Numerical Methods in Civil Engineering
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Lecture - 12
Solving Nonlinear Equations

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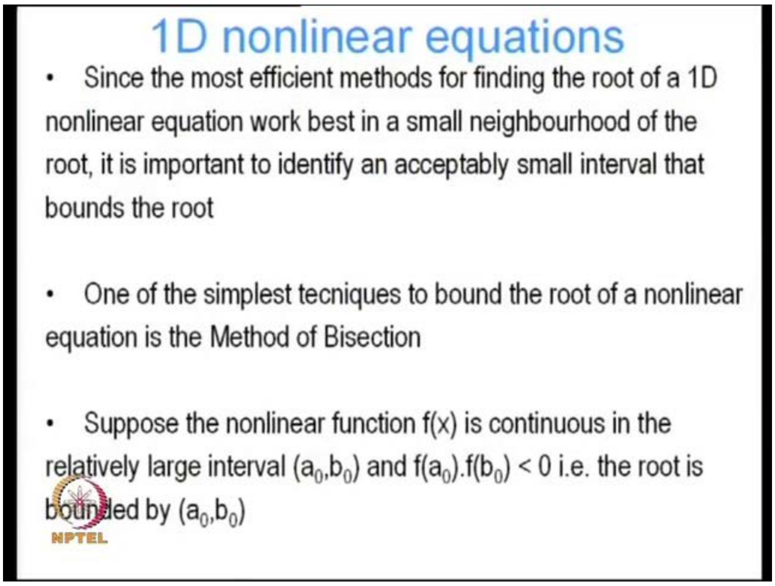

((Refer Time: 00:15)) Series on non linear methods in civil engineering, we are going to focus on solving and methods to Solve Non linear Equations. Last time in our lecture, we briefly introduce this topic. And it said that for non linear equations, typically we use two methods, usually two methods are used to two iterative schemes, two algorithms are generally used for solving the equations.

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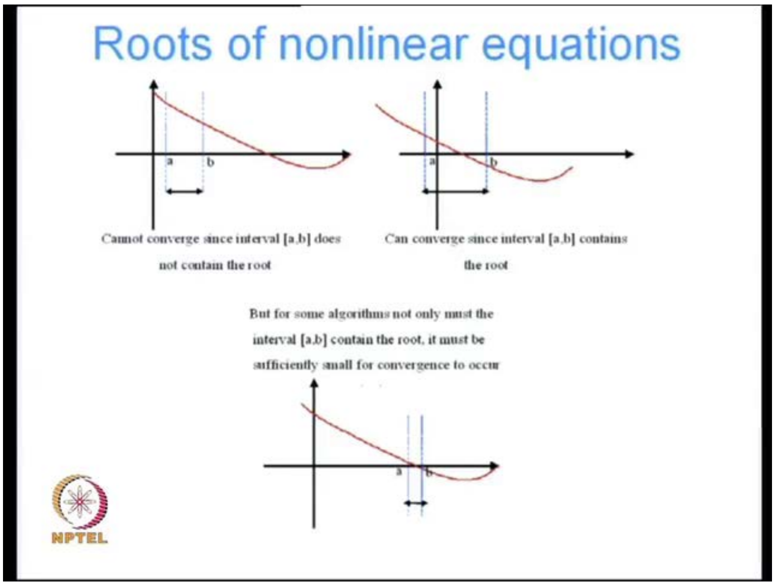

While, well we looked at this these two pictures right. And we said that any iterative scheme has to bound, we must choose an interval in which to iterate, which bounds the true solution. If we chose an interval which is outside, which does not bound the route right, if you choose an interval like this a, b which does not bound the route. In that case we have no hope of finding a solution.

However, if you choose a very large bound, then it make sense to use a relatively cheaper technique, which is less expensive to narrow down the bound, and then use a more sophisticated technique. Once, the bound has been sufficiently narrowed, the sophisticated techniques are more expensive. So, it make sense to first narrow the bound and then once we have narrowed down the bound sufficiently use a more advanced algorithm, which has got better convergence properties to solve the equation in that narrow bound.

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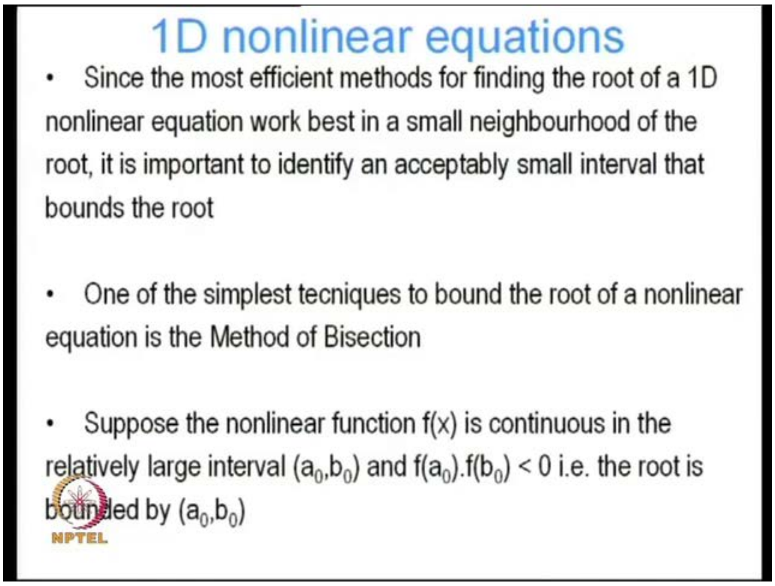

Since, the most efficient methods for finding the route of a 1D non linear equation work best in a small neighborhood of the root, it is important to identify an acceptably small interval that bounds the root. So, that to make sure that we achieve higher convergence as well as efficient computation, as well as ensure that my computational effect is not too expensive.

One the simplest methods for bounding the roots of a non linear equation, uses what is known as the bisection method. And let us, suppose that you have a non linear function effects, which is continuous in a relatively large interval a 0, b 0; where a 0, b 0 bounds the solution because f of a 0 and f of b 0 have opposite signs. So, product of f a 0 times product of f b 0 is negative; that means, that the root is bounded by a 0, b 0.

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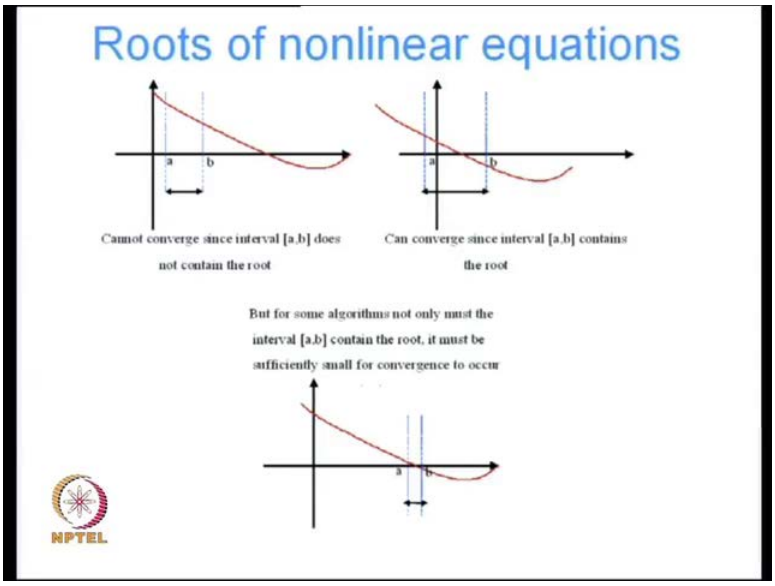

Basically, what I am saying is that if I look at a 0, b 0 here, then here f a 0 is positive, here f b 0 is negative, so f a 0, f b 0 is going to be negative.

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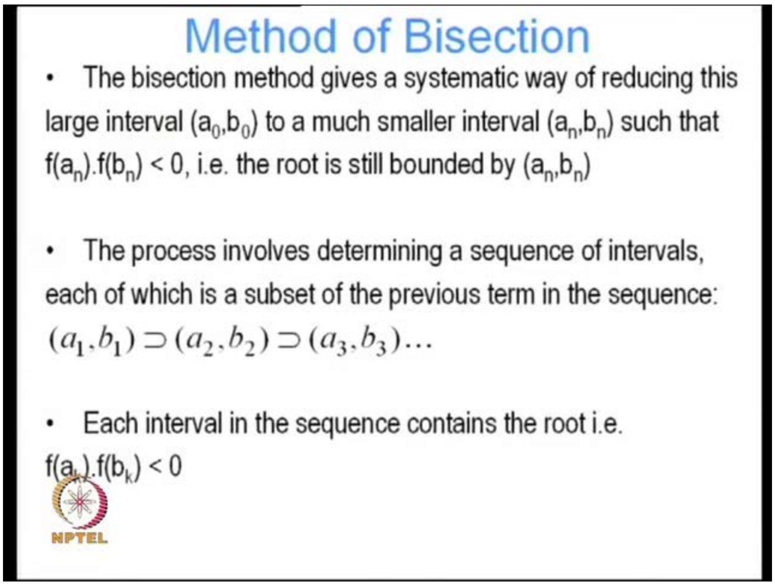

The bisection method gives the systematic way of reducing this large interval a 0, b 0 to a much smaller interval a n, b n such that f a n, f b n is less than 0 that is the root is still bounded by a n, b n. So, we started with an interval a 0, b 0 and that, interval bounded the root because sin of f of a 0 was different from the sign of f of b 0, but that interval was too big. So, you want to find an algorithm like the bisection method to systematically reduce that interval.

So, that my size of the interval becomes small, but at every iteration I want to make sure that my new a i, b i the new values of a i, b i which are the two bounds of the interval, are actually have opposite signs right they bound the roots. So, f of a i dotted with f of b i and that will be negative. So, this process involves determining a sequence of intervals, each of which is a subset of the previous term in the sequence, that is a 1, b 1 is greater than a 2, b 2 is greater than a 3, b 3. So, a 2, b 2 is a subset of a 1, b 1, a 3, b 3 is a subset of a 2, b 2 and so on and so forth.

So, we systematically make the size of the interval smaller and smaller, but every time we make sure that the interval contains the root, how can I make sure that is true, by making sure that if I evaluate the function values at the two end points, those have opposite signs. So, f a k times f b k is less than 0.

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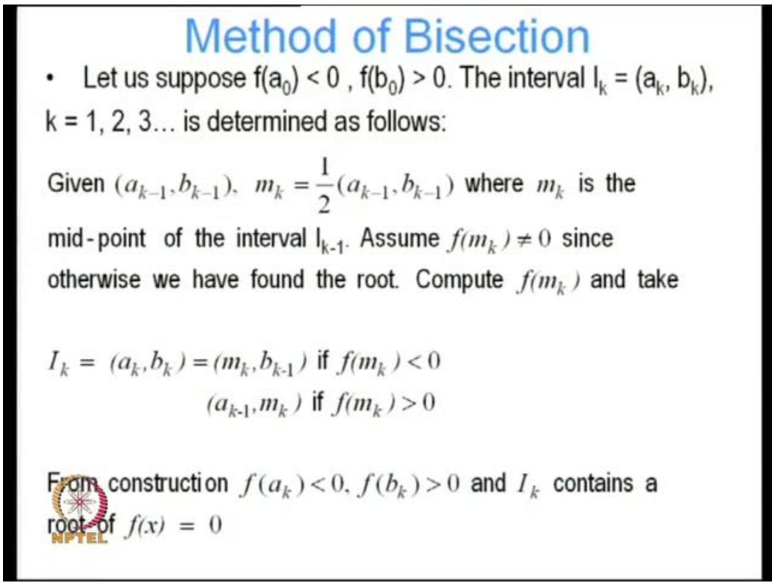

So, let us suppose we start with a very large interval with a 0, b 0. And let us suppose, that the interval a 0, b 0 bounds the root why because f of a 0 is negative and f of b 0 is positive. The interval I k which is a k, b k, k is equal to 1, 2, 3 or 3 and so on and so forth is determined as follows. So, given a k minus 1 and b k minus 1, where again f of a k minus 1 is less than 0 and f of b k plus minus 1 is greater than 0, I find the midpoint of that interval n k, which is half of a k minus 1 b k minus 1.

And then assuming that f of m k is not equal to 0 because after all if f of m k is equal to 0; that means, I have already found the root, but suppose that f of m k is not equal to 0, then I look at the sign of f at m k. And then check whether f of m k is positive or f of m k is negative. So, if f of m k is negative in that case, my new interval is going to be m k, b k minus 1 why because I knew that f of b k minus 1 is positive right. So, if f of m k is negative; that means, m k and b k the root must lie between m k and b k minus 1 right.

Similarly, if f of m k is positive I know that since f of a k minus 1 is negative; that means, the root must lie between a k minus 1 and m k right. So, depending on the sign of f of m k I choose my new interval, which is going to be a k, b k it is going to be m k, b k minus 1 or a k minus 1 m k. So, from construction f of a k is less than 0, f of b k is great because this is how I constructed my interval.

So, f of a k is going to be less than 0, f of b k is going to be greater than 0 and I k is always going to contain a root of f of x is equal to 0. So, every time I make I half it, basically that is why it is called the method of bisection, so initial interval, next iteration that size becomes half, next iteration becomes 1 forth, 1 eight. So, every time it becomes half, so that is why it is known as the method of bisections.

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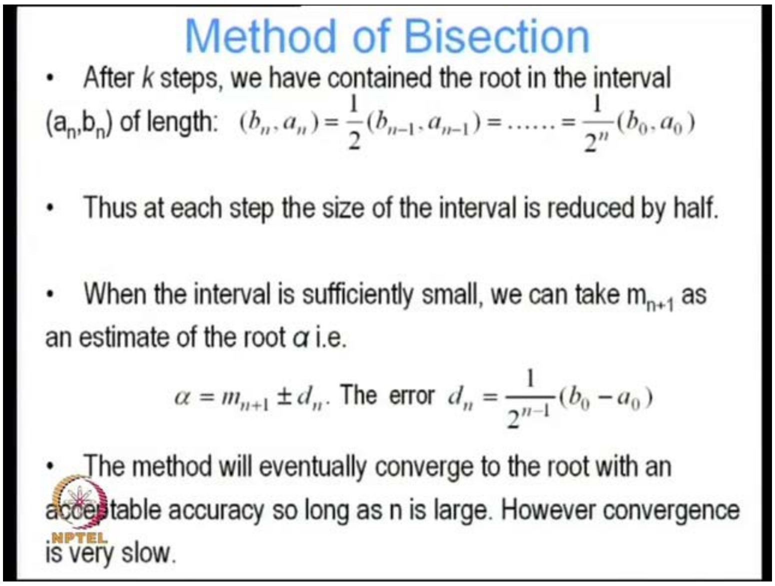

After k steps, we have contained the root in the interval a n, b n. So, now my root after n steps, basically it should be n steps really. After n steps I had contained the root in the interval a n, b n and the length of b n, a n is half of b n minus 1, a n minus 1, b n minus 1 a n minus 1 is again half of b n minus 2, a n minus 2 and so on and so forth. Until I can write b n, a n is equal to 1 by 2 to the power n b 0, a 0 thus at each step the size of interval is reduced by half.

So, if I take sufficiently large number of steps, so 2 to the power n the size of the interval is going to become, very small. And that interval is be going to contain the root right, then the size of the interval is very small is sufficiently small, we calculate the midpoint of the interval m n plus 1. Suppose, my a n, b n interval has reached the sufficient size restriction right, that my interval is now sufficiently small. Then I say that my root, my root is actually at the midpoint of the interval.

And error in the root is given by plus minus d n, where the error in the root is d n is equal to 1 by 2 n minus 1 b 0 minus a 0. So, I know the root up to this, up to the accuracy given by d n right and suppose, if my interval size, if I have sufficiently large intervals then since d n goes as 1 by 2 n minus 1 say suppose, I have like 16 bisections right 1 by 2 to the power 16 minus 1 that is a very small number right. So, my d n my error is going to be sufficiently small and I can say that m n plus 1 is my root up to this accuracy.

So, the method will eventually converge to the root with an acceptable accuracy. So, long as n is large, so what is the good thing about this method, good thing about this method is that, it is always going to converge right. If, I take sufficiently large number of n’s it is always going to converge; however, the convergence is relatively slow, compared to some of the other methods that we are going to look at right.

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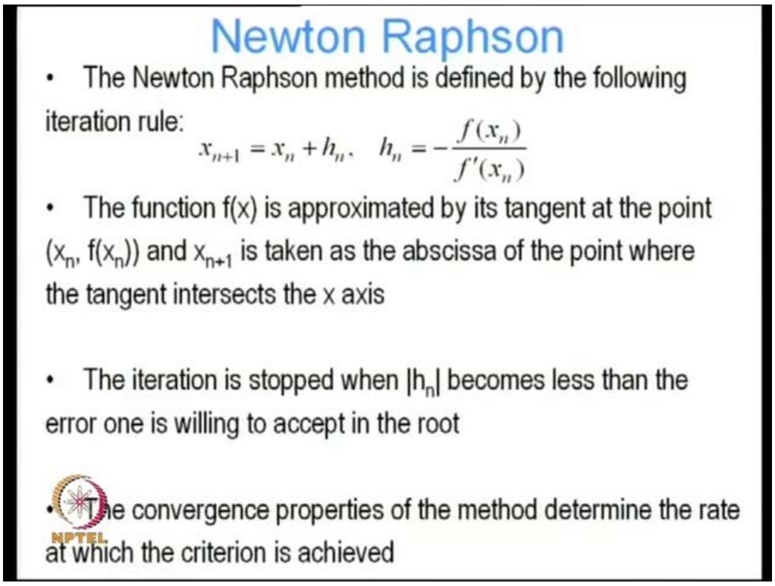

One such method, we are going to look at is the Newton raphson method which is defined by the following iteration rule, where the iteration rule says, that at a new increment at a new iteration I calculate my x n plus 1 using my old value x n plus an update which is given by h n. And what is the update, that update is given by the quotient of the function evaluated at x n and it is derivative evaluated at x n taken with a negative sign. Why do we have this update formula, well basically it is because the function f x is approximated by it is tangent at the point x n, f x n an x n plus 1 is taken as the abscissa of the point, where the tangent intersects the x axis.

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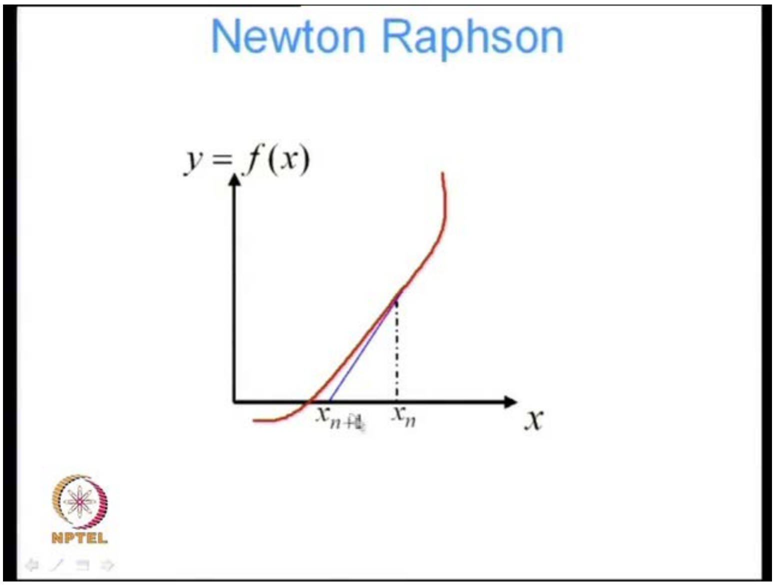

Basically, if you look at this picture given x n, we say that I am going to do a linearization of my function. So, my function is non linear, I am going to assume that within the small range it behaves like a linear function right. And what is that linear function going to be that linear function is going to pass through x n, f x n and it is going to have a slope which is equal to this slope at x n right.

So, this is my straight line, blue line is my straight line which approximates my non linear function which is given by the red line. And what is the criteria, that it passes through this point x n, f x n right it satisfies the function value at x n and it is slope is given by the slope of the function at x n right. And the point at which it intersects the x axis that is going to be my new update, my new iteration value x n plus 1.

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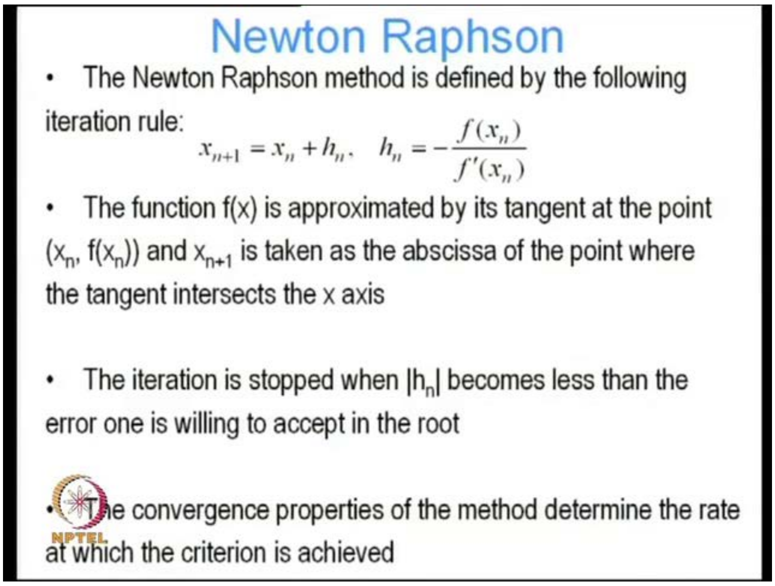

So, x n plus 1 is equal to x n minus f of x n by f prime of x n. The iteration when we stopped the iteration, well when we stop the iteration then my update becomes smaller and error that I will willing to accept in the root right. So, if I find that my changes, my solution, my update from iteration to iteration, my solution change is, so small that it is negligibly small, then I say that I have converged right and that is my solution. The convergence properties of the method determine the rate at which the criterion is achieved.

So, we said that, the bisection method well it is always going to converge that works bad about it is that it converges very slowly right. So, that is why we went for this method

Newton Raphson and we say therefore, Newton Raphson method, the convergence properties are comparatively better right. And we shall see why the convergence properties of Newton Raphson method are better.

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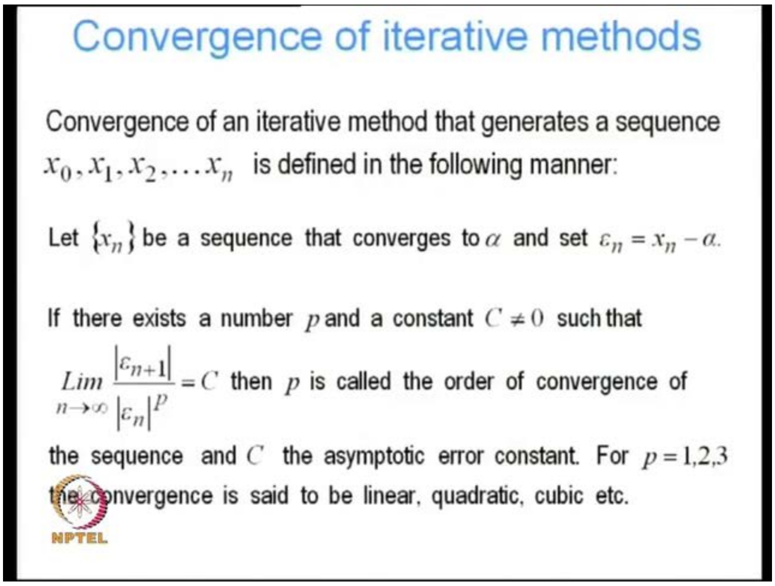

So, convergence of an iterative method that generates a sequence is defined in the following manner. So, this is stated it in a somewhat formal way, so let us suppose x n be a sequence. So, that x n, x 0, x 1, x 2, x 3, so these are my iterates right in my iterative method and this is a sequence, that converges to the true solution alpha. So, that as n goes to infinity x n is going to tend to alpha right.

And let me set epsilon n is equal to x n minus alpha. So, I want to define the error, at any iteration n and how am I going to define that I am going to define that by, saying epsilon n is equal to x n minus alpha. So, whatever is my iterate value, I subtract the true solution from that that gives me the error epsilon n. Now, if there exist a number P and a constant C which is strictly not equal to 0.

Such that, limit of mod of epsilon n plus 1 by mod of epsilon n to the power P when n tends to infinity is equal to the constant C. Then we say that my iterative method has ordered of convergence equal to P and C is my asymptotic error constant. So, when I have very large, so what in words what does this mean, it means that when I increase the number of iterations right, when if I sufficiently Largent n, if I look at iteration n and I if I look at the iteration n and I look at iteration n plus 1.

I calculate the error at iteration n, how do I calculate error iteration n, by taking the value of the iterate minus the true solution. And taking the normal of that how do I calculate the error iteration n plus 1, I again calculate the value of the iterate minus the true solution x n plus 1 minus alpha right. So, that gives me my epsilon n plus 1 and epsilon n and if it turns out that mod of epsilon n plus 1 is equal to some constant, times mod of epsilon n to the power P then I say P is called the order of convergence. So, let us think about it like this in just using numbers see suppose, at iteration n I have an error epsilon n is equal to 0.1 right.

And then at iteration n plus 1 I have an error which is 0.1. So, in that case what is going to be my P and what is going to be my c it is obvious that P is going to be 2 and c is going to be 1 because 0.1 square 0.1 is my error at n. So, 0.1 is my error at n plus 1, so 0.1 square is equal to 0.01, so the error has is reduced by squared the magnitude of the previous error. So, it is converged it has got quadratic convergence right, so P is equal to 2.

So, it has got quadratic convergence, which is very good actually and for P is equal to 1, 2 we can p can be anything P can be 1, 2 or 3 convergence is said to be linear quadratic, so for the bisection method, we have typically linear convergence P is equal to 1 for the Newton Raphson method, we have P is equal to 2. But, this P is equal to 2 we get quadratic convergence only near the root right, when it sufficiently close to the root we get quadratic convergence.

We will look at the secant method and secant method is somewhere between linear and quadratic, it is 1.6 something right. So, the order of convergence is better than my bisection method, but worse than my Newton Raphson method right. So, the larger the value of P, the better is my better are the convergence characteristics of my algorithm; that means, if I start with an error 0.1, if I have convergence, If I have quadratic convergence then it is going to go to 0.1 times, 0.1 forth root of that right. So, that is going to 10 to the power minus 5. So, 1 iteration it was 10 to error was 10 to the power minus 1, the next iteration it is going to be 10 to the power minus 5. So, that is wonderful right, so I need to take fewer iterations, when my error is going to converge my solution is going to converge faster.

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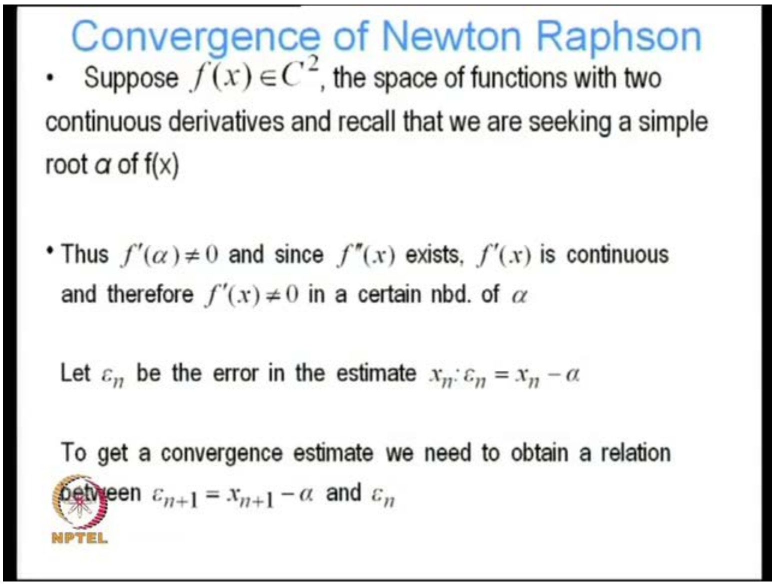

So, let us suppose, f of x the function f of x belongs to the class of functions C 2 which is basically a mathematical nomenclature, for saying that this function belongs the class of functions, which have continuous second derivatives right. C 0 means, only the function is continuous and it is partial, it is derivatives are piecewise continuous, C 1 means that function is continuous, as well as the derivative C 2 means the function is continuous as well as it is second derivative right.

So, suppose f of x belongs to C 2 the space of functions, with two continuous derivatives and let us, we call that we are seeking a simple root alpha of f x. Last class we talked about simple roots and multiple roots, let me try to or we will we are going to talk about that later again. So, since it is a simple root, I know that f prime of alpha is not equal to 0 right and since, this is the definition of a simple root right, a simple root says that at the root where, the function f of alpha is equal to 0, f prime of alpha is not equal to 0 that is it is not a stationary point right.

The function is going like this or like this, it is not a stationary point at the root right. So, it because at alpha what does f prime alpha equal to 0 means; that means, there are multiple roots right. So, f prime alpha not equal to 0 means, that there are that is a simple root right it is just crossing like that, right and since, we said that f of x belongs to C 2 to the space of functions with, continuous second derivatives; that means, f double prime of x must exist, right since if something is continuous it had better exist right.

Since, the second derivative is continuous f double prime x must exist, right and f prime x is continuous, why is f prime x continuous well. So, if f prime x is not continuous, there is no chance of f double prime x being continuous, so f prime x must be continuous. And therefore, f prime x is not equal to 0 in a certain neighborhood of alpha, why because f prime of alpha is not equal to 0 and I know that, f prime of x is continuous.

Therefore, that what is continuous means, there is a non 0 I can draw always a non 0 sphere, around alpha where f prime of x is not equal to 0 right. Because, f prime of alpha is not equal to 0 right, at alpha f prime of alpha is not equal to 0 and I know that f prime of alpha f prime of x is continuous; that means, there is a non 0 interval, centered around alpha at which f prime of x is not equal to 0 right I may. So, there is there must exist a neighborhood, centered around alpha in which f prime x is not equal to 0.

So, let epsilon n be the error in the estimate x n. So, epsilon n is equal to x n minus alpha, so that is the error at iteration n, is x n minus alpha, to get a convergence estimate we need to obtain a relation, between epsilon n plus 1 and epsilon n. So, what is epsilon n plus 1, it is x n plus 1 minus alpha.

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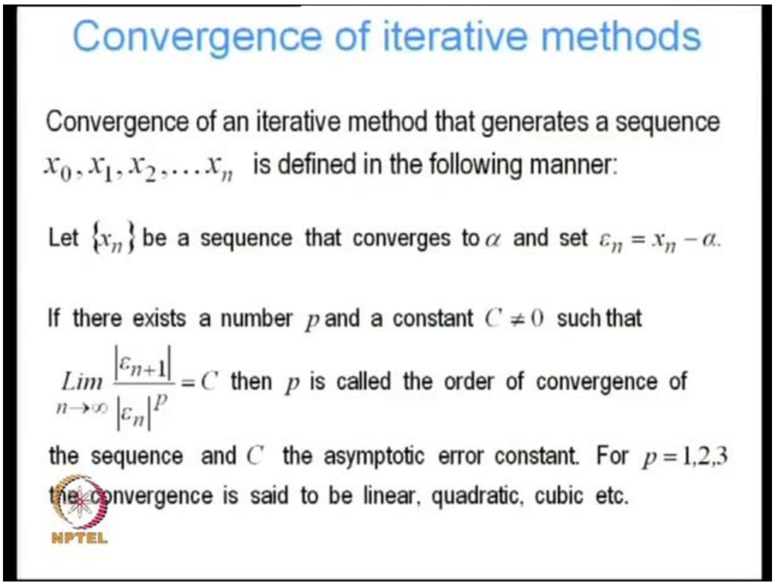

So, we saw what did, we see that this is how we obtain a convergence estimate right, by obtaining a relationship between the error at the n plus 1'th iteration and the error at the n’th iteration. If we can write a relation between epsilon n plus 1 and epsilon n, I can find out what is my rate of convergence right.

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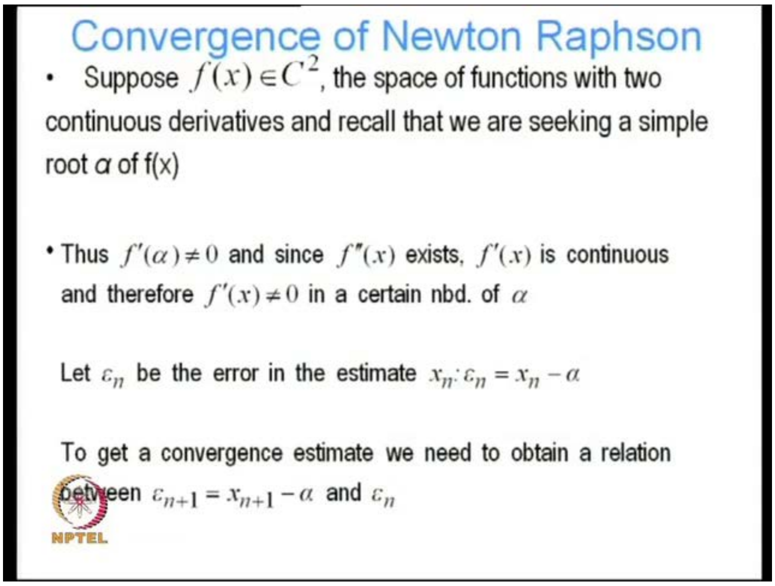

So, to get a convergence estimate we need to obtain a relation between, epsilon n which is equal to x n plus 1 minus alpha and epsilon n.

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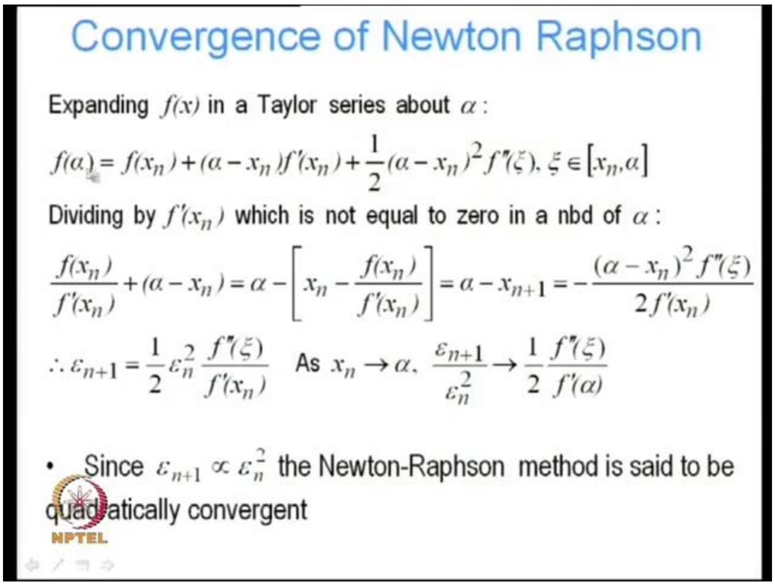

So, expanding f of x in a Taylor series about alpha, so what we do, we will expand it f of x in a Taylor series about alpha, so we get. So, basically we are writing it as f of x n plus alpha minus x n plus, so this is my perturbation alpha minus x n, so f of x n plus alpha minus x n plus f prime of x n plus I am putting this as a remainder term, right like a Taylor series expansion this is the remainder, provided that xi belongs to the interval x n and alpha right.

So, basically I just do a Taylor series expansion of f about alpha. Then I divide throughout by f prime of x n and how can I divide because I know that f prime of x n is non 0 in a neighborhood of alpha, if it I cannot divide anything by 0 right. So, because I am assured that f prime of x n is non 0 in a neighborhood of alpha, so I can divide it by f prime of x n. So, I get f of x n plus x prime of x n plus alpha minus x n and this f prime x n goes away, so this I can rewrite as alpha minus x n minus f of x n by f prime x n just by rearranging terms right.

So, I am just pulling this within the bracket and putting the x n here and bring out alpha there right, this with this, what is this, this is exactly my expression for x n plus 1 right it is the iterate at n plus 1. So, this is my expression for the new iterate, so this becomes alpha minus x n plus 1 and this is equal to this term, right this is equal to this term which is equal to alpha minus x n square f double prime i and I have divided throughout by f prime x n. So, f prime x n appears at the bottom right.

So, what do we have, so that not, but what is this alpha minus x n plus 1 is just epsilon n plus 1 right, this is the error at the n plus 1'th iterate, this alpha minus x n is nothing, but epsilon n right. So, I can write epsilon n plus 1 is equal to half epsilon n square f double prime of xi divided by f prime of x n right, and as x n tends to alpha f prime of x n is going to tend to f prime of alpha. So, I can write epsilon n plus 1 divided by epsilon n square tends to half f double prime x i by f prime alpha this is a constant right, because I am evaluating it at xi and alpha. So, again I have got this error estimate this ratio of these errors, I have got it in that form.

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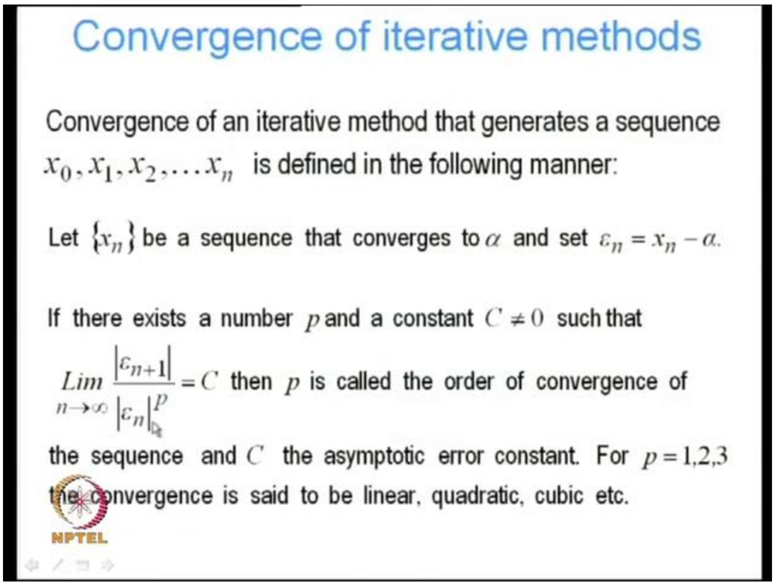

Epsilon n by 1 plus epsilon n to the power P is equal to C I can clearly identify.

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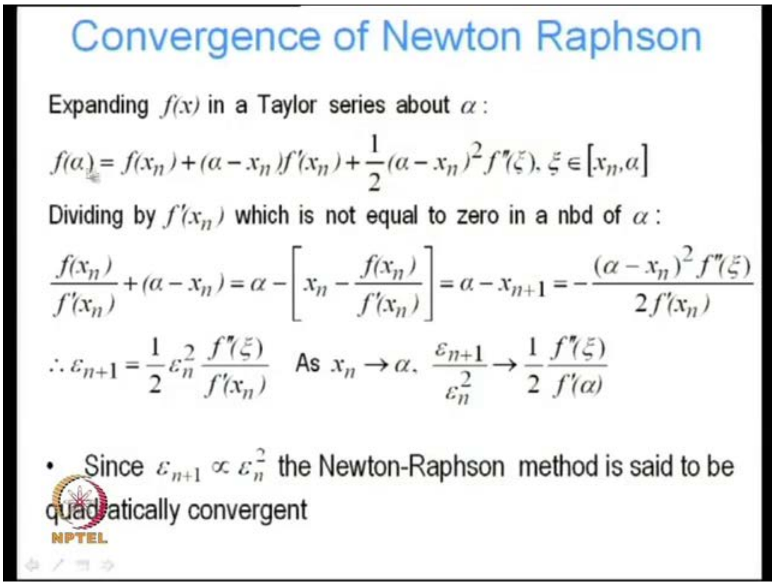

That in that case P, is in this case P is equal to 2 and my constant C is given by this. So, since epsilon n plus 1 goes as epsilon n square, the Newton Raphson method is said to be quadratic ally convergent.

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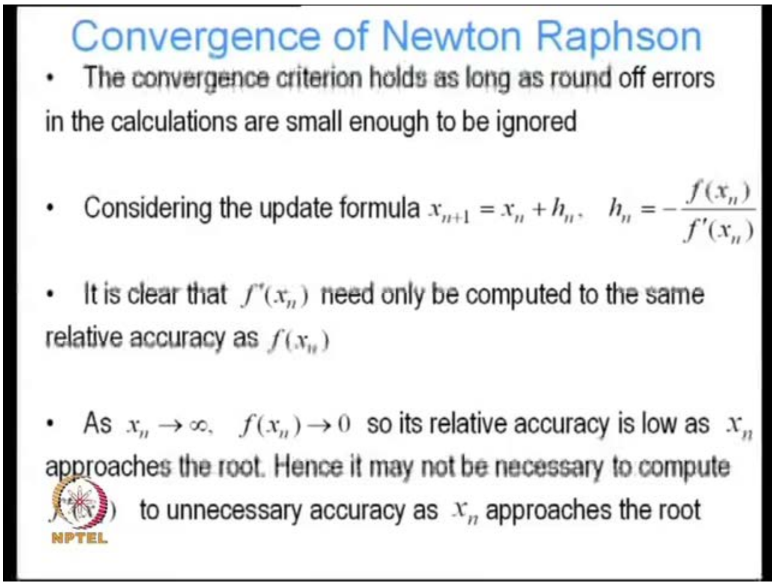

So, the convergence criterion holds as long as round off errors. So, what are the criteria, first criteria is that we have assumed everywhere that, we are close to the root right, we have assumed that in a neighborhood, we have looked at a neighborhood where, f prime of x n is not equal to 0. So, we have to stay within sufficiently close to the root, right because far away from x, far away from the root I have no guarantee that f prime of x n is equal to going to be non 0 right.

Because, f prime of alpha is not equal to 0, I know that if I look at a sufficiently close interval centered around the root, I can always be sharpened at f prime of x n is not going to be 0. But, if I move away from that from alpha and there is no guarantee, f prime of x n may be, may not be equal to even may be equal to 0 right may not be non 0. So, when that whole proof is going to break down, right my whole proof of convergence is going to break down.

So, you can see why it is very important to keep in mind, that the Newton raphson method convergences quadratic ally near at the root right. So, that assumption is built in there, far away from the root there is no guarantee that it is going to converge linearly right. The other assumption that, we made is that round off errors in the calculations are small enough to be ignored right.

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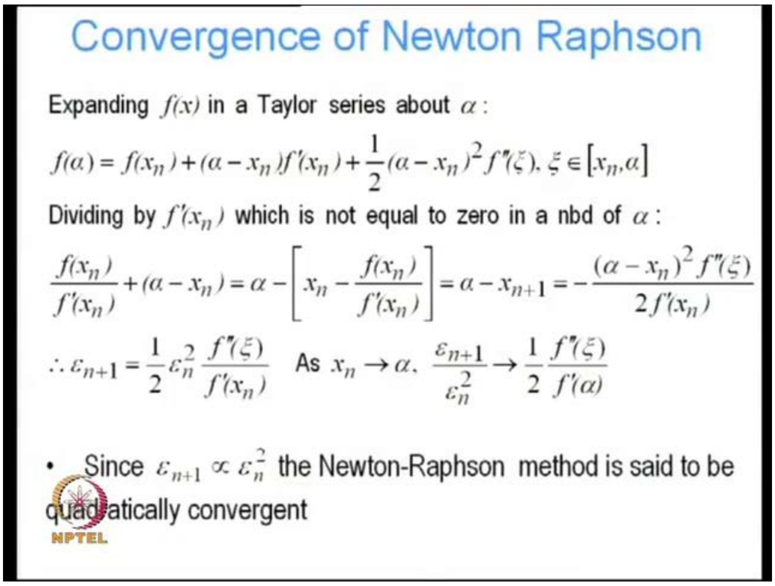

Why did you make that assumption, well we said that all this epsilon n plus 1 which is equal to alpha minus x n plus 1 and epsilon n which is alpha minus x n, can be calculated to infinite precision right. So, there are no errors due to round off, with any finite precision machine right, any finite precision computer I know that when I compute epsilon n plus 1 is equal to alpha minus x n plus 1, I am not exactly going to get exactly alpha minus x n plus 1 if I going to be round off errors.

And what are those round off errors going to depend, well we looked at that lots of times in previously in the course, this rounds off errors are going to depend on my machine precision right on the precision of my computer right. So, everywhere the rounds off errors are unavoidable, so I cannot, so every time I do numerical computations, I have to leave that round off errors, but the important thing to remember is that this estimate that we have caught, assumes this quadratic convergence result assumes, that there are no round off errors right.

So, if there are significant round off errors am I going to get full quadratic convergence no, but if my round off errors are sufficiently small, am I going to get sufficiently close to quadratic convergence yes right. But, the it is I know I never going to get, full quadratic convergence because round off errors are never going to be 0.

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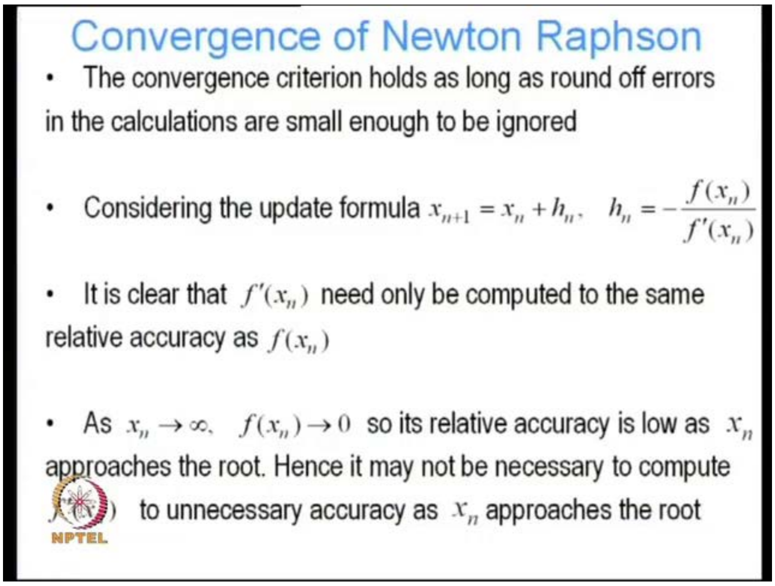

So, the convergence criterion holds as long as round off errors in the calculation as small enough to be ignored. Let us consider again the update formula x n plus 1 is equal to x n x n plus h n, h n is equal to minus f of x n divided by f prime of x n. It is clear that f prime of x n needs only be computed to the same relative accuracy as f of x n, this is a very important result, which has got very important implications, for particularly, formality dimensional approaches for Newton iterations.

What am I saying, I am saying that when you compute your update h n, there is no point in computing f prime x n to very, very high accuracy. If your f of x n is not very accurate right because there is no point in computing only one part of this thing, very accurately while if f of x n is not accurate that is going to put all the reason. So, my h n is also going to be inaccurate right, so there is no point in computing the derivative with a high degree of accuracy, if I cannot compute my function value at an iterate with a sufficiently high accuracy.

Why when can I not compute my function value with a sufficiently high accuracy. When my f of x n is when I am near the root right because near the root at near the root I know my function value is going to be 0 right. So, when I approach the root, my function value is going to become smaller and smaller, so as I become as the function value becomes smaller and smaller, it becomes harder and harder to compute it with sufficient accuracy right, because numbers becomes small. So, if I have numbers which are of the order of

10 to the power minus 5 and my machine precision is 10 to the power minus 6, it becomes harder to compute f of x n accurately right. So, there is going to be, more errors in the function evaluation closer to the root, so there is no point in computing the derivative with a great deal of accuracy, if my function value has got a lot of errors built into in.

Simply because my function, my iterate is sufficiently close to the root. So, my function value is very small, so these numbers are very small, so the round off is going to become more and more important right. So, it is clear that f prime x n needs only be computed, to the same relative accuracy as f of x n right. So, as x n goes to infinity, as I go as and the number of iterations increases f of x n goes to 0 because I am approaching closer and closer to the root.

So, it is relative accuracy is low as x n approaches the root because f of x n becomes closer and closer to 0. So, it is relative accuracy is low, hence it may not be necessary to compute f prime of x n to unnecessary accuracy as x n approaches the root. So, as I go close to the root, it may not be necessary to compute f prime x n with a lot of accuracy sometimes, you can get away with not computing f prime x n at every iteration, which is a very tremendous value, when we are looking at a multi dimensional problem.

We are computing f prime x n basically, involving, evaluating an n by n matrix right and that is extremely expensive. So, if we can avoid doing that every iteration that saves a lot of computational time right hence, it may not be necessary to compute f prime x n at each iteration, the rate of convergence will only be slightly slower. So, we do not need to converge to compute f prime x n, we do not need to compute the derivative at each iteration with suppose, I compute the derivative at n, I can use the same derivative that apply at n plus 1.

Why because the error I get by using the derivative at n, at n plus 1 is less or comparable with a error in the function value, error in the evaluation of the function itself right. So; that means, that error is not going to govern right. Hence, it may not be necessary to compute f prime x n at each iteration, the rate of convergence will be slightly slower, so if we do this approximation, we do not compute the derivative at each iteration, we can be guarantee that we are not going to get quadratic convergence. But we if my error in f prime x n is not too high right, is not too high it is small compared to my it is comparable to my error in x n, I am not going to get too far away from quadratic convergence right.

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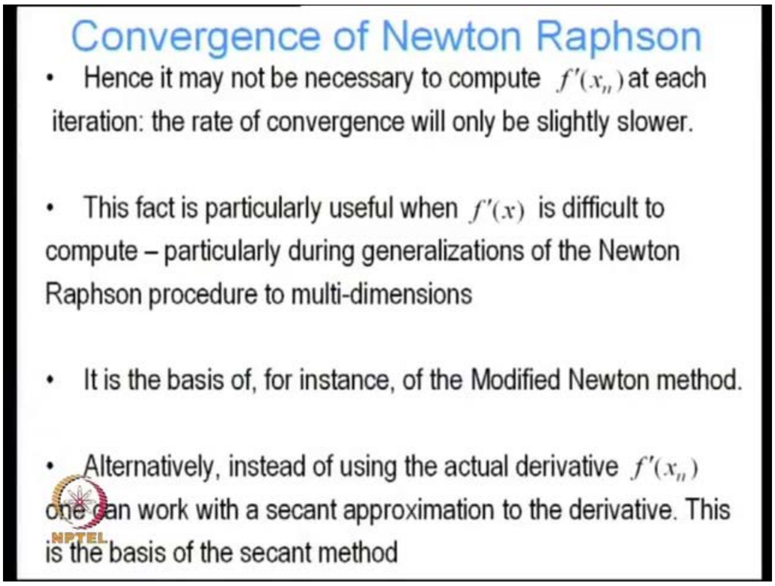

This fact is particularly useful when f prime x is difficult to compute, particularly deriving generalizations of the Newton Raphson method to multiple dimensions as I mention, as soon as you go to multiple dimensions, when have f as a function of x 1, x 2, x 3, to x n I have to compute the gradient matrix right. I have to compute terms like del f 1, del x 1, del f 1, del x 2, del f 1, del x n, del f 2, del x 1, del f 2, del x 2, del f 2, del x n and so on and so forth. And terms at each iteration right that is very expensive.

So, if I can avoid doing that, if I can persist with my old tangent I can persist with my old derivative, then that saves a lot of computational expense right. And that is the basis or what is known as the modified Newton Raphson method, called multidimensional non linear equations right. So, the idea is becomes clear, if you look at a one dimensional problem.

So, alternatively that is what we wanted to say about the full Newton Raphson method. Let us switch over now to the secant method, when instant of actually evaluating the derivative, evaluating the tangent at f x at x n, I try to get an approximation to the tangent, how do I get an approximation to the tangent, by constructing a secant right. And I say that the slope of a function at x n is given by the slope of a secant, which connects the function values at x n and x n minus 1 right.

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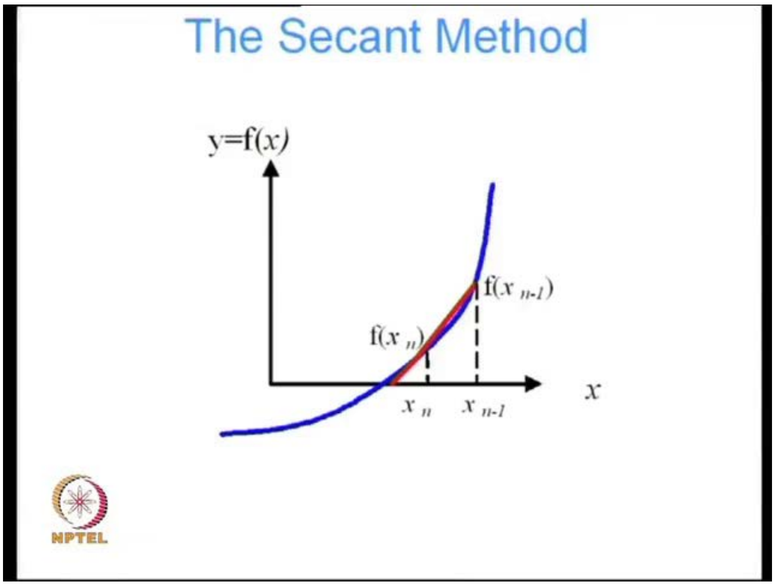

So, this is a picture I hope I have a picture of the secant method which is here, which says that earlier around in a Newton method I was actually constructing the accurate tangent here. Now, I do not construct the tangent, I see that I approximate the tangent by the secant, what is the secant it is the function value evaluated at an this point, minus the function value evaluated at that point, divided by the interval right.

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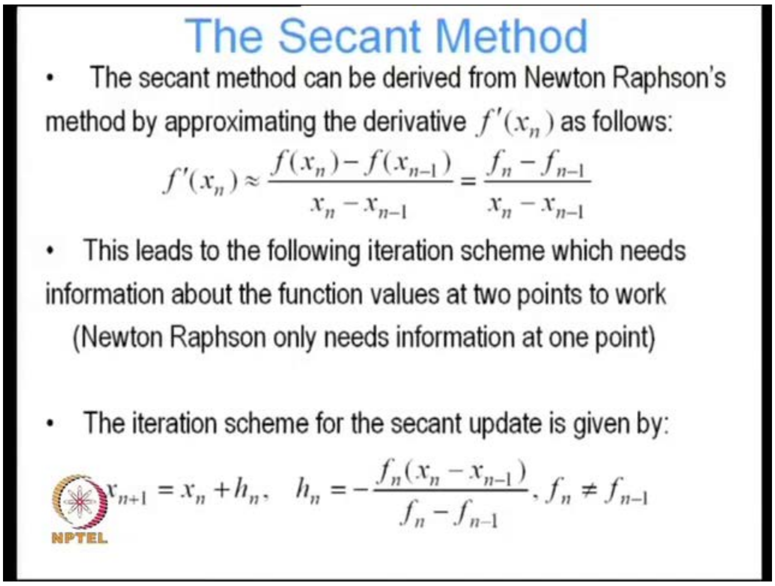

So, that is the secant method, which can be derived from Newton Raphson's method by approximating the derivative f prime x n as follows. So, f prime x n is approximately equal to f of x n minus f of x n minus 1 divided by x n minus x n minus 1, I am using this slightly simpler notation, I can write it as f n minus f n minus 1 divided by x n minus x n minus 1. So, this leads to the following iterative scheme, which needs the information about the function values at 2 points to work.

Remember, that for the Newton Raphson method I only needed evaluation I note only needed to evaluate the function, and it is derivative at one point x n. But, for the secant method to work, I need to evaluate the function at two points, at n and n minus 1 and this is the iteration scheme for the secant update which of course, resumes that f n f of n is not equal to f of n minus 1, we looked at that before.

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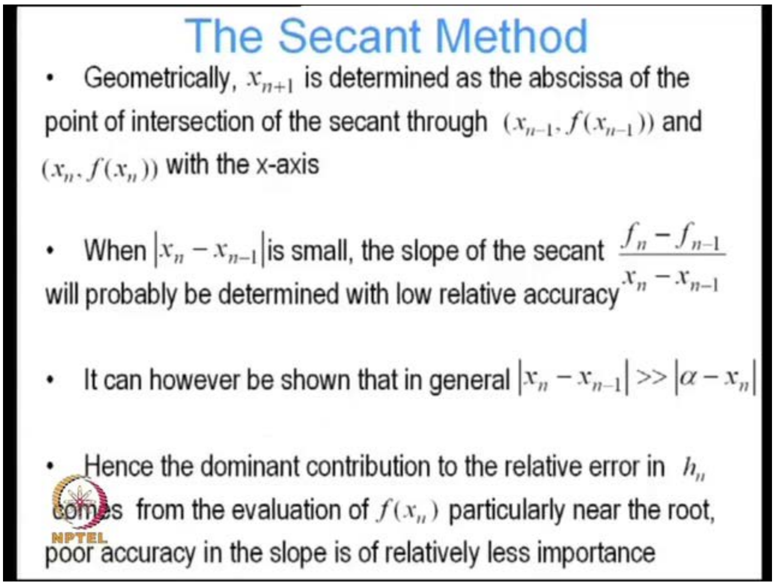

So, geometrically x n plus 1 is determined as the abscissa of the point of intersection of the secant through x n minus 1 f x n minus 1 and x n f x n with the x axis.

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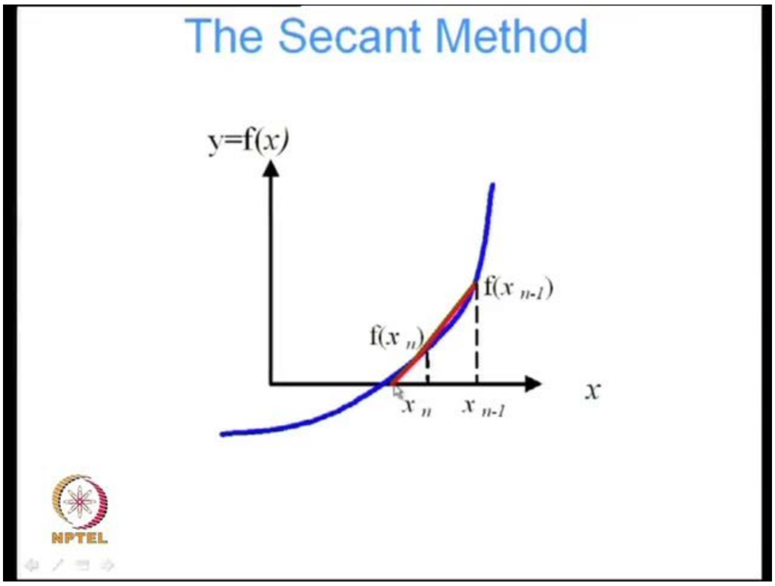

So, basically it is determined my new iterate, is determined by the point of intersection, with the x axis right of the secant right.

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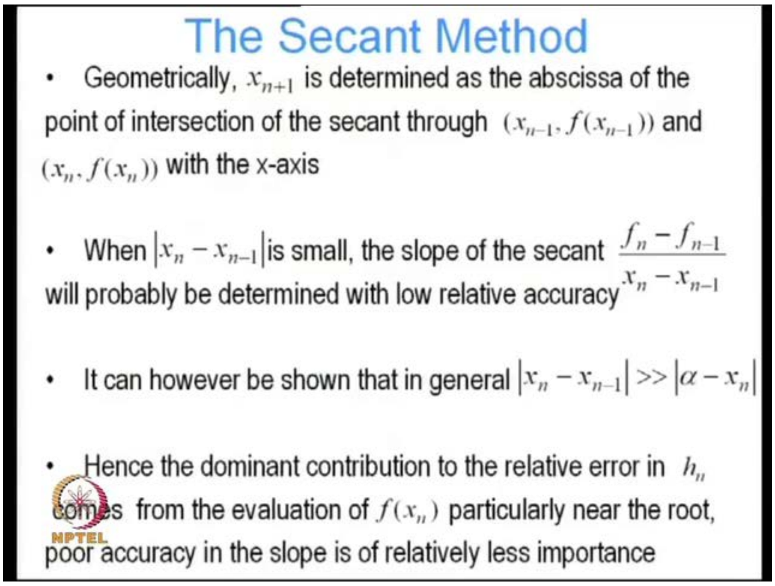

Now, when bound of x n minus x n minus 1 is small, the slope of this secant the slope of the secant being this right.

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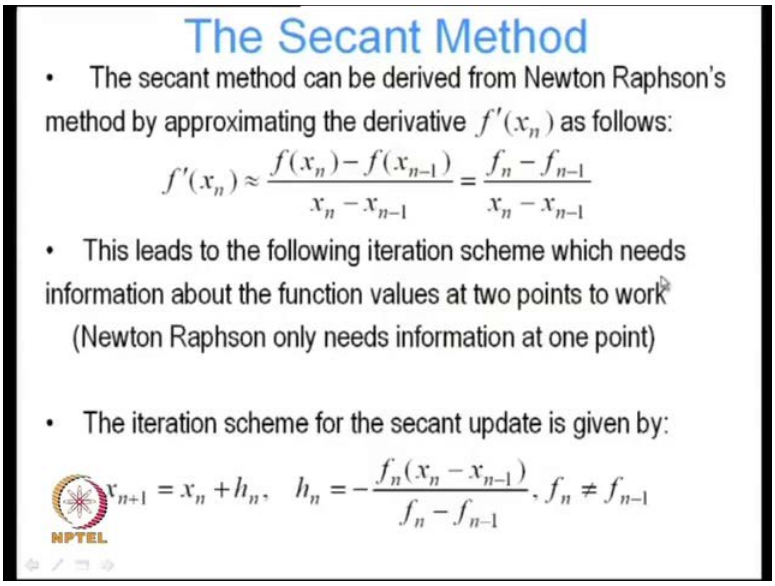

The slope of the secant being this, this slope when x n minus x n minus 1 is small you can see that, this slope is going to be determined with relatively low accuracy right, there are going to be no errors in the evaluation of the slope.

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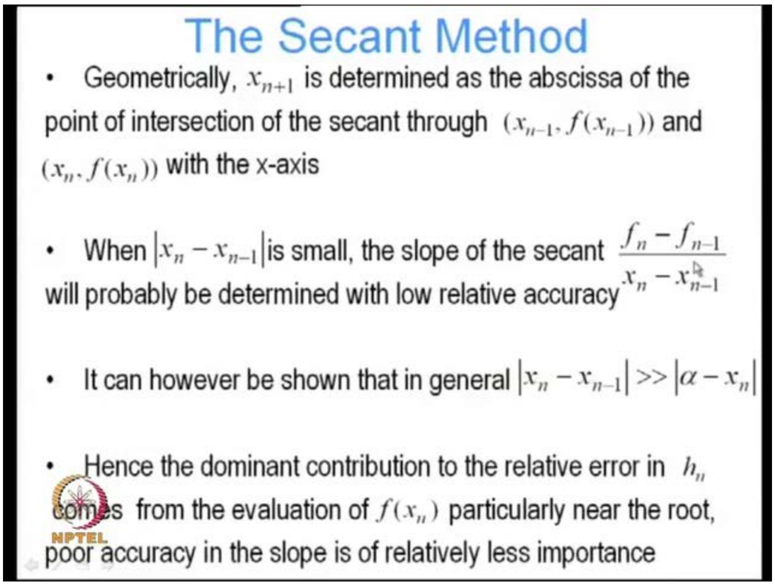

However, it can be shown that a mod of x n minus x n minus 1 is always greater than alpha minus x n. So, the value of the two, value of the iterates between x n and x n minus 1 is, always going to be much greater than alpha minus x n. Hence, the dominant contribution to the relative error in h n comes from the evaluation of f of x n particularly near the root. Poor accuracy in the slope is of relatively less importance.

As the dominant contribution to the relative error, comes from geometrically x n plus 1 is determined as the abscissa of the point of intersection of the secant to the x n minus 1 f of x n minus 1 n x n f of x n with the x axis. We can see, that one f mod of x n minus x n minus 1 is small, the slope of the secant will be determined with low relative accuracy, this term is going to become smaller and smaller. So, those errors are going to start becoming significant.

However, it can be shown that in general, mod of x n minus x n minus 1 is always going to be greater than alpha minus x n. So, now, here, so the what does alpha minus x n correspond to since, the dominant contribution to the relative error in h n comes from the evaluation of f of x n particularly the near the root. Poor accuracy in the slope is relatively less important why is that, well let us go back and take a look.

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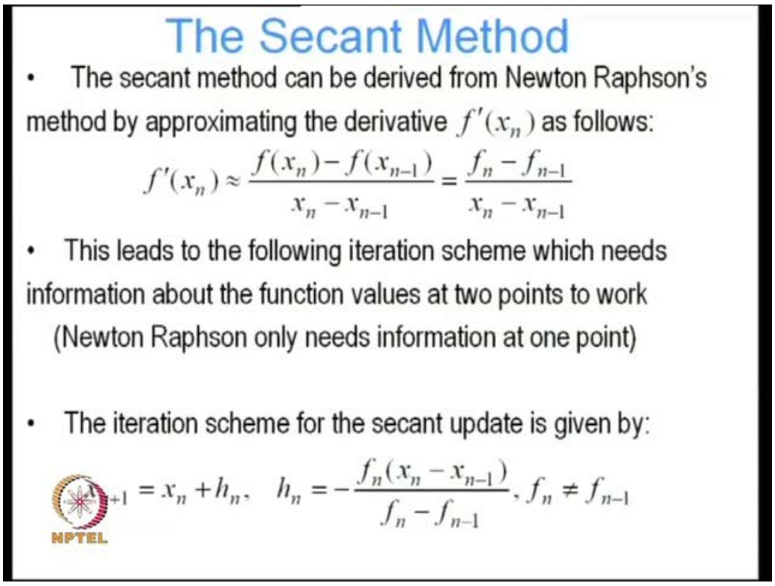

So, this error, the error due to this right this becoming small is going to be less important than the error due to the numerator becoming small right. Because, the error in the function is probably going to govern right because as the function goes to alpha, the error is going to the function values are going to become closer and closer to 0. So, here again as with the Newton Raphson the error in the function evaluation is going to govern.

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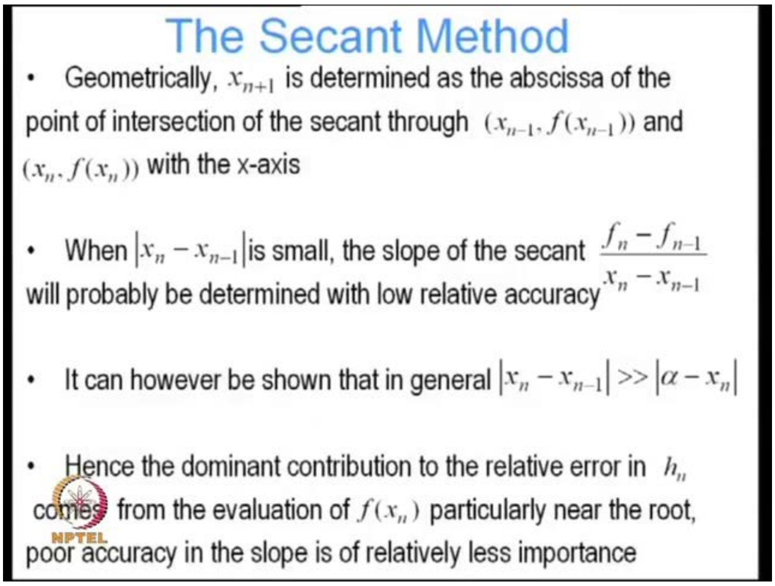

Hence, the dominant contribution to the relative error in h n comes from the evaluation of f of x n particularly near the root, poor accuracy in the slope is of relatively less importance.

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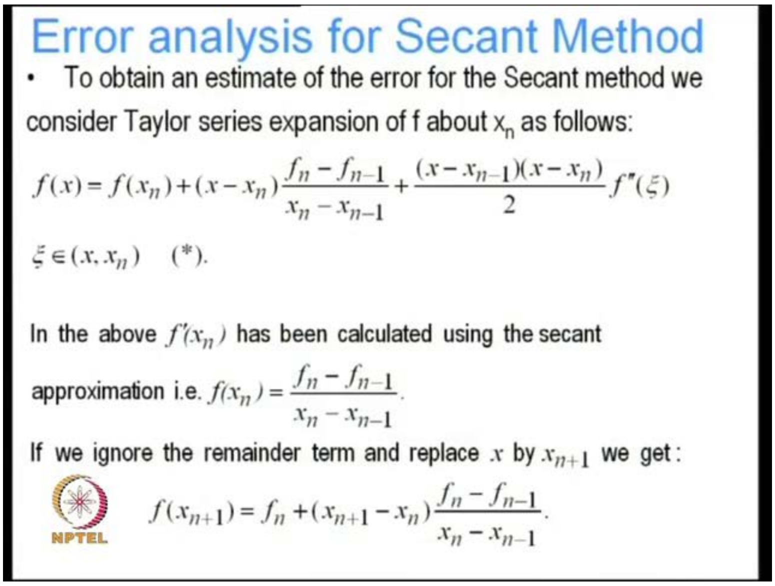

Totally similarly to what we did for the Newton Raphson method, we would like to obtain an estimate of the error for the secant method. So, we consider an expansion of f about x n, which is similar to a Taylor series expansion I have just said that it is not exactly a Taylor series expansion, you can see from the following it is called using something which is known as Newton in peculation formula. But, it is sufficiently close to the Taylor series expansion I did not want to talk about Newton interpolation formula.

So, it is sufficiently close to the Taylor series expansion for you to get the idea. So, what we are doing is we are expanding this f of x right and we are expanding it about f of about x n right. So, we are writing f of x using Newton's interpolation formula, so we are writing f of x is equal to f of x n plus x minus x n right, times the slope which are now evaluating assuming the secant formula right. So, now, I am assuming the slope evaluated by the evaluating is using the secant formula.

And then there is this abdicate. In the abdicate, we will see it is different from the traditional Taylor series right why because now I am no longer using x minus x n square right, I am using x minus x n minus 1 times x minus x n by factorial two times f double prime of xi where, xi belongs to the which is the typical remainder term consists in a Taylor series, but only difference between a Taylor series and this term is that I am using x minus x n minus one times x minus x n. I am not using x minus x n square right.

This is known as Newton's interpolation formula. So, I have to take it for me un trust because we are not going to cover that in detail right because it can be written like this. So, in the above f prime of x n has been, you have evaluated the slope using the secant assumption that is we have used f of x n is equal to f of n minus f of n minus 1 divided by x n minus x n minus 1. So, if we ignore the remainder term here, if we for the timing if I get the remainder term and we replace x by x n plus 1.

So, we replace x by x n plus1 we get f of x n plus 1. So, we replace x here by x n plus 1 where f of x n plus 1 is equal to f of x n which I write as f of n plus x n plus 1 minus x n times f of n minus f n minus 1 divided by x n minus x n minus one. Therefore, the time do not resume that my remainder is, so small that I can throw it away.

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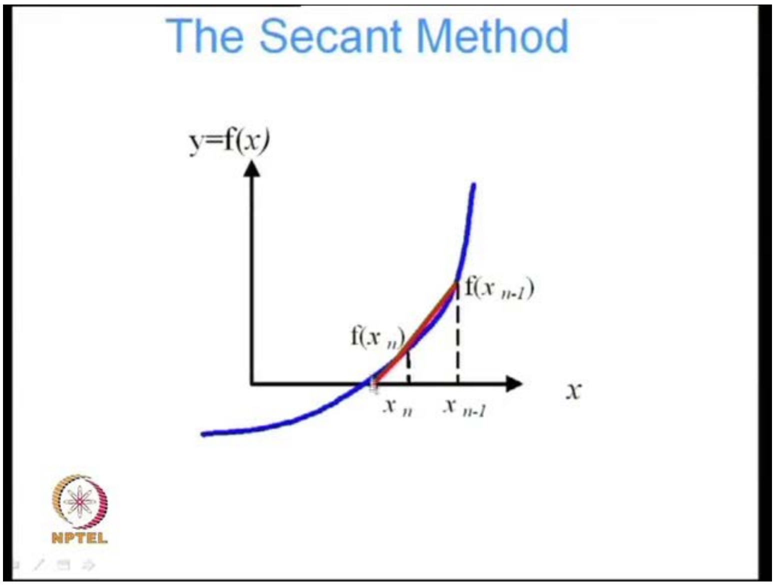

And since, in the secant method we know that f of x n plus 1 is equal to 0 right, why is that , let us go back and take a look at a secant method picture again. So, the assumption is that f of x n plus 1 is equal to 0 right.

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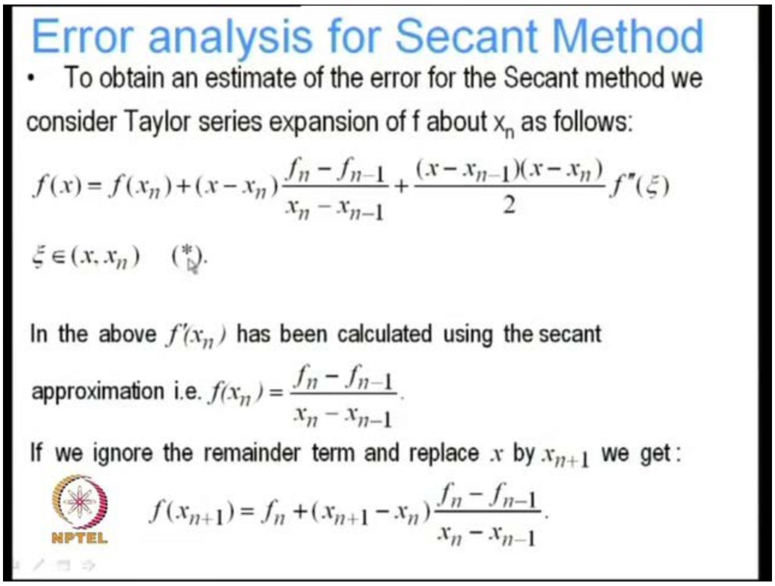

So, secant method we assume that, so we have from this expression this term becomes 0.

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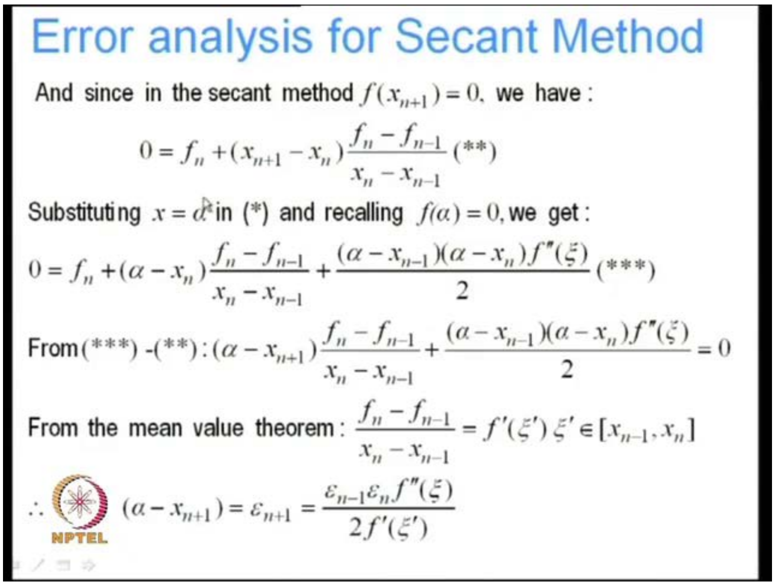

So, f 0 is equal to f of n plus x n minus 1 minus x n times f of n minus f n minus 1 divided by x n minus x n minus 1. And substituting x is equal to alpha, then we leave it like that and then we substitute x is equal to alpha in this expression and we call that f of alpha is equal to 0 because alpha is my root right. So, we have 0 is equal to f of x n which I am writing as f of n plus x minus x n becomes alpha minus x n. So, as alpha minus x n times f of n minus f n minus 1 x n minus x n minus 1 x n divided by x n minus x n minus 1 plus alpha minus x n minus 1 times alpha minus x n by 2 times f double prime of xi right.

So, alpha minus x n minus 1 alpha minus x n f double prime of xi by 2. So, you get that from my previous equation by substituting alpha and take remainder that f of alpha is equal to 0. So, and then if I subtract this equation minus that equation, I have alpha minus x n plus 1 times this, term this term cancels out f n, f n cancels out. So, at this term plus this term is equal to 0 right alpha minus x n minus 1 alpha minus x n f double prime xi by 2 equal to 0.

And then from the mean value theorem, I can write f n minus f n minus 1 by x n minus x n minus 1 is equal to the derivative f prime of xi times f prime of xi because this is in denominator right provided xi prime belongs to x n minus 1 times x n. So, I can write this expression as alpha minus x n plus 1, is which is equal to by definition this is equal to epsilon n plus 1. And let us look here, what is this alpha minus x n minus 1 it is epsilon n minus 1 right.

It is the error at the n minus 1'th iterate alpha minus x n is the error at the n'th iterate. So, as epsilon n minus 1 epsilon n times f double prime of xi divided by 2 f prime of xi, this way bringing this term to the denominator right, this term I am representing my f prime is xi prime and let me bring this to the denominator.

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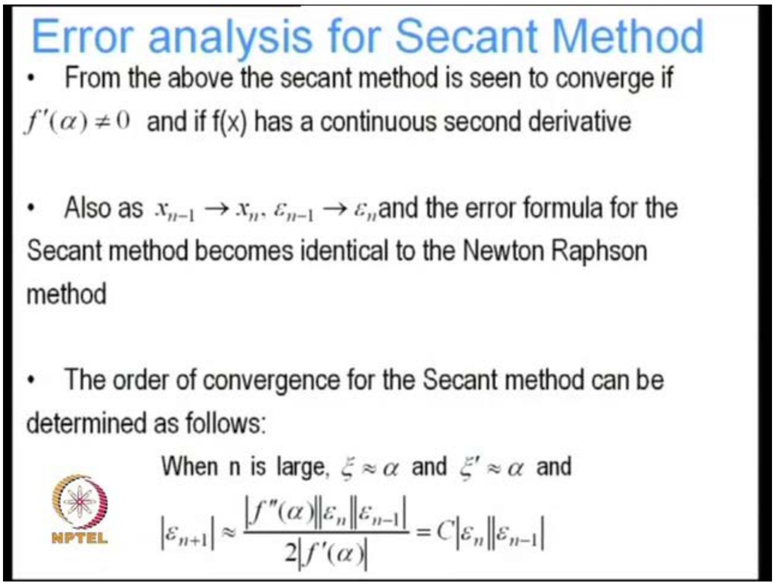

So, what do we have, so from the above the secant method is seen to converge, if f prime of alpha is not equal to 0 and if f of x has a continuous second derivative why is that.

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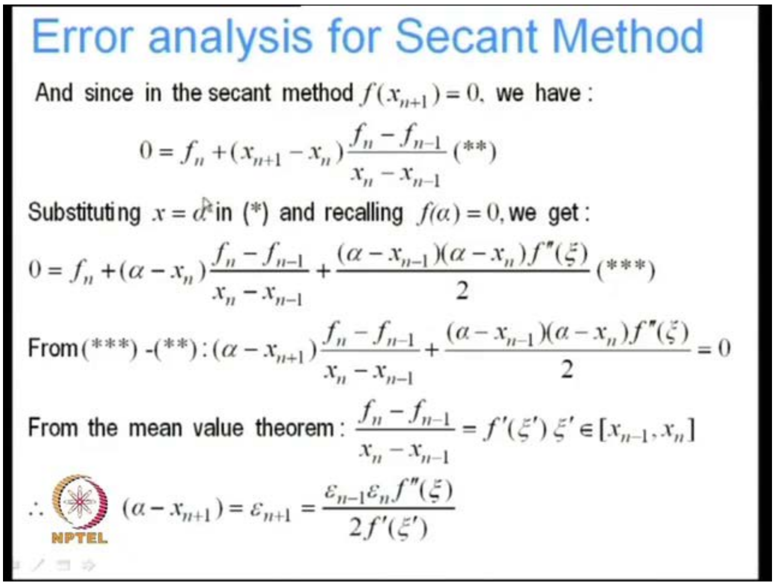

This is formed to converge when f prime of this is not equal to 0 right. So, the converge this is going to hold, only when this term is not going to be 0 right. So, again what is the criteria, that criteria is that there is sufficiently close to my root because only when it is sufficiently close to the root is my as I am guaranteed, that f prime of alpha is not equal to 0.

Because, f prime of x n is f prime of xi prime is not equal to xi I am always guarantee that f prime of alpha is not equal to 0 because it is a by definition it is a simple root right why is it is a simple root f prime of alpha is not equal to 0, but I am guaranteed that f prime of xi prime is not equal to 0, provided that xi prime lies in a small neighborhood about alpha, where f prime of x n is not equal to 0.

Secant method is seen to converge if f prime of alpha is not equal to 0 and if f of x has a continuous second derivative right. Since, f of x has a continuous second derivative f prime of x must be continuous, which tells me that in a sufficiently small neighborhood around alpha, my derivative is not going to be 0. So, also as x n tends x n minus 1 tends to x n epsilon n plus 1 epsilon n prime x 1 is going to epsilon n and this secant method is going to secant convergence formula is going to collapse to the Newton Raphson formula. Why is that let us go back and take a look again.

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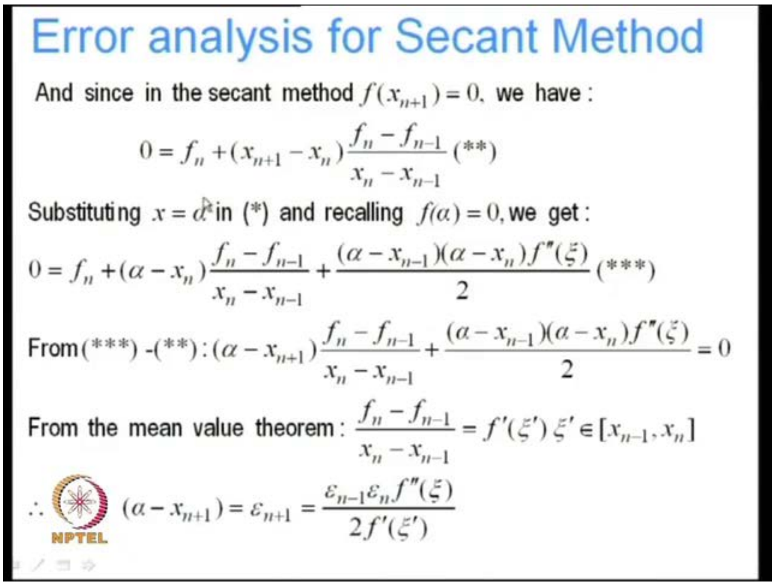

So, when my iterations are converging my x n minus 1 and x n are going to be very close to each other. So, my epsilon n minus 1 and epsilon n are also going to be very close to each other. So, this term I can basically represent epsilon n minus 1 epsilon n, I can write it as epsilon n square right and then I get back my Newton Raphson update formula.

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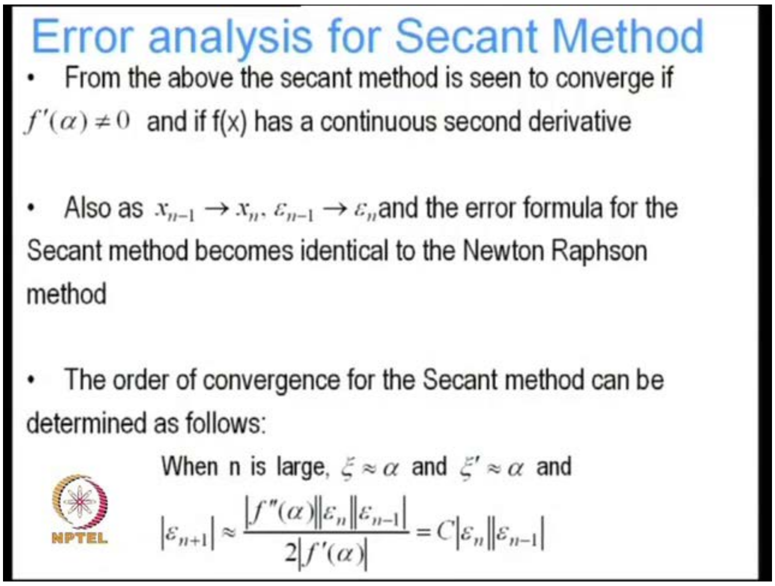

What is the order of convergence for the secant method, well we can find it out quite easily. Suppose, when n is large then xi is approximately equal to alpha and xi prime is approximately equal to alpha why is that well.

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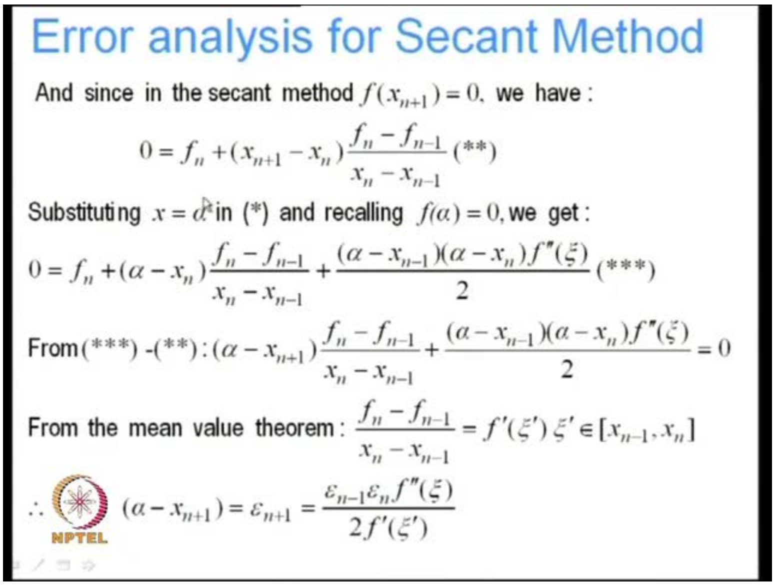

What is xi, prime xi prime belongs to x n minus 1, x n right then when we are close to the solution both x n minus 1 and x n are also very close to alpha right. So, I can replace. f prime of xi prime as f prime of alpha right, sufficiently close to the root I can replace the f prime xi prime in this formula, my f prime of alpha. Similarly, I can replace f double prime xi, xi again sufficiently close to the root I can replace it by it is value of the root right.

Because, it is continuous right I can and if I am sufficiently close to the root if even if I am not exactly at the root the value is going to be infinitesimally different from the value of alpha because the function is continuous right. So, I can replace these values by the values of alpha.

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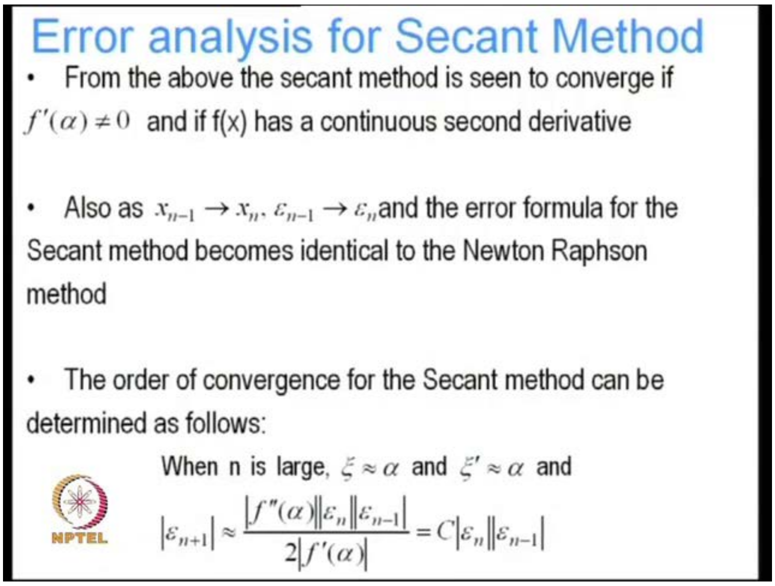

And in that case, I can get this. So, I am just taking mod of both sides, I am taking mod of both sides of this expression right. So, mod of epsilon n plus 1 is approximately equal to mod of f double prime mod of epsilon n mod of epsilon n minus 1 divided by twice mod of f prime of alpha. So, this I can treat it as a constant f double prime of alpha because the root is known the derivative of the function, double derivative of the function at a certain point they are constants right. So, I can replace this term by a constant and write it as mod of epsilon n plus 1 is approximately equal to C times mod of epsilon n times mod of epsilon n minus 1. So, let us you how we can use it to find the order of convergence.

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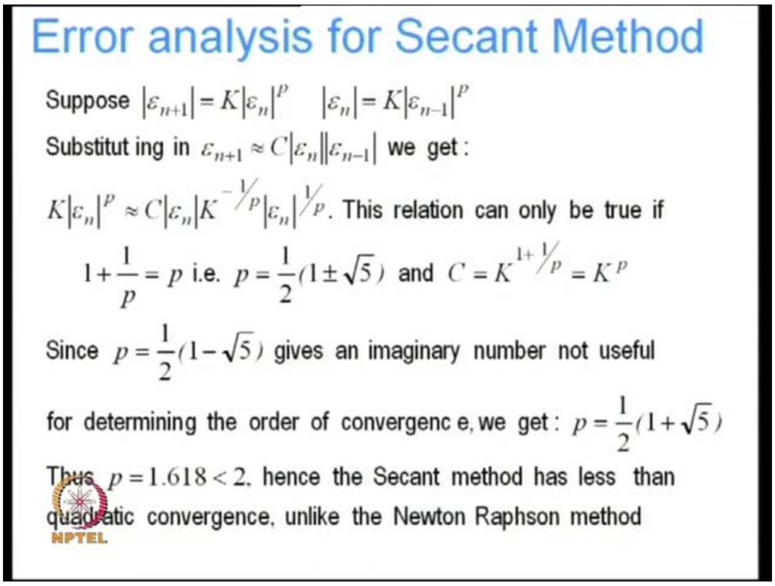

So, let us suppose that my order of convergence is P, then I can I know that I can write mod of epsilon n plus 1 as come constant times mod of epsilon into the power P. And again, we can write mod of epsilon n as K is of same constant, times mod of epsilon n minus 1 to the power P right because this is the order of convergence of my algorithm right.

So, this asymptotic value k and that order of convergence is going to be the same whether, I am looking at n plus 1 or n right. So, let us substitute use these expressions and substitute it in my this expression, if I substitute that in that expression what do I get, I get replacing mod of epsilon n plus 1 is K mod of epsilon into the power P is approximately equal to C times mod of epsilon n. And then mod of epsilon n minus 1 I have again replace by mod of epsilon n to the power 1 by P and K to the power minus 1 by P.

I taking the P'th root of both sides of this equation right. Now, this relationship can only be true look at the powers of mod of epsilon n, so in this side I have mod of epsilon n rise to the power P and this side has mod of epsilon rise to the power 1 and mod of epsilon n rise to the power 1 by P. So, this is only going to be true, if 1 plus 1 by P is approximately equal to P right, if 1 plus 1 by P is equal to P, then this equation is going to be satisfied right.

So, what does this gives give me, this gives me a quadratic equation in P, it is a quadratic equation in P I find the roots of that right, P is equal to half 1 plus minus root of 5 right. So, if I have something if I have P negative, it is not really an imaginary number if I take this negative sign, then I get a negative value for P which does not tell me matched out convergence. So, I take the positive root of this right, half plus 1, 1 plus root of 5, which gives me P is equal to 1.618 right.

And again, if we look at the constants what do we have, we have K here, we have C here, and we have K to the power minus 1 by P. So, what does this tells me, this tells me that K to the power 1 plus 1 by P, must be equal to C and I know that 1 plus 1 by P is equal to P. So, C must be equal to K to the power P right, so from evaluating this expression, we can find out that P is equal to 1.618, which is less than 2, which basically means that the secant method has less than quadratic convergence, unlike the Newton Raphson method right. So, it is not linear convergence, it is P is not equal to 1, P is not equal to 2, P is somewhere in between right. So, that is what is happens for the secant method.

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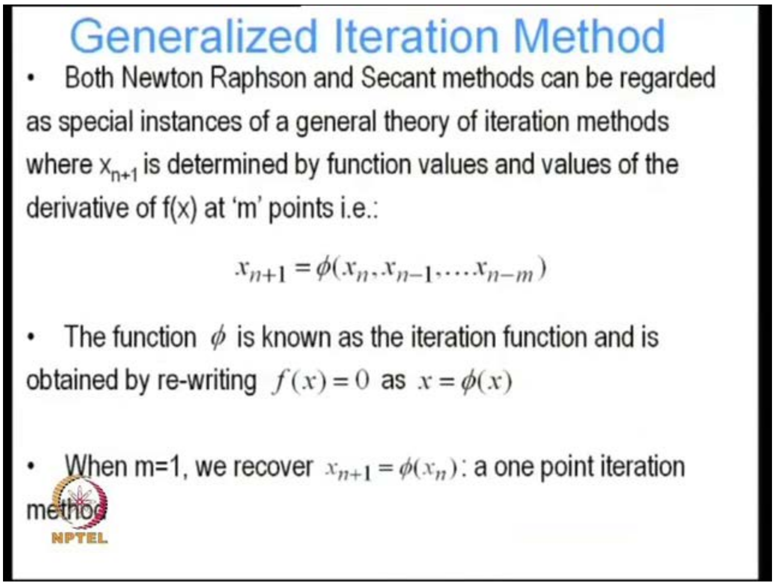

So, that is all I have to talk about one dimensional equations and finding roots of one dimensional equations. That in reality in most engineering problems, in most engineering problems, in most civil engineering problems, we deal with multidimensional situations right. So, we deal with non linear equations, which are not one dimensional that is f is no longer a function of x only, f is a function of x 1 through may be n variables and we have f n times n f equations we have n equations and n unknowns and each of those equations is non linear.

So, we have to find ways of solving those systems that non linear equations, it turns out the ideas that we have developed for one dimensional equations, they are very useful to carry over for non linear, for multidimensional equations as well. Particularly the idea that we talked about relating to the, relative accuracy in the derivative right I mean the relative accuracy in the, there is no point in achieving a in calculating the derivative with a high level of accuracy.

Then my function evaluation, when my function has a significant amount of error right. So, those things become very useful, when we generalize to multiple dimensions, so we are going to continue with that in next class.

Thank you.