right. So, r prime x one plus x prime x two must be equal to r which gives me x one i can write it has r prime inverse by minus r prime inverse this x two are arbitrary right because my matrix has my e matrix has got rank r right we have to assume x two arbitrary value x two arbitrary remember anytime we have less than full rank we are going to have infinitely many solution right infinitely many solution and we have to assume the as many solution as the null space right in this case the null space is of size n minus r right. So, x two is of size n minus r these r arbitrary values and i can write my remaining expand in terms of this x two right and my y which i found during the orthogonalasation.

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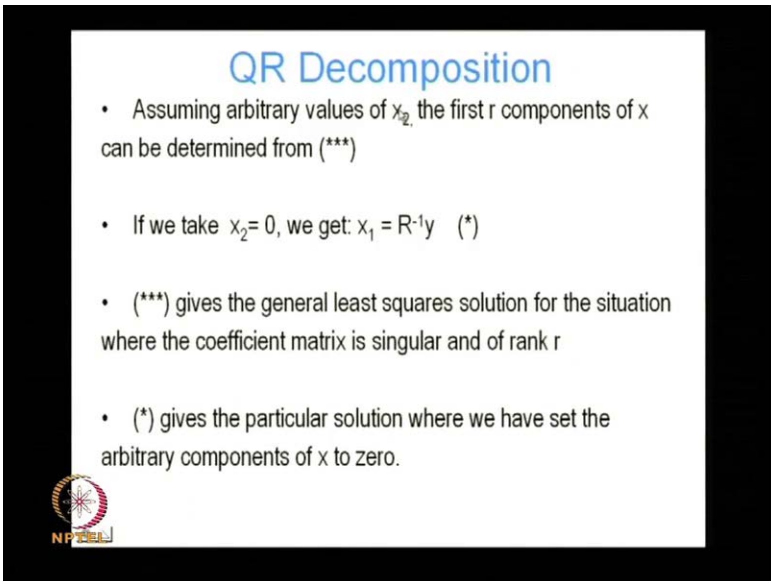

So, assuming the obituary value of x two the first r components of x can be determined from this expression takes one component similarly if we take x two is equal to zero since z two is arbitrary we can take any value if we put x two is equal to zero we get x one is equal to r inverse y.

So, this equation gives the general least square solution for the situation where the sink co efficient matrix is singular and of rank r right. So, there are infinity may recall now there are infinitely many least square solution right now why because my x two are arbitrary right those n minus r components of x are arbitrary i can make them zero i can make them anything i like right, but they all satisfy the least square solution. So, they all minimize the residual they all minimize the residual they all satisfy the least square criteria. So, infinity many least square solution to the problem right unlike the case when transpose a was non singular were we had only one solution in this case the infinite many least square solution, but in all of them make sure that the residual is all them make sure that residual is orthogonal to the to the space span by the column of a in this case the space span by column of q.

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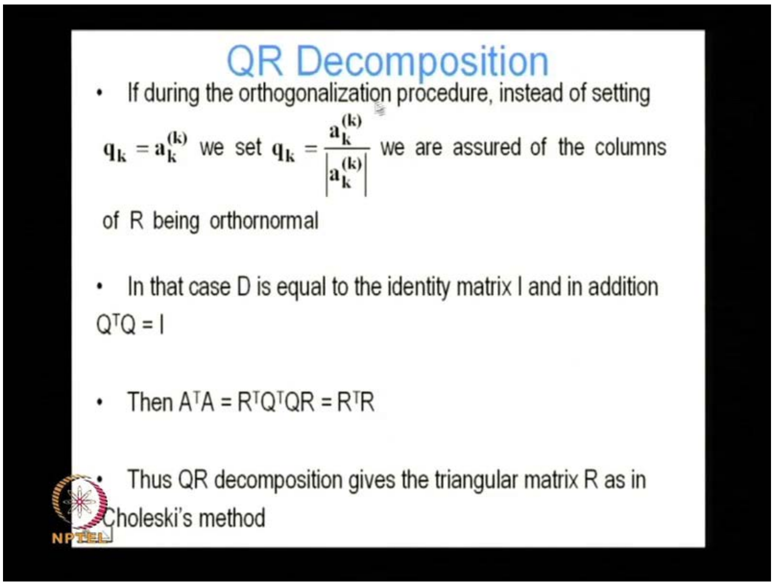

So, this is the just the diversion, but during the orthogonalization procedure instead of setting q k is equal to a k k in addition to orthogonalize we also normalize right we ortho-normalize. So, we said the q k vector. So, that each of them have norm one. So, not only the orthogonal to each other they unit norm also then we are assured that my r my q matrix is this is again a typo i assured the columns of q being.

Orthogonal apologize this is q right the columns of q are ortho normal right in that case d transpose d a sorry q transpose q is going to be identity matrix. So, d is equal to the identity matrix and in that case a transpose a can be written as r transpose q transpose q r which is equal to r transpose rwe have encountered this decomposition before when we talked about the choleskismethod right. So, this q r decomposition is a triangular matrix r as in choleskis method in case the ortho in addition to orthogonalizing q we normalize the columns of q we get a we get the we recover the choleskis method.

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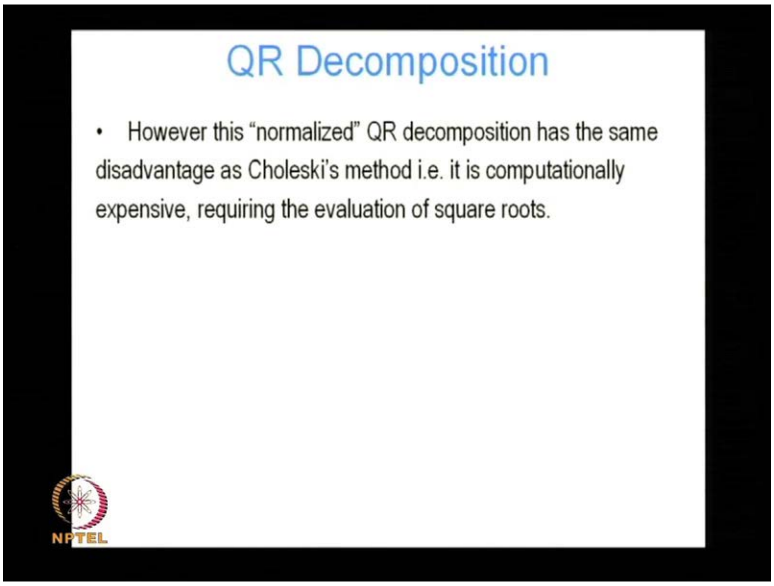

This normalized q r decomposition has the same disadvantage as choleskis method it is computationally expensive requiring the valuation of square roots which we find in case of choleskis method as we have r transpose r that going to be square terms and we have to evaluate square roots. So, it become computationally expensive ok.

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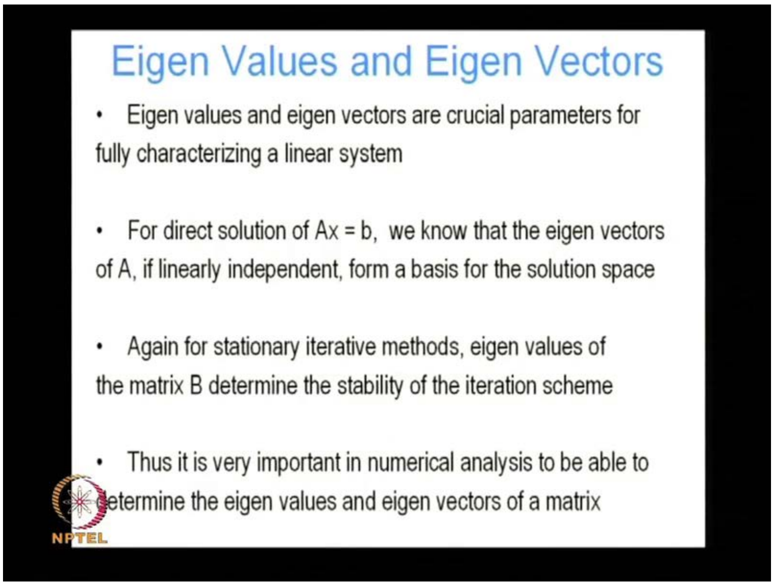

So, that was the discussion on over determined linear system u r decomposition to solve the two determined linear system the q r decomposition.

Is very very it is very important tool in the numerical method. So, it is not only usefulness not restricted to over determine linear system it can use it has lots of other uses it can be acutely used to find probably very powerful tool for finding the eigen value also all though we are not talking in detail about use q r in finding eigen value, but it is very use full tool in finding eigen value also it is very important and power full tool, but next lets switch over to finding out how can we calculate how can we solve eigen value problem basically if you have given an n by n matrix and we are interested in finding its eigen value in eigen vector how can we solve it if it is the small matrix if it is the two by two if it is three by three its four by four probably matrix we can do it by hand we can set up the characteristic equation and solve it right, but in case we have very large matrixes then solving that problems is it becomes difficult it requires very sophisticated numerical method and i can assure you that solving eigen value problem are essential if you are going to do any reasonable numerical solution if you want the stability of the analyses we need to find the eigen value countless other applications were eigen value is very important. So, eigen value and eigen vectors are crucial parameter for fully characterizing linear system. So, for direct solution of A x is equal to b we know that since the eigen vectors of a if linearly independent form basis for the solution space we can use the eigen vectors to solve the problem right again for the stationary iterative method which we have encountered just little while ago we found that the whole stability of the method depends on the eigen value of my b matrix right which was my iteration matrix right. So, the eigen value of the b.

Matrix is determined whether my iteration method is going to be stable or not after i do the n iteration whether i am going to converge to a solution or i am going to divert and get go somewhere totally different right. So, all that depends on my on the spectrum on the spectrum of my b matrix right thus it is very important in numeric analyses to be able to determine the eigen values and eigen vectors of a matrix.

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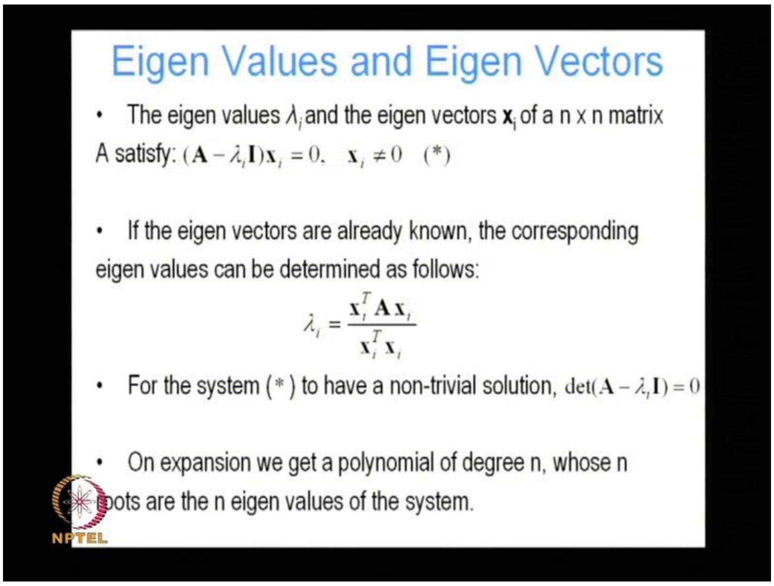

Eigen values lambda i and the eigen vector x of n by n matrix satisfy the relation a minus lambda x is equal to zero provided that x i is not equal to zero if we know the eigen vector we can find the eigen value sterilely we just compute a is equal to lambda i x. So, x i transpose x i x i transpose x i divide x i transpose x i x i transpose x i we get my egien value right. So, that can be done trivially right provide the eigen vector are known; however, that is not. So, easy finding the eigen vector is not. So, easy. So, we have to find ways of finding the eigen values typically the eigen values are found first and.

Simultaneous eigen vectors are found typically we do not know that eigen vector are purring beforehand. So, we know that for this system this is the homogeneous system for it to have nontrivial solution, but meaning that for not all x i components of x i to be not equal to zero then the determinant a minus lambda i must be equal to zero on expansion we get the polynomial of degree n. So, if it is three by three matrix we get the polynomials in third degree cubic equation in lambda if it is two by two matrix we get the quadratic equation in lambda if it is four by four matrix we get the quintet equation in lambda. So, we get the polynomial of degree n whose n root are the eigen values of the systems.

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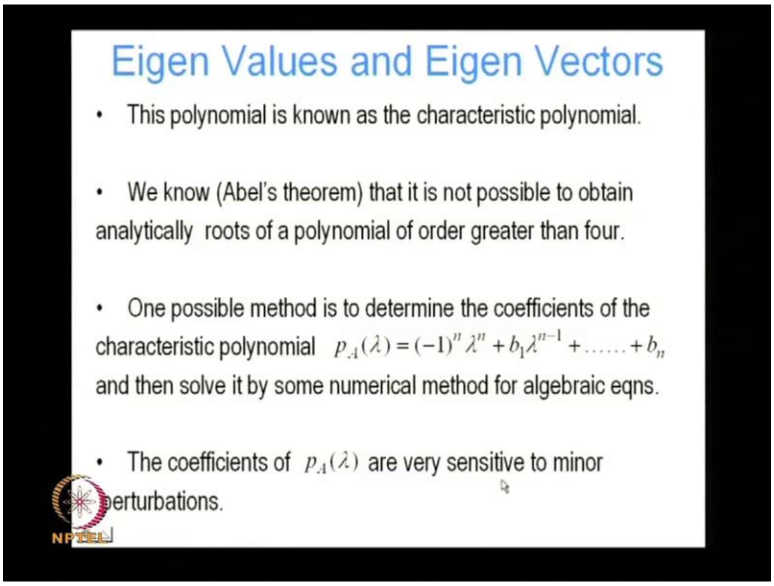

This polynomial is known as the characteristic polynomial again there is theorem which is known as abels theorem it tells that it is not possible to obtain analytically the roots of the polynomial of order greater than four. So, if this polynomial become greater than four it is a five by five matrix as the quintet polynomial right polynomial lambda to the power five in that case there is ables theorem which tells me that you cannot find the roots of this polynomial using any analytical method; however, hardly try right. So, the only way you can find the roots of that equation are through numerical methods right. So, one possible method is to determine the co efficient of the characteristic polynomial p a lambda which is like this lambda to the power n lambda to the power n minus one and. So, on and then solve it by some numerical method algebraic equation why numerical method because for order greater than four we cannot use an analytical methods and even for orders less than four it is not it is not the good idea sometimes to.

Use analytical methods i talk about the reasons just now the coefficient of p a are very sensitive to minor perturbations if you have your original matrix a and i give minor perturbation to the elements of a. So, we add to a matrix epsilon time b were epsilon is the very small number. So, i give perturbations to a. So, you except the eigen values of a to change by epsilon right because i change the components of a by epsilon. So, i would accept the eigen values a also change slightly by epsilon, but it does not happen like that why is that because the coefficient of my characteristic equation are very sensitive to minor perturbations right. So, they are not very well condition right minor perturbation in a result in large changes in the roots of my polynomials of my characteristic equation.

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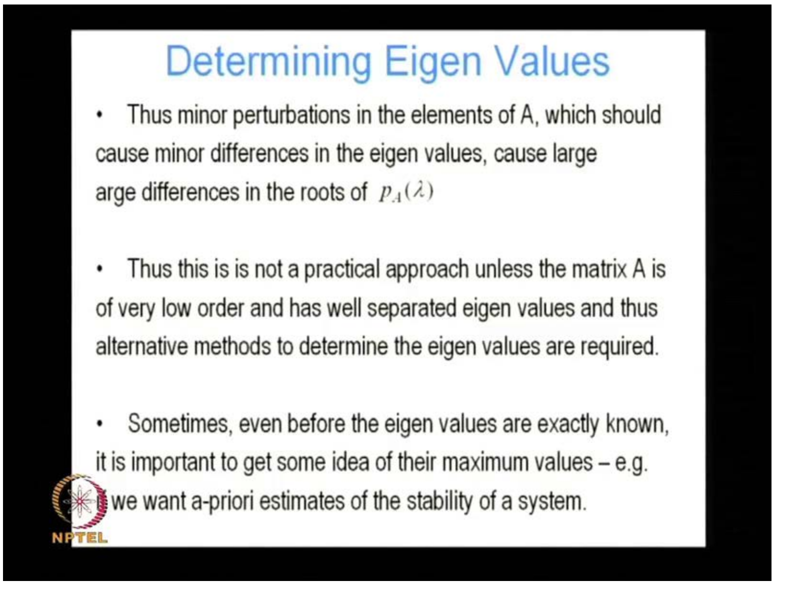

Thus the minor perturbation of elements of a which should cause minor difference in the eigen value cause again there is large differences in the roots of p a lambda thus this is not the particle approach until the matrix a is a very low order and a well separated eigen values and has the well separate if a is of low order and separate eigen values then this is.

Fine; however, otherwise we need to think of alternative method for determine the eigen values. So, before talking about the alternative method for determining the eigen values we are going to talk about certain important theorem in linear algebra which is known as the gerschgorins theorem which is very use full although it is not as use full as (refertime:39:00)it would like to hope it because it is use full because even determine because it allows us to determine the bounds on the eigen we call our stability criteria what was our stability criteria for our jacobians method or for galls syghal method the stability criteria was that the spectral radius must be less than one which means that my largest eigen value must have magnitude must have norm less than one. So, this gerschgorins theorem allows us even before actually computing the eigen values to find out bound on the eigen value it tells that what can be the maximum possible eigen values of this matrix given in a matrix without actually computing the eigen values by using gerschgorins i can tell i can find out what can be the maximum absolute value g it g beyond which we never be exceeded by the eigen value of that matrix right. So, before that allows us to get some idea of stability of the system even before computing the eigen value it allows us to get half clear estimate of stability of the system which are use full, but not. So, use full because the bounds predicted by the gerschgorins theorem are. So, wide right. So, they are since my eigen value is five suppose and gerschgorins theorem tells me that eigen value will never exceed that value five hundred that bound is not very use full right, but still use full for sometimes, but it is a very very wide bounds. So, narrow bounds are more use full than wide bounds right so.

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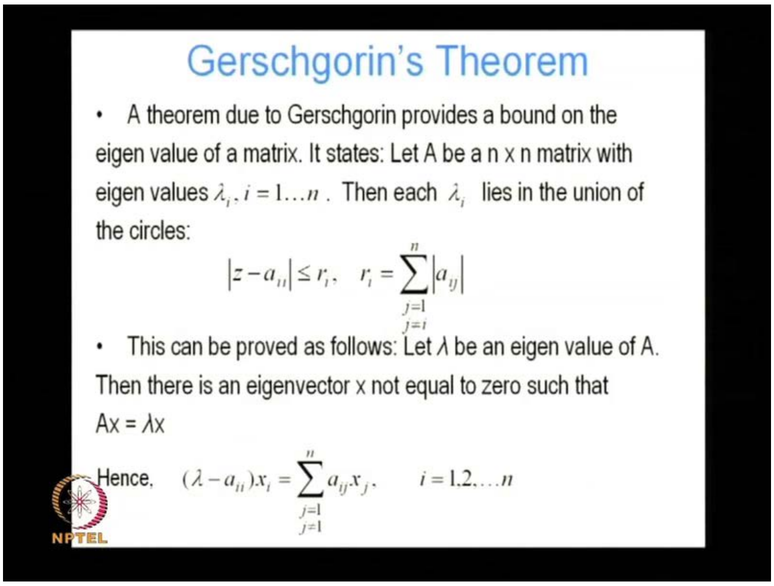

So, let us look at gerschgorins theorem the theorem due to gerschgorins provides a bound of.

The eigen value of a matrix it states that let a be an n by n matrix with eigen value lambda one lambda i i is equal to m then each lambda i lays in the union of the cycle from like that. So, what does; that means, it means that if i form a circle which centre i form a series of circles right and each circles circle one for instants has centre with diagonal element one one right it has center with a diagonal element one one and it has got radius which is some of the absolute values of all the diagonal all the non diagonal elements in that row right if i am looking at the first row alright the radius of the circle is sum of all the non diagonal element in the first row s it tells me that if i compute circles right this look at the first row look at its diagonal element make that the centre of my circle and draw a circle with a radius which is equal to the sum of norm of my half diagonal elements i do that for the second row again draw a circle centered with the diagonal term and radius equal to the some of the.

Absolute value of diagonal elements i go on doing this then this union of all this circle right the union of all this circle defines my gerschgorins bound. So, if i draw this series of circle and then i draw a circle which circumcise all this circle which is basically the union of all these circle union of all those circle the eigen value will lay within that circle the eigen value cannot lay beyond that circle the maximum eigen value cannot lay beyond that circle this can be proved as follows let lambda be eigen value of a then there is eigen value of x is not equal to zero such that A x is equal to lambda x right hence we can write lambda minus a i i x i is equal to i x a basically i kept the diagonal term on the left hand side and i have moved all of the diagonal term to the right hand side.

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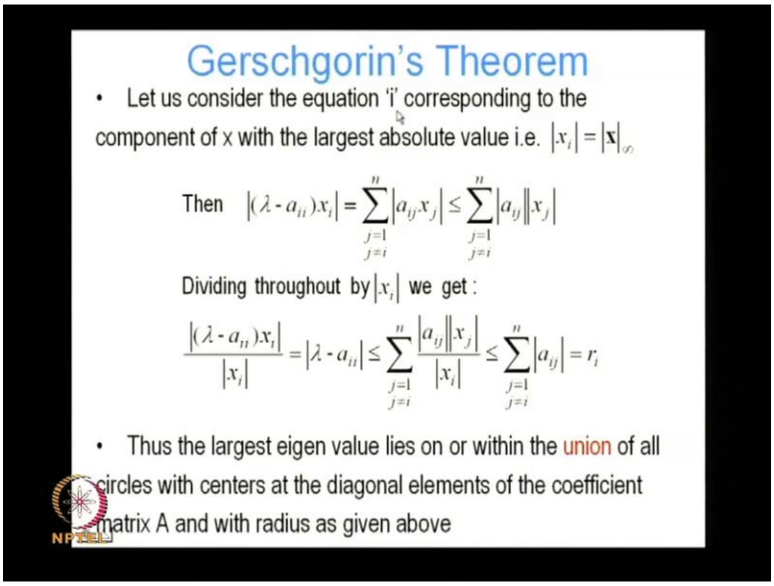

So, let us consider. So, if i do that i can do that for each of my equation i equal to one through n assuming that a is the matrix of size m by n. So, i can do it for each of those rows next let us consider the equation i corresponding to the component to the x largest absolute value right. So, let us consider i th row such that i th row x has the maximum no has a maximum absolute value right in that case for that row i can write mode of lambda minus a i i x i is equal to sigma mode of a i j basically am just taking the mode of both sides right am taking the mode on both side in this equation and mode of sum of a i j x j i can write as this is going lesser than or equal to sum of mode of a i j x we have seen this many times before and dividing throughout by mode of x i. So, we divide throughout both side by mode of x i. So, bond of lambda minus a i i x i divided by mode of x i is going to give me mode of lambda minus a i i this is going to be lesser than or equal to mode of a i j x j by mode of x i sum from j equal to one to j.

Not equal to i which is going to be lesser than or equal to sum of mode of a i j why because we have chosen i to be the largest component in the x vector. So, all the x j and these j are expanding from j equal to one to n j not equal to i. So, all these x j are going to be less than x i. So, this is going to be this is going to be less than one this factor mode of x j mode of x i is going to be less than one. So, this is going to be less than mode of a i j which is equal to r i right. So, for the row which corresponds to the largest absolute value of x this is true right this is true for that row thus largest eigen values lays on or within the union of all circle with centers at the diagonal elements of the coefficient matrix a and with the radius as given above this is slightly we have to use our imagination basically we are looking for the for the row which has the largest entry in the x this is true right, but we do not know our priority what is my row with the largest entry of x right i do not know what are my eigen vector. So, how do i know which is going to become my row in the eigen vector with the largest entry of x.

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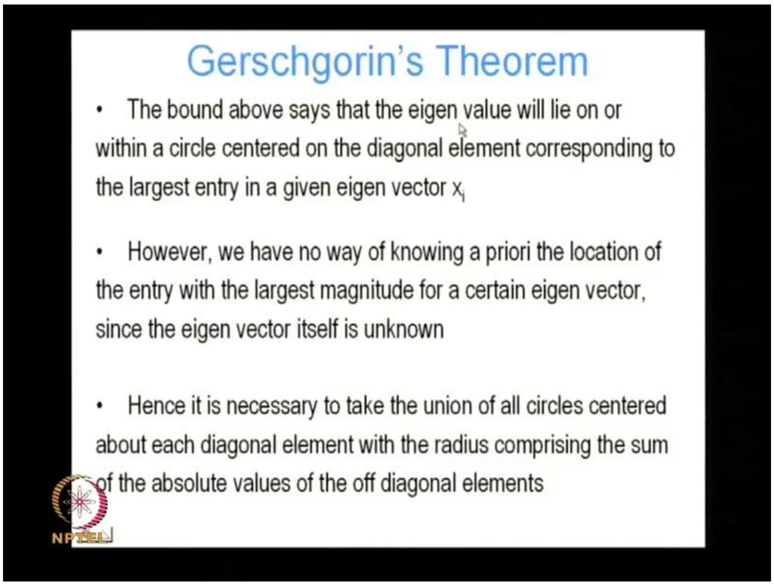

So, i have to assume that any of those row in x can be the row with largest entry and have to compute this this circle and the radius for each of those rows right and the union of all those circles is going to give me my upper bond right is that clear.

So, bound above says that eigen value will lay on or within the circle centered on the diagonal element corresponding to the largest entry in the given eigen vector x i; however, we have no way of knowing our priory the location of the entry with the largest magnitude for the certain eigen vector since the eigen vector itself is unknown hence it is necessary to know union also we have to take all the rows right you have to take the union of all the circle scented about each diagonal element with the radius comprising the sum of absolute values of the of diagonal element.

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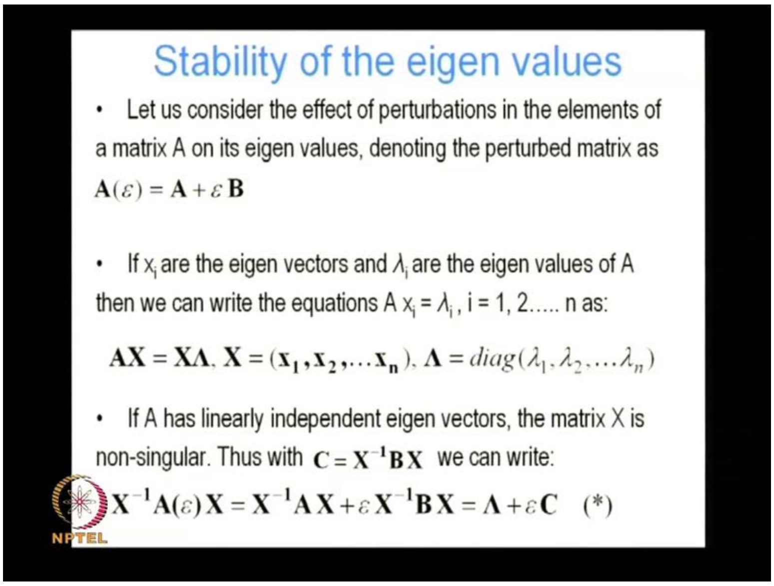

So, that was gerschgorins theorem. So, let us use we will use gerschgorins theorem look at stability of the eigen value let us consider the affect of perturbation the elements of matrix on its eigen values denoting the perturbed matrix a epsilon is equal to a plus epsilon b. So, i have by original matrix i apply some perturbations on the elements of original matrix by scaling a matrix b with the small number epsilon and adding into my matrix a. So, i am going to know if i know my eigen values of.

A if i know my eigen value of a how what what does these tells me about the eigen values of a epsilon which is a plus epsilon b first let us assume that a has linearly independent eigen vectors. So, the matrix x is non singular what is the matrix since x i eigen vectors x i the eigen values of a we can write the equation A x i equal to lambda i as a operating on x x comprises columns n columns each column being in eigen vector of a. So, a operating on x is equal to x operating on lambda right basically we are putting all the writing all the eigen value equation together right. So, we are creating a matrix x each column of which is an eigen vector and we are creating the matrix lambda diagonal which is the diagonal matrix and each diagonal matrix is eigen vector we can write it like this and since a linearly independent eigen vector the matrix x has full rank because each of its column is a eigen vector in each eign vector is being linear independent. So, x has got full rank x is invertible right x is non singular. So, we can write. So, we can define a matrix c is equal to x inverse b x. So, that we can write x inverse a x. So, we are operating on a operating on x with a right and then operating on resultant with x inverse.

So, x inverse A x is going to be A x inverse A x right this part plus epsilon x inverse b x, but x inverse A x is from this x inverse A x is nothing, but lambda right x inverse A x is going to give me diagonal matrix lambda. So, lambda plus epsilon x inverse b x i define x inverse b x is equal to c. So, x inverse A x is equal to lambda plus epsilon c.

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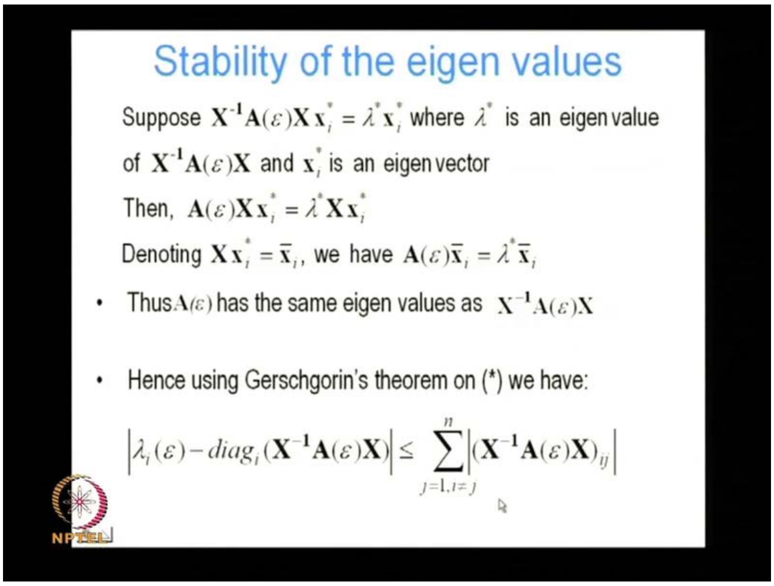

Suppose x inverse A x has got eigen values lambda star and eigen vector star. So, we are now writing the eigen value problem for x inverse A x eigen a epsilon x and we are saying that suppose x inverse a epsilon x has eigen vector x star and eigen value lambda star then A x x i is equal to lambda star x x i is basically you are bringing this x to the right hand side right lambda star x x i and we have a epsilon x x i right. So, if we denote x x i bar x i star as x bar A x bar is equal to lambda star x bar. So, what does this mean this means that a epsilon has the same eigen value as x inverse a A x x inverse a epsilon x why i because the eigen value of x inverse a epsilon x lambda star and we have just proved that the eigen value of a epsilon are also lambda star thus a epsilon has the same eigen value of A x inverse a epsilon x and eigen vector of a epsilon are related to the eigen vector of x inverse A x by this operation eigen function of x inverse A x by this operation x x i star. So, if i have the eigen vector x inverse A x if i operate that with this matrix x i am going to get the eigen vector of a epsilon hence using gerschgorins theorem on star basically i am going to use gerschgorins theorem on this equation and what does gerschgorins theorem tell.

Us it tells us that the largest eigen value of this matrix right satisfy this bound right lambda i minus diagonal element of this matrix right must be lesser than or equal to sum of the of diagonal terms right it has to satisfy this equation from gerschgorins theorem diagonal minus of diagonal from which lambda i epsilon which is this right lambda epsilon why do you right lambda epsilon because we know that lambda i of a lambda is nothing, but the eigen values of a epsilon we know that the eigen value of a epsilon as same as the eigen value of x inverse A x right.

So, lambda i a epsilon is equal minus lambda i epsilon c r why do we do that well because we call that x inverse A x is equal to lambda plus epsilon c lambda is the eigen value of my original a matrix and epsilon c where c i defined as this right. So, we can write this as lambda i minus lambda i epsilon c i i because this is basically the eigen value of diagonal term of this matrix right and this must be lesser than or equal to the sum of the diagonal term which is equal to epsilon c i j right because again lets go back again this matrix is diagonal right. So, all the of diagonal term of the x inverse A x are given by epsilon c right. So, we have this must be lesser than or equal to epsilon mode of c i j which implies lambda i epsilon minus lambda i must be of order.

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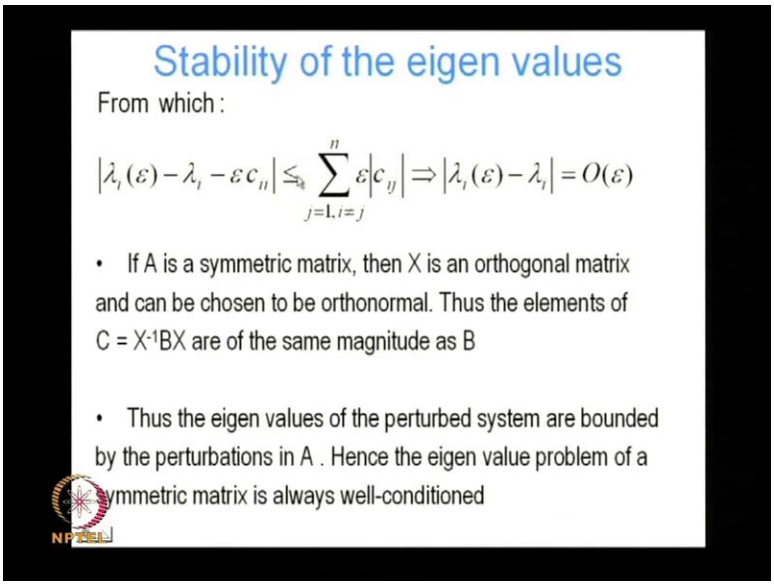

Epsilon. So, these term are order epsilon. So, different between lambda i epsilon and lambda i must be of order epsilon. So, what does that mean this mean that eigen value of my perturbed matrix lambda i epsilon are different from the eigen value of my original matrix by are by order epsilon right. So, they are of the same order as the perturbations right. So, if lets if a is the symmetric matrix then x is an orthogonal matrix and we can choose x to be a orthogonal matrix by normalizing the eigen vector thus the element of c x inverse b x are of the same magnitude as b why because c is equal to x inverse b x component of column of a each of x each have norm one unit norm.

So, x inverse magnitude norm of x inverse b x of c must be of the same order of b because these thing have each column as unit norm right. So, this c has to have the same norm as must be of the same order as b. So, the eigen values of perturbed system are bounded by the perturbations in a hence the eigen value problem of symmetric matrix is always well condition why do we say it is well condition because i have applied perturbations to a right small perturbations epsilon times b, but that hasn’t change my eigen value by a large amount it has only changed it by order epsilon by the order of whatever perturbations i applied right so; that means, the problem is well condition that is the definition of well condition problem a minor change in the input values does not change in does not result in large differences in the solution. So, minor change in my a matrix does not change my eigen value significantly. So, we will continue our discussion of the eigen values next class next lecture. Thank you very much.