Numerical Methods in Civil Engineering
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Lecture - 13
Solving Nonlinear Equations - II

((Refer Time: 00:18)) series and non and Numerical Methods in Civil Engineering, we are going to continue our discussion on solving non-linear equations.

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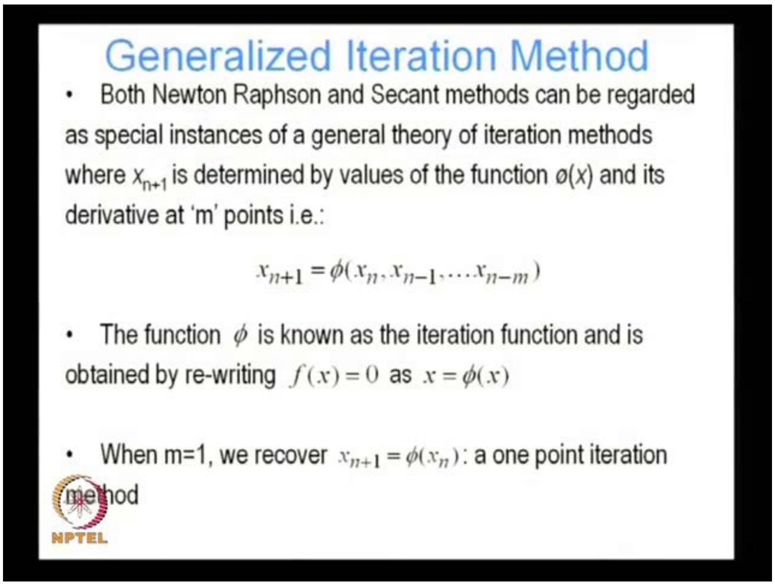

Last time we looked at the Newton Raphson and secant methods for solving one dimensional non-linear equations, this time we want to generalize the concept and talk about some general iteration methods for solving non-linear equations. We are going to first talk about equations with one variables, one dimensional non-linear equations, and then we are going to extend our approach to multidimensional non-linear problems.

So, both Newton Raphson and Secant methods can be regarded as special instances of a general theory of non-linear equation solving, and for instance non-linear iterative method of equation solving. For instance, when we are interested in finding the value of x at by doing say series of iterations, so we are interested in finding the value of x at step n plus 1 provided we know the values of x at the previous step.

Now, for instance the most general method iterative method x n plus 1 is a function of the values of x at n points, if it is not only the function of the value of x at n minus 1 or n it is actually a function of the values of the function at m point. So, it is a value of x at n, n minus 1 up to n minus n, so this is an m'th order iterative method. So, in that case x n plus 1 is a function of x the value of x at the nth iterate at the n minus 1th iterate and so on up to n minus mth iterate right and that is why it is called a mth it has got m points.

So, that means, that x n plus 1 is a function of the value of x of the value of the function as well as it is derivative at those m points. When m is equal to 1 we recover the one point iteration method basically the Newton Raphson method is a one point iteration method, the secant method is a two point iteration method, because it looks at the function values at n as well as n minus 1. To find the value at n plus 1, we will look at the values at n as well as n minus 1, so the Secant method is a two point iteration method whereas, the Newton Raphson method is a one point iteration method, but we can have in general iteration methods with n points.

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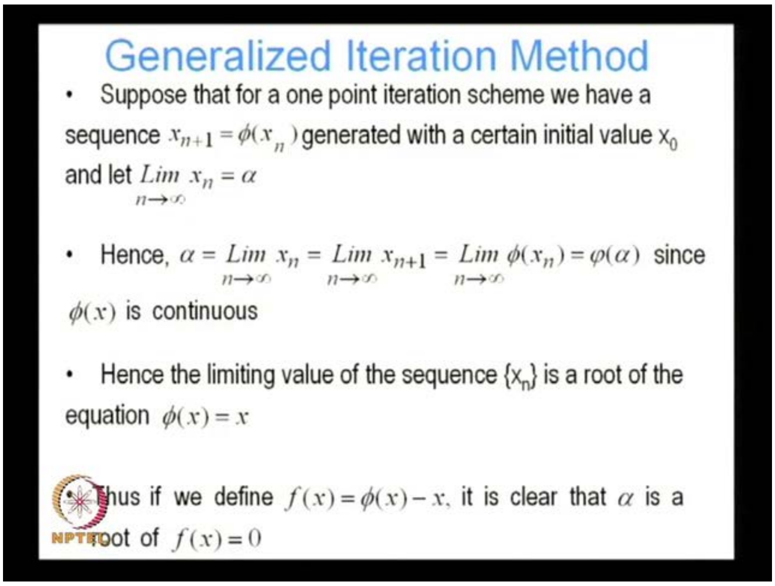

Suppose, that for a one point iteration scheme we have a sequence x n plus 1 is equal to phi of x n generated with a certain initial value x 0, and let us suppose that this sequence is convergent, so as we increase the number of iterations as n goes to infinity, x n goes to or known solution which is unknown solution which is alpha. So, we can say alpha is the limit of x n as n goes to infinity, and since the sequence x n is convergent we can as well say that it is the limit where x n plus 1.

It is the limit of x n plus 1 if n goes to infinity since the sequence is convergent, it does not matter whether we take x n plus 1 on x n because both of them are going to tend to alpha. But, since we once we write it as limit x n plus 1 in the limit n goes to infinity we can write it as limit of phi of x n as n goes to infinity since our iterative scheme says that x n plus 1 is equal to phi of x n.

And since phi of x is continuous we know that as n goes to infinity x of n goes to alpha, we knew that right, but since this function phi of x phi is continuous. So, as x n goes to alpha phi of x n will also go to phi of alpha, which is not necessarily true if the function phi is not continuous, but since the function phi is continuous, we know that as x n goes to alpha phi of x n is going to phi of alpha. Hence, the limiting value of the sequence x n is a root of the equation phi of x is equal to x, so when x n reaches alpha right then we have phi of alpha is equal to alpha, we have alpha is equal to phi of alpha.

So, what does that mean; the limiting value of this sequence x n is a root of the equation phi of x is equal to x, because in that case alpha satisfies this equation phi of x equal to x exactly, so alpha is a root of this equation. Thus if we define, if you have a given function f x and from that function, we define f of x is equal to phi x minus x, so finding the root of f of x, finding f of x is equal to 0, is equivalent to finding the root of phi x equal to x. So, it is clear that if we find the root of since we have found the root of phi x equal to x with a sequence x n tending to alpha, we have also found a root of f of x is equal to 0. So, we are converting this equation f of x equal to 0 to finding the root of f of x equal to 0, to finding the root of this equation phi of x is equal to x.

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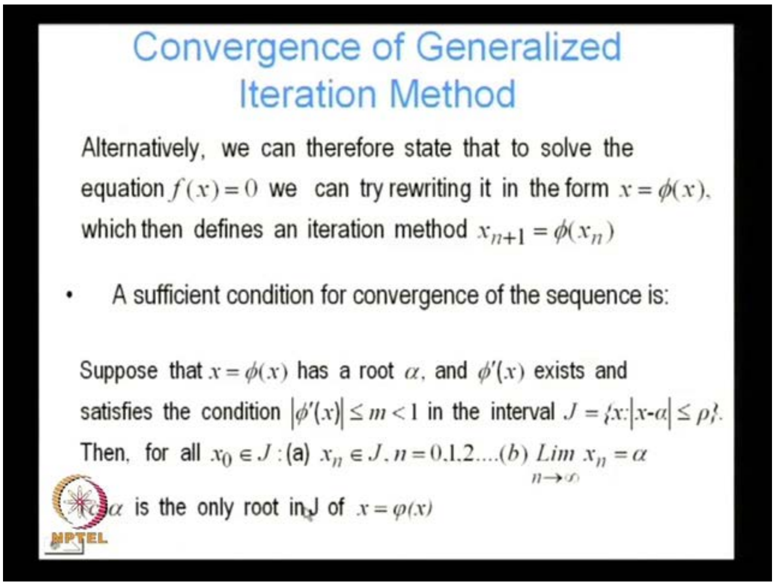

Alternatively, we can therefore, state to solve the equation f of x equal to 0 we can try rewriting it in the form x is equal to phi of x, and then which defines an iteration method. Automatically, once you write this you have to naturally get an iteration method like this, and a root of and the once this iteration method converges that is going to be a root of f of x is equal to 0.

Next let us look at conditions for convergence, that what when will this sequence phi of x is equal to x as x n plus 1 is equal to phi x n, when is that sequence going to converge, what are the sufficient conditions for that sequence to converge. Suppose, that x is equal to phi of x has a root alpha, and let us suppose that the derivative phi prime of x exists and it satisfies the condition mod of phi prime of x is less then m which is less than 1.

So, that means, the derivative is bounded the derivative of phi of x is less than some number m, which is less than 1, in a certain interval neighbor in a neighborhood of my solution. So, in a certain interval neighboring my solution alpha, let us suppose that the derivative phi prime x exists and also the derivative phi prime x is bounded, that is it is value is less than some specified value in all case we are saying that specified value m is less than 1.

So, let us suppose those conditions hold, so phi prime x is less than one in the interval j where j is such that any x belonging to j is it is distance from alpha, is less than some value rho. So, if I take this take alpha as the center, draw a you are looking draw a circle with radius rho all my x is we are considering all the x's which are going to lie within that circle.

And we are saying that foil all the x's which lie within that circle the derivative phi prime of x is bounded, and it is less than some constant m which is less than 1. Then in that case if we start with some value x 0 within that interval, so we are starting with some value x 0 within that circle with center alpha and radius rho, then if we keep if we then we start with that value and then put that in our iterative scheme. Let us put that value here, in phi x 0 get x, x 1, you can substitute again x 1 here get x 2 and so on and forth.

If you continue our iteration scheme starting with a value in that interval, then we are guaranteed that all our new iterates are going to lie within that interval. So, what it says that is that for all x 0 belonging to j x n also is going to belong to j, for all n 0 1 2 , so all x 1 x 2 x 3 and so on and so forth. They are all going to lie within that circle with center at alpha and radius rho and I, secondly and the limit as I increase the number of iterations x n is going to tend is going to go to alpha.

So, it is a very beautiful concept it says that basically I start with a one, if when I am sufficiently close to the root convergence basically means that if I start with any value, which is within a certain neighborhood of the root. And the derivative epsists within that in that neighborhood, and the derivative is bounded, then I am bound to get convergence because of my limit my sequence x n in the limit n goes to infinity is going to tend to alpha. Number 1, number 2 number 3 alpha is going to be the only root in j of x is equal to phi of x. So, if I satisfy these conditions then these are going to follow.

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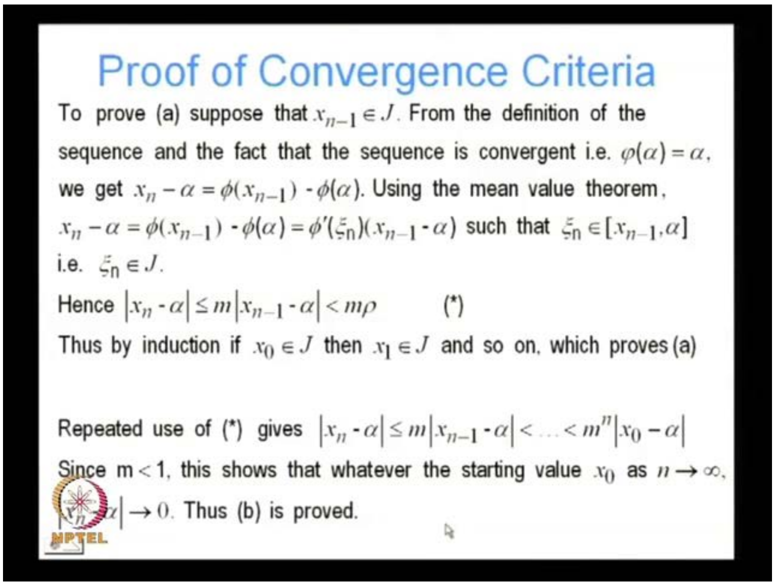

How can you prove that well let us suppose we start with a point within j, and let us see if we can prove that if we start with a point within j, then if all those conditions are satisfied then any new iterate is going to lie within j. How can we do that? From the definition of the sequence and the fact that the sequence is convergent that is phi of alpha is equal to alpha.

We can write x n minus alpha is equal to phi of x n minus 1 minus phi of alpha, why because x n is equal to phi of x n minus 1 according to my iterative scheme and since it is alpha is the root, so alpha is equal to phi of alpha, so I can write this like this and then I use my mean value theorem. If I use my mean value theorem, I can write x n minus alpha is equal to phi of x n minus 1 minus phi of alpha is equal to phi prime the derivative evaluated at a point psi n, such that psi n lies between x n minus 1 times alpha times x n minus 1 x n minus 1 minus alpha.

It is just the mean value theorem this difference is approximated by the derivative evaluated at an intermediate point, times the interval size which is x n minus 1 minus alpha. This is going to be true such if there is a psi n which belongs to j, belongs to this interval, psi n is going to belong to j because x if psi n belongs to x n minus 1 in alpha, and I have already assumed that x n minus 1 belongs to j. So, it since psi n lies in the in that interval it; obviously, means that psi n also belongs to j, and then if I take mod of both sides I have x n minus 1 x n minus alpha is lesser than or equal to m times mod of x n minus alpha. Why is that m?

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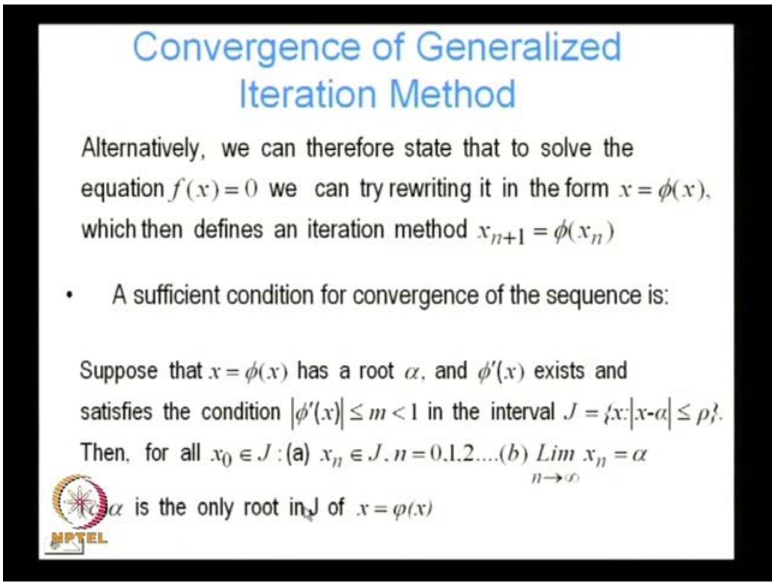

Because, I know that anywhere in that interval j right my derivative is bounded, if I my derivative is going to be less than m, that is that was our assumption that phi prime exists and phi mod of phi prime is less than or equal to m less than 1.

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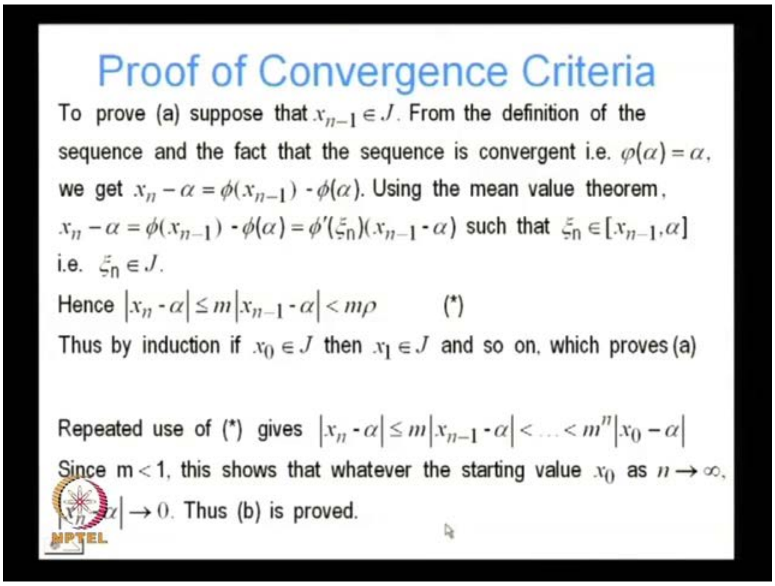

So, that means, that this thing this phi prime psi n has to be less than m, so we have m here, and since we have taken mod of both sides we have mod of x n minus 1 minus alpha and since x n minus 1 belongs to j this thing must be less than rho. So, mod of x n minus alpha is less than m rho what does that mean; that means, that x n also lies within j, because x n minus alpha is less than m rho and what is my j.

J is such that x minus alpha is less than or equal to rho, and since x n minus alpha is less than or equal to m times rho some fraction m is less than one. So, since this is less than this; obviously, means that mod of x n minus alpha must be less than m rho, so x n must be lying within that circle which I drew with center alpha and radius rho. Thus by induction if x 0 belongs to j x 1 is going to be, so we started with any arbitrary x n x n minus 1 which belong to j.

So, if I start with x 0 I knew that x 1 is going to belong to j since x, and then if x 1 belongs to j same argument x 2 is going to belong to j and so on and so forth. So, all my iterates are going to lie within that circle, so that is the first requirement of convergence, so when I am within at within a sufficiently close neighborhood of my root, and then f I satisfy those conditions on the derivative then in that case my all my iterates will belong within that circle.

So, repeated use of this expression repeated use of this expression this is my x n minus alpha is lesser than or equal to this I already got, x n minus alpha is lesser than or equal to m times mod of x n minus alpha. Then if I use the same procedure and this I can show that mod of x n minus 1 alpha is going to be less than n times mod of x n minus 2 alpha and so on and so forth.

So, eventually I can show that this is going to be less than m n m to the power n times mod of x 0 minus alpha, and remember x 0 minus alpha is within that circle with center alpha and radius rho so; that means, that as I keep on iterating, I am going to get x n minus alpha is going to less than m to the power n and remember m is less than 1. So, m to the power n is going to become smaller and smaller as n increases, so eventually my x n is going to converge, so since m is less than 1. So, this shows that whatever with be the starting value x 0 as n goes to infinity x n minus alpha is going to 0.

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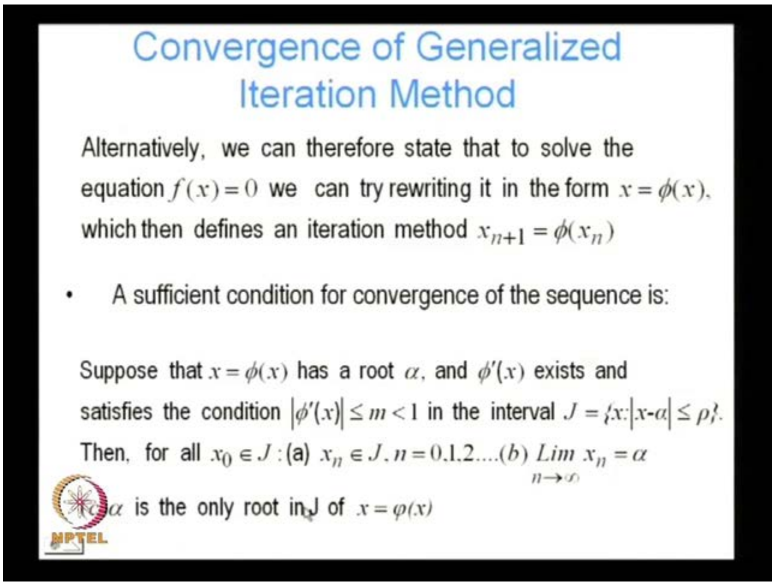

Thus we have prove the second part which was that this is going to converge, that this sequence x n is going to converges and goes to infinity.

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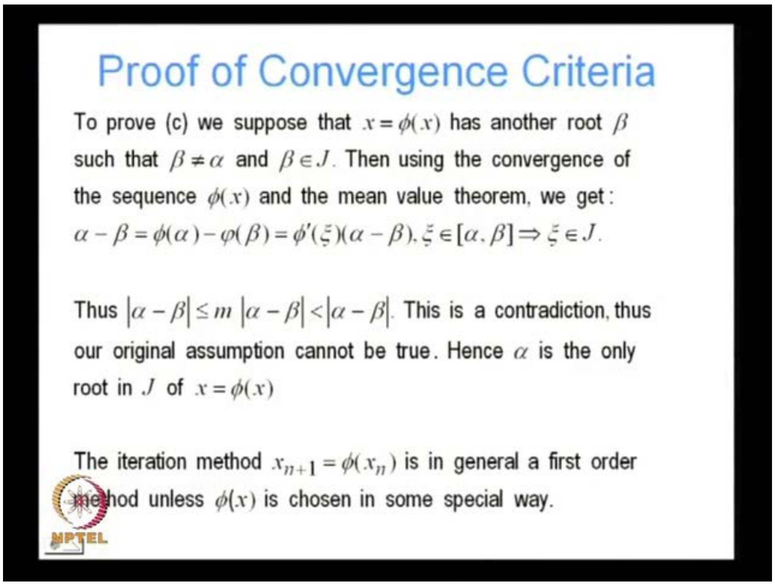

And finally, to prove c we said that in that case alpha is going to be the only root in that interval in that neighborhood j, so if you let us see what happens if case in case there is another root, is that possible to proof prove see. This suppose that x is equal to phi x has another root b within that interval j such that b is not equal to alpha; that means, the b also satisfies the phi of b equal to b.

It satisfies that equation, sorry not b phi of beta equal to beta, but beta is not equal to alpha. Then using the convergence of the sequence phi of x and the mean value theorem we get alpha minus beta is equal to phi of alpha minus phi of beta, because both alpha and beta are roots of this, phi alpha minus beta, then I again use my mean value theorem. So, this I can write is phi prime of psi times alpha minus beta, so long as psi belongs to alpha beta, which means that psi belongs to j since both alpha and beta belong to j.

So, I will this lies in that interval function is continuous, so psi must belong to j. So, thus if then again we take the mod of both sides mod of alpha minus beta is lesser than or equal to mod of phi prime of g which is m which has to be less than m, which is the condition we said times mod of alpha minus beta which is less than mod of alpha minus beta.

So, m since m is less than 1, this has to be less than mod of alpha minus beta which is not possible, alpha minus beta cannot be less than alpha minus beta, which is impossible this is a contradiction. Thus the original assumption that phi of x has another root beta in that interval is false, and hence alpha is the only root of j only root in j of x is equal to phi of x. The j iteration method x n plus 1 is equal to phi of x n is in general, a first order method unless phi of x is chosen in some special way, what do I mean by a first order method; that means, that the order of convergence is normally first order.

Unless there are some special conditions of on phi of x we know that Newton's Newton Raphson is a second order method, while the secant method we found was not clearly a first order method it might had a convergence value, it had rather somewhere between 1 and 2 something like 1.6 something. So, in case we choose, so in case we have to improve the order of convergence of that method, in that case the function phi of x must satisfy certain additional properties. what are those additional properties? And how do those additional properties? Influence the order of convergence of the function, so that is what we want to talk about next.

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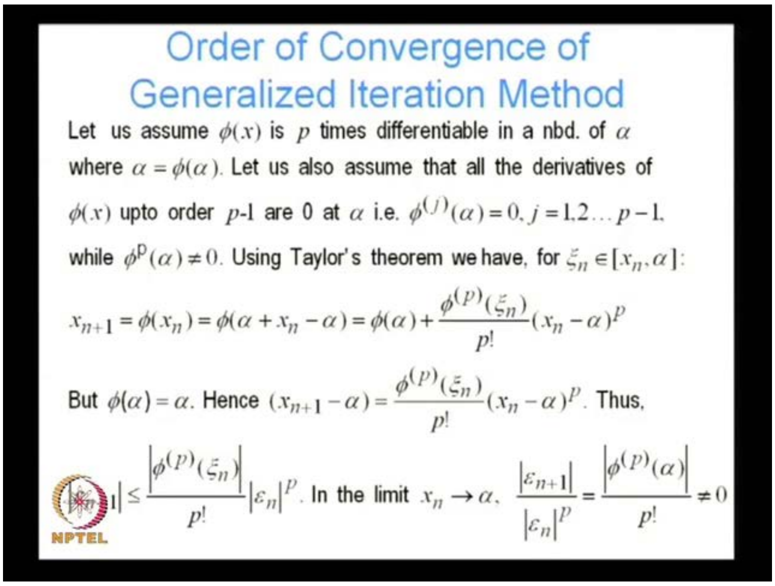

So, let us assume that phi of x is p times differentiable in a neighborhood of alpha, so if I take p derivatives all those derivatives exist, so p derivatives exist in a neighborhood of alpha, where alpha is equal to phi of alpha, alpha is a root of that equation. So, let us also assume that all the derivatives of phi of x up to order p minus 1 are 0 at alpha, so let us this is the crucial part this is the additional assumption that we have to make.

We are making the assumption that all the derivatives of phi up to the order p minus 1 at 0 at the root alpha right and the derivatives of order p is not equal to 0, so if under those conditions we can show that the order of convergence can be higher, and how high will it be well let us take a look. So, in that case since the all the derivatives up to order p minus 1 are 0 if we use Taylor's series expansion for x n plus 1 equal to phi of x n this is from an iteration algorithm, x n plus 1 is equal to phi of x n it I can write this as phi of alpha plus x n minus alpha.

And then I do a Taylor's series expansion about alpha, so if I do a Taylor series expansion about alpha what do I get, I get phi of alpha plus in the normal course of events. I would have first derivative phi prime evaluated at alpha times x n minus alpha plus the second term would be phi double prime evaluated at alpha times x n minus alpha square by factorial of two and so on and so forth.

But, remember what is our condition our condition is that all derivatives up to order p minus 1 at 0 at alpha, so all the p minus 1 derivatives, they vanish, because that they are all 0 by assumption. And then we are left with only the pth derivative and we can evaluate the 2 pth with which this as the remainder term in the Taylor's series, we can then cut off our Taylor's series at that point.

If we can write this as phi, pth derivative of phi evaluated at psi n divided by factorial of p times x n minus alpha to the power p, where psi n belongs to this interval. Where, psi n belongs to this interval, but phi of alpha is equal to alpha since alpha is a root of this, this equation phi x equal to x, so this and this, so this I can replace by alpha. So, I get x n plus 1 minus alpha by bringing phi of alpha to the left hand side I have x n minus 1 alpha, and this is phi to the power p psi n by factorial p times x n minus alpha to the power p, so what does this remind you.

So, here on the left hand side we have x and x n plus 1 minus alpha, which is nothing but the error at the n plus 1th iteration, because this is epsilon n plus 1 it gives me it gives me how much if I take the mod of this, it gives me by how much the n plus 1th iterate differs from the root. And that is my error at the n plus 1th iteration, well on the right hand side I have this term, what is this term it is the error at the nth iterate x n minus alpha is equal to epsilon n.

So, now, I have a relation which says that the error the mod of the error at the nth n plus 1th iteration is less than this term multiplied by the mod of the error of the nth iteration to the power p, so this is we call this is we emphasize this. When, we talk about Newton's method and the Secant method that if you have to establish the order of convergence you have to be able to write the n plus 1th error in terms of the nth error.

So, we have just done that and in the limit as x n goes to alpha we have epsilon n plus 1 to the divided by epsilon n plus 1 mod to the power p phi p alpha because now x n is going to alpha, so psi must be psi has to tend to alpha 2, because everything is converging to alpha. So, we have this thing which is a constant and which is not equal to 0 why is that not equal to 0, because of I requirement write here which says that phi prime of alpha is not equal to 0. So, this term is a constant this is not equal to 0 and we have a error at epsilon n plus 1 is equal to this constant, times p times the error of epsilon n. So, just by looking at this we can say lets think of Newton Raphson, Newton Raphson we know that p is equal to 2, that is the order of convergence is equal to 2.

So, what does that mean what does that require in terms of this requirement, so for Newton Raphson to be quadratically convergent number 1 the function must have first derivative, which is equal to 0 at alpha. And the second derivative p is equal to second, so the first derivative must be 0 at alpha and the second derivative must not be 0 at alpha, so that is the requirement under which Newton Raphson is going to give me second order convergence.

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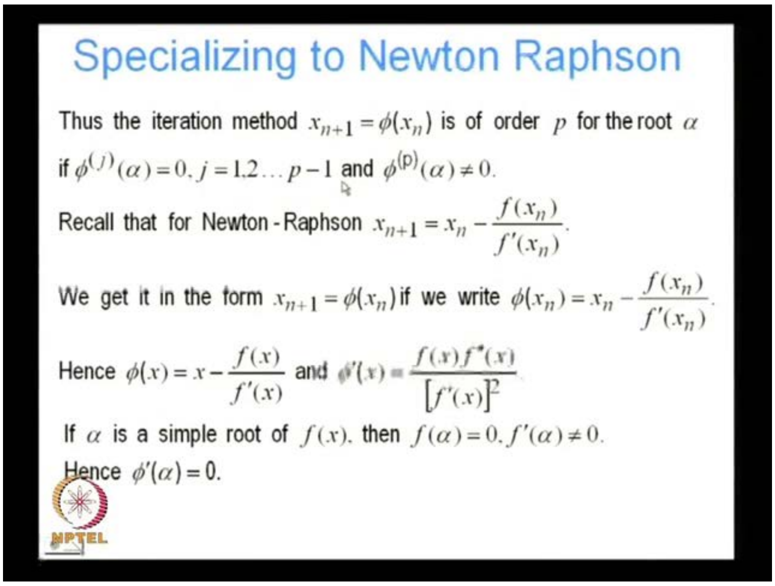

Thus the iteration method x n plus 1 is equal to phi of x n is of order p for the root alpha if phi over j alpha, that is the jth derivative of alpha for j is equal to 1 to p minus 1 is equal to 0 at alpha and the pth derivative phi p alpha is not equal to 0, at alpha. Let us recall that for Newton Raphson, we can write x n plus 1 is equal to x n minus f of x n by f prime of x n.

So, in that case what is our phi our phi is nothing but this function on the left hand side, so we can write phi of x n is equal to x n minus f of x n by f prime of x n. Hence, phi of x is nothing but this and phi prime of x, if I take the derivative of this, this is equal to this. And let us recall again what is our definition of a simple root of f of x, alpha is a simple root of f of x and f of alpha is equal to 0 and f prime of alpha is not equal to 0.

Since, f prime of alpha is not equal to 0 and f of alpha is equal to 0, what can we say about phi prime of alpha phi prime of alpha is equal to 0. Why? Phi prime of alpha is equal to f of alpha f of alpha by definition is 0, because alpha is a root of that and phi prime f prime of alpha is not equal to 0. So, this is never going to blow up, this is going to be bounded, this is going to be bounded, but on the top we have so; that means, that phi prime of alpha must be equal to 0. And if phi prime of alpha is equal to 0 and phi double prime of alpha is not equal to 0, then we are going to have quadratic convergence of Newton Raphson.

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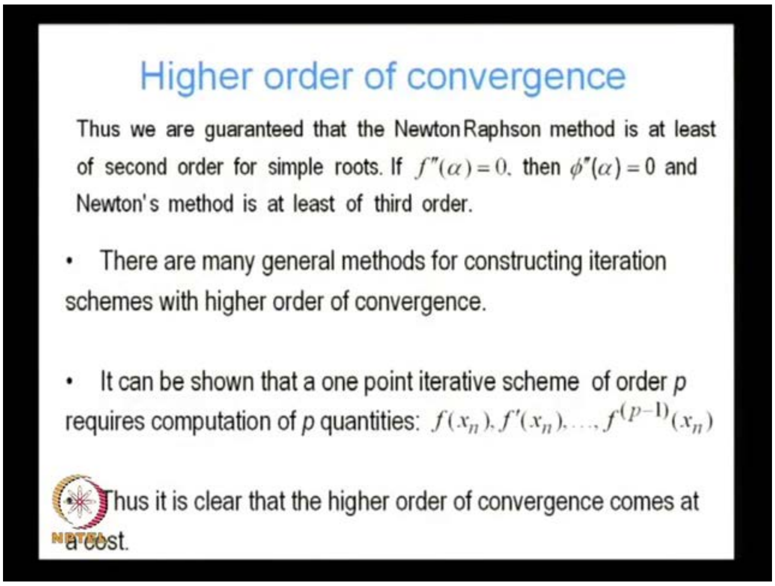

Thus we are guaranteed that the Newton Raphson is at least quadratically convergent for simple roots, if in addition f double prime alpha is equal to 0, then we can show that phi double prime alpha is equal to 0. In that case the Newton Raphson convergence is going to be at least of third order, because the second derivative is 0, third derivative may or may not be 0.

If it is not 0 then it is at has to be at least third order, if it is 0 then we are going to have even higher order convergence. So, in that case if phi phi double prime alpha is equal to 0 then the Newton Raphson method is also going to be convergent at least up to third order. There are many general methods for construction, so this we can see that we can make the Newton Raphson scheme. We can make the incase this condition is satisfied Newton Raphson is going to be third order convergent, but that is incase that is satisfied, but by construction the Newton Raphson method is second order convergent, because we saw that whatever be the function f of x.

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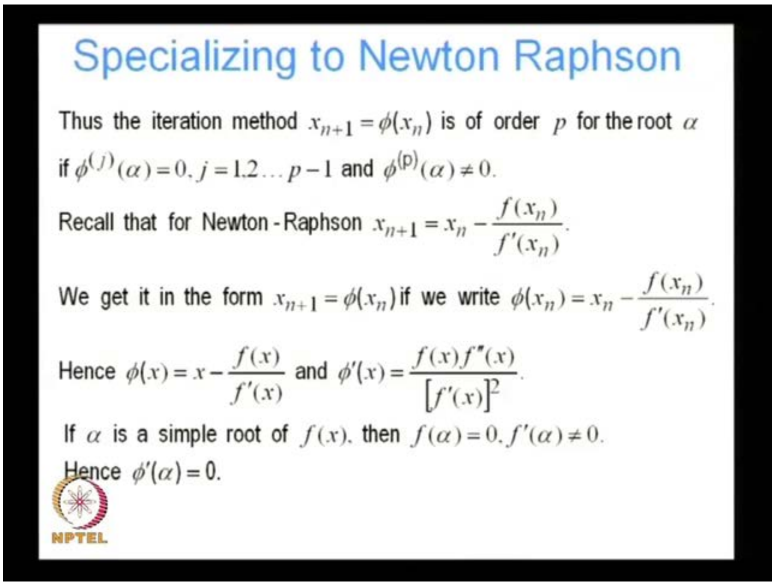

So, long as this condition is satisfied, which is a prerequisite, f prime of alpha is not equal to 0 right in that case the Newton Raphson method is going to be second order convergent, but in case in addition in addition if we have f double prime equal to 0 in that case we can have higher order convergence of the Newton Raphson method.

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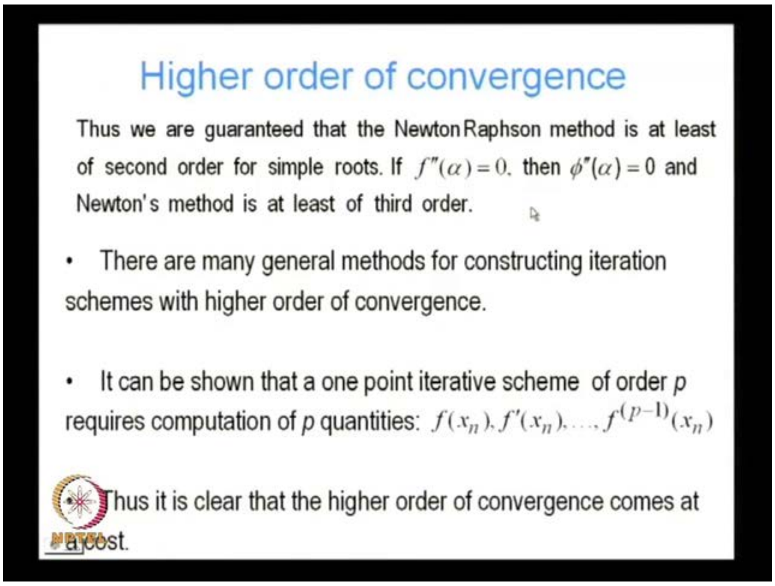

There are many general methods for constructing iteration schemes with higher order of convergence, it can be shown that for a one point iterative scheme of order p requires computation of p quantities basically we need to compute p quantities. what are those quantities? The function and it is first p minus 1 derivatives therefore, as you can see to improve the order of convergence there is a cost, because each time you take the you need to calculate a derivative it is expensive.

So, if you can you can improve the order of convergence, you can have as many high order of convergence as you like, but ultimately it becomes it does not it is no longer economical computationally economical, but the cost of taking all these derivatives. You can say you can lower the order of convergence, so by lowering the order of convergence what we are going to get we are going to converging fewer number of iterations.

But, the cost of taking all those derivatives and converging in a fewer number of iterations is going to be more than the cost of calculating less derivatives there the order of convergence is less, but the number of iterations is more. So, if you take a lower order method you need more iterations, but you need to compute only a limited number of derivatives, so it all depends on a tradeoff. So, it turns out that if you have to compute more and more derivatives then the cost of reducing the number of iterations is much, much is offset by the cost of calculating the derivatives. So, people will prefer to take a lower comparatively lower order method do more number of iterations and that is come computationally more efficient.

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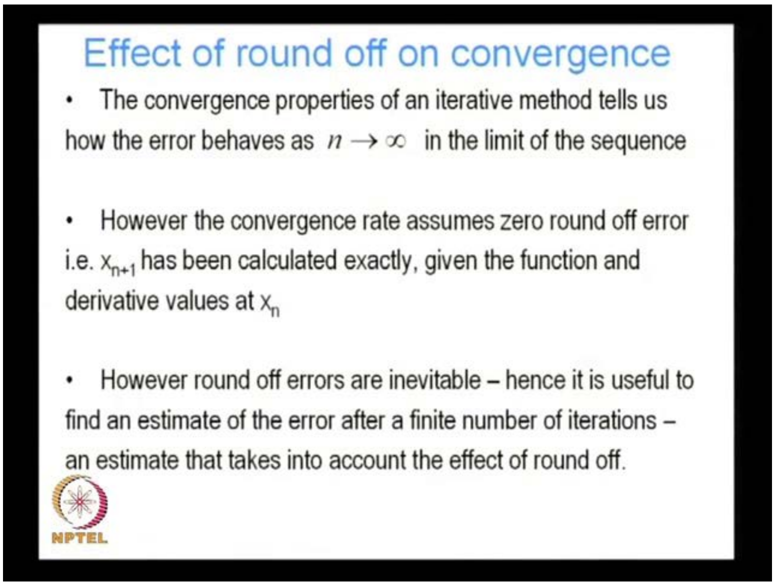

So, the convergence properties of an iterative method tells us, how the error behaves as n goes to infinity in the limit of the sequence; however, in all our derivations, so far we have assumed that there is zero round off error. So, when we calculate phi of x n plus 1 from phi of x n, we are we are assuming that there is no round off error, but in reality. There is going to be; obviously, going to be round off error, because whenever we compute phi of x n we are not going to compute phi of x n exactly we are only going to compute it up to our machine accuracy.

Because, we are using finite precision arithmetic I compute that has that many limited floating point numbers, it can store for my mantissa. So, I have to I am bound to get some round off errors and how are those round off errors going to affect my convergence, so that is that also is interesting. Hence, round off errors are inevitable, hence it is useful to find an estimate of the error after finite number iterations an estimate that takes into account the effect of the round off. So, up till now all convergence loss, convergence relationships they have all assumed that round office non existing.

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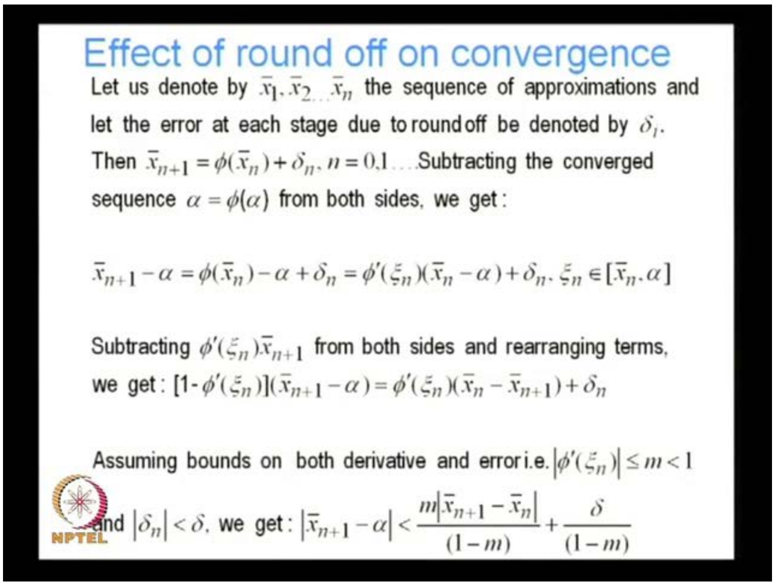

So, let us denote by x 1 x 1 bar x 2 bar x n bar the sequence of why we are using a bar and term, because we want to denote them as the real sequence of approximations meaning they have round off in them. While, x 1 x 2 x three x n did not have any round off, a sort of a ideal iterative values, which assume that there is no round off.

Let us now consider x 1 bar x 2 bar x n bar, where there is round off, and let the error at each stage due to round off will be denoted by delta I, then we can write x bar n plus 1 is equal to phi of x bar n plus some delta n, because there is this round off, so delta n is my round off for n is equal to 0 1 and so on and so forth. So, this is going to be true at every stage in our iteration, so subtracting the converged sequence alpha is equal to phi of alpha from both sides, so we subtract alpha equal to phi of alpha from both sides.

So, we have x bar n plus 1 minus alpha is equal to phi of x bar n minus alpha plus delta n, again I use my mean value theorem. So, phi of x bar n minus alpha I am going to write it as because alpha is equal to phi of alpha, so phi of x bar n minus phi of alpha I am going to write as phi prime of psi n times x bar n minus alpha, where psi n belongs to x bar n and alpha. I can only do that because alpha is equal to phi of alpha, I can use my mean value theorem here.

Then if I subtract phi prime psi of n x bar n plus 1 from both sides, and rearrange terms I am going to get something like this, which is basically 1 minus phi prime psi n times x bar n plus 1 minus alpha is equal to phi prime psi n times x bar n minus x bar n plus 1 plus delta n. So, let us assume bounds on both the derivative and the error, so let us assume that the derivative is bounded, that phi prime psi n is lesser than or equal to m less than one which is what we assumed earlier remember.

So, that is what we assume earlier and let us assume that a round off error is also bound which is true, because round off error is bounded by the we have seen that into a detail earlier in the in the beginning of the course round off error is bounded by things like machine precision and things like that right. So, the round off error is also bounded, and let us suppose that bound is delta we therefore, get by taking mod of both sides mod of x bar n plus 1 minus alpha.

And then if you take mod of this what is this going to be, this is going to be 1 minus n, because mod of phi prime psi n is less than equal to n minus 1. Then if I divide both sides by 1 minus n this has also got bound m, so for n times mod of x bar n plus 1 minus x bar n divided by 1 minus n plus delta by 1 minus m. So, this gives me a bound on my x bar n plus 1 minus alpha, this gives a bound on my error at the n plus 1th iteration that takes into account the effect of round off, because of this term.

So, this says that at any stage in my iteration at stage n plus 1 for instance, my error is going to be bounded by these values right and this, these are two parts you can see, so the first part is what is known as the truncation error. In any numerical method in any this we talked about also right in the first class in any numerical method, there are two there are bound to be two types of errors any numerical algorithm for solving a problem, there are going to be two types of errors.

One type of error is going to be the truncation error, the second type of error is going to be the round off error. Why do we have the truncation error? The truncation error is because when you are approximating an infinite sequence by a finite number the terms, in this case we are approximating alpha which is actually the limit of the sequence phi of x n it is the limit. When n goes to infinity by the term at the after the n plus 1th iteration, so an approximating the infinite term on the sequence by the n plus 1th term, because of that there is an error and that error is my truncation error, and this error is my round off error.

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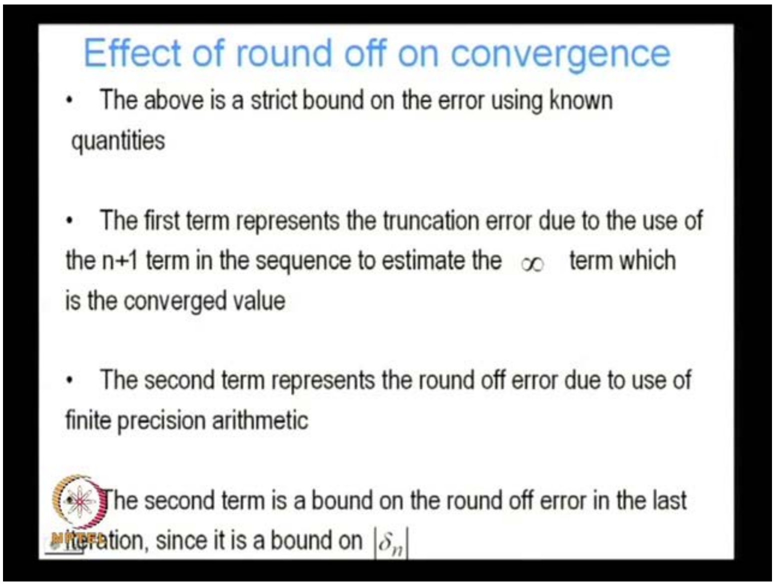

So, the above is a strip bound on the error using known quantities, the first term represents the truncation error due to the use of the n plus 1th term in the sequence to estimate the infinite term, which is my converged value alpha. So, that is my truncation error the second term represents the round off error due to user finite precision arithmetic any prime, I do not have infinite precision, finite precision arithmetic that is going to give me that error. The second term is a bound on the round off error, this is very important term the second term is a bound on the round off error in the last iteration.

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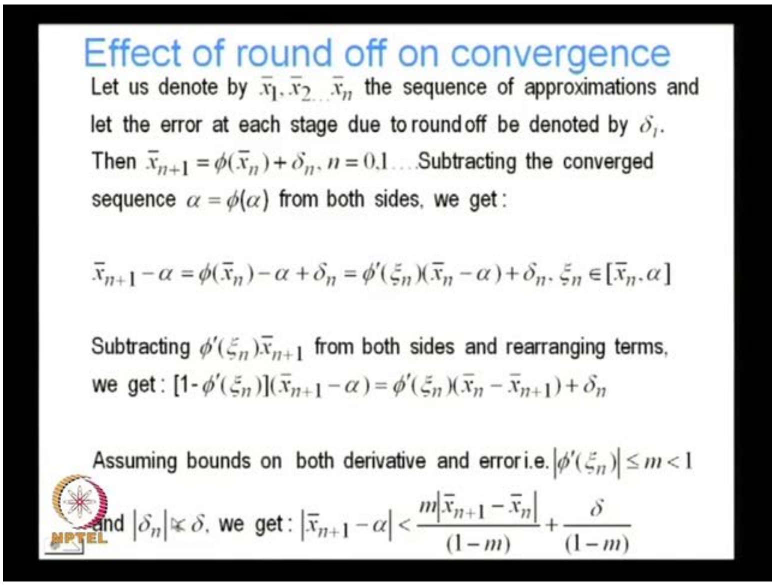

Since, it is a bound on delta n look at this delta is delta is a bound on mod of delta n, delta is not a bound on delta n minus 1 or whatever any time in that it is a bound on the error in the in the round off error in the last iteration right which is very, very crucial. Let us see why?

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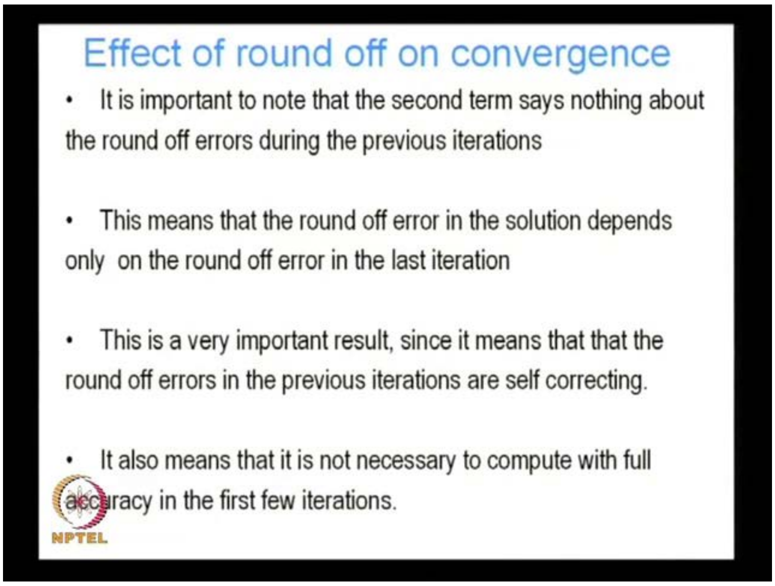

It is important to note that the second term says nothing about the round off errors during the previous iterations, it deals with the round off error in the nth iteration. This means that the round off error in the solution is going to depend on the round off error only in the last iteration, which is very, very important, because what does that mean; my round off errors in the previous iterations are not contributing to my final round off error. Why?

Because, those round off errors are cancelling each other out, it is very otherwise you can see that if they do not cancel each other out then my error is going to be significantly larger. So, the round off error in the solution depends only on the round off error in my last iteration, since it means that the round off errors in the previous iterations are self correcting that is they cancel each other out.

It also means that it is not necessary to compute with full accuracy in the first this is a very, very important result, because a in lots and lots of places it is used out all in multi dimensions this has got very important implications. So, it is not necessary to compute with full accuracy in the first few iterations, so we will see I mean if I talked about this here that if I am going to talk about it later on it has got very important implications.

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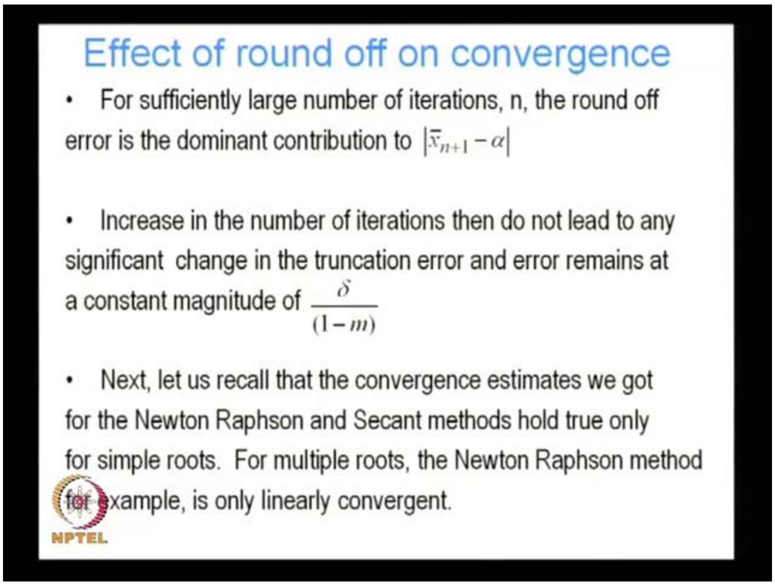

For sufficiently large number of iterations n the round off error is the dominant contribution to x bar n plus 1 minus alpha, you can see that, because as the number of iterations goes up the truncation error is going to, become smaller and smaller.

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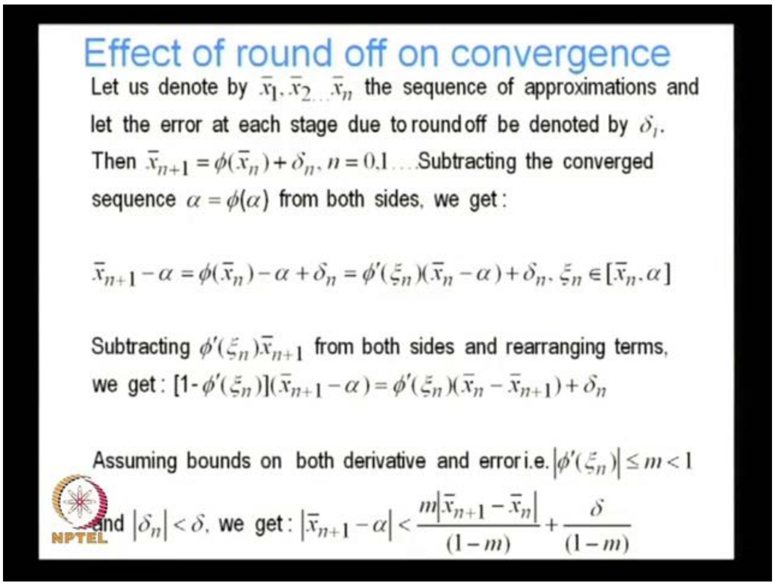

Because, my solution my n plus 1 has an n increases my x bar n plus 1 becomes a better and better appropsimation to my root alpha. So, as I increase the number of iterations the truncation error is going to become smaller and smaller, it is actually as I go to alpha the truncation error is going to go to 0. But, my round off error is not going to go to 0, my round off error is also always going to be there.

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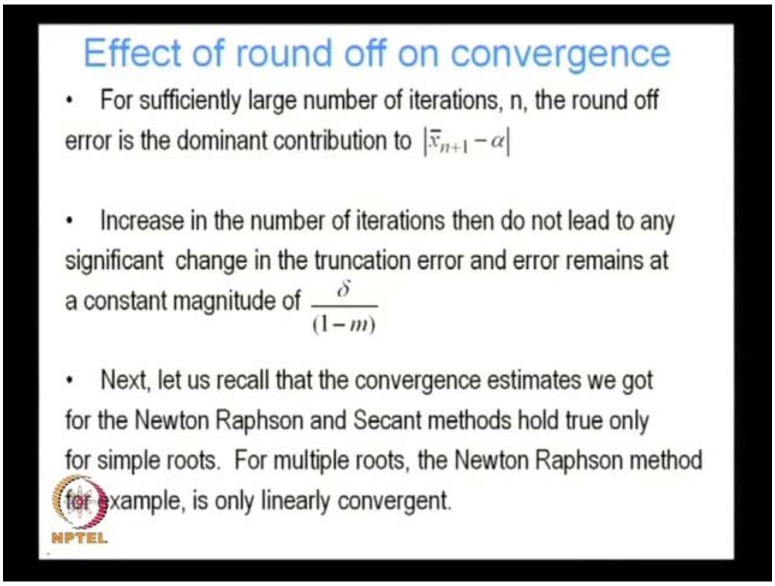

So, increase in the number of round off error is the dominant contribution to x bar n plus 1 minus alpha, so increase in the number of iterations, then do not lead to any significant change in the truncation error, because as I am reaching the limit of the sequence the contribution of the truncation error is becoming negligibly small. But, my round off error remains at a constant magnitude of delta 1 minus m, where delta is a bound on the round off error in the last iteration.

Next, let us recall that the convergence estimates we got for the Newton Raphson and Secant method hold true only for simple roots, for multiple roots the Newton Raphson method for example, is only linearly convergent. So, let we will talk about improving the convergence of Newton Raphson for multiple roots, but at this point I want to talk a little bit about this error bound a little bit more.

So, basically we saw that because this error bound depends only on the error in the last and the round off error in the last iteration, so it is not important to for the first beginning iterations to be calculated with great accuracy, because their accuracy is not going to determine the round off error. A round off error is going to be, so this has lots of implications because for instance, when you have multiple, when we are solving a non-linear problem in multiple dimensions and we want to use a Netwon Raphson update formula.

Newton Raphson update formula as you can see in the we will see that in more detail later, but that mean cost of the update is the cost of calculating the derivatives. So, if we have a multi dimensional function the derivative the Jacobean is no longer going to be a simple a single term it is going to be a n by n term, so the Jacobean forming that Jacobean and if necessary inverting it, so those are extremely, extremely expensive operations.

So, what does that mean; so the accuracy is not that important in the first iterations, first few iterations accuracy only becomes important towards the end; that means, that towards the beginning it is also not necessary for me to calculate all those derivatives to a great deal of accuracy. And because of that there are things like modify Newton quasi Newton and things like that we try to approximate those derivatives, so that they are easier cheaper to calculate, to reduce the cost.

Because, we know that what we do in the beginning of that beginning of a iteration it is not going to be that crucial, unless I do something absolutely weird that is not going to be which makes my solution diverge and that is not going to be that crucial, so that is why this is very important. Now, let us let us look at multiple roots, we recall that the convergence estimates we got for the Newton Raphson and secant methods hold true only for simple roots, for multiple roots the Newton Raphson method for example, is only linearly convergent.

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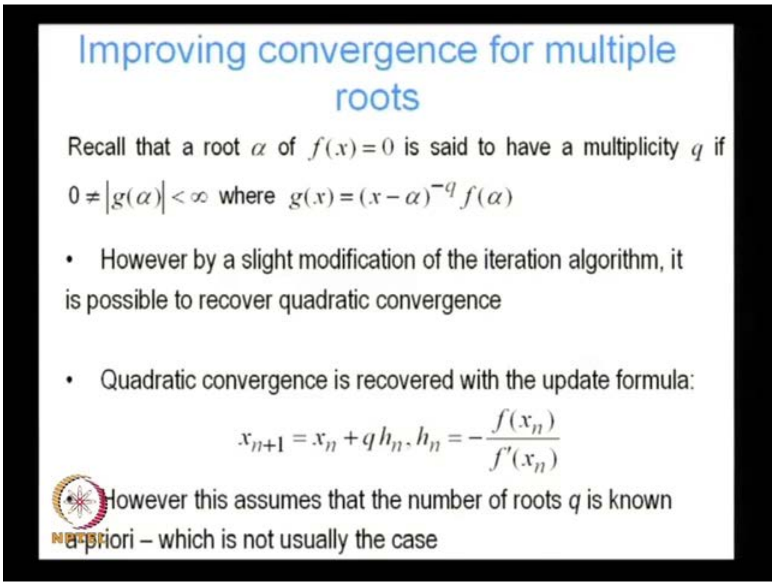

So, is it possible to improve the convergence properties at multiple roots well let us take a look at that, so we call that a root alpha of f of x is equal to 0 is said to have a multiplicity q, if we can write there exists a function g of x, which is equal to x minus alpha to the power minus q times f of x. Where, q is the multiplicity of the root alpha and g of alpha is bounded, that is mod of g of alpha is less than infinity is greater than 0.

So, what this basically means that there exists a function g of x which can be written as if I take out the q roots of alpha, if I take my original function of f of x I divided by it x minus alpha to the power q, I take out the q roots of f of x. The function what is means let me call that g of x and that function g of x exists, what does it mean exists; that means, that g of alpha is not zero and is not infinity it is bounded right and it is not zero so; that means, there exists a function g of x, which I can write like that.

In case f of x has root of multiplicity q; however, so we saw that for the Newton Raphson method if it has multiple roots if it has two roots for instance, then we are going to get linear convergence. However, by slight modification of the iteration algorithm it is possible to recover quadratic convergence, what is that slight modification well originally we had x n plus 1 is equal to x n for the Newton Raphson method, where x n plus 1 is equal to x n minus f of x n by f prime of x n.

So, now, I am saying that instead of having x n plus 1 equal to x n minus f of x n by f prime of x n I am going to have x n plus 1 is equal to x n plus q times f of x n by f prime of x n, where q is the number of is the multiplicity of the root alpha. So, that is well fine and good that, but assume is that I knew a priori beforehand, how many roots there are right it assumes that I know beforehand what is the value of q, , but suppose we do not know that then what can we do.

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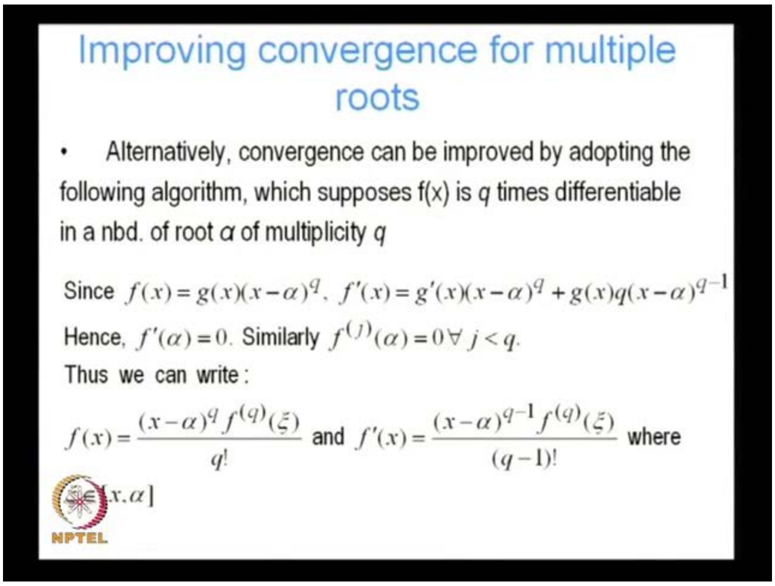

Alternatively, convergence can be improved by adopting the following algorithm, which supposes that f of x is q times differentiable in a neighborhood of root alpha of multiplicity q. Set let us suppose that the function is differentiable at least q times, q being the multiplicity, it assumes that q is f is differentiable. In that case since f of x is equal to g of x x minus alpha to the power q, since alpha is a root of a multiplicity q.

So, I can write f prime of x is equal to g prime of x x minus alpha to the power q plus g of x times q times x minus alpha q minus 1 just taking the derivative, so f prime of alpha is equal to 0. Why? Substitute alpha here both sides these two terms go to 0, so f prime of alpha is equal to 0. Similarly, f to the power j alpha is going to be equal to 0, for any j less than q you can see that, because if you take derivatives up to less than q, there is always going to be a term x minus alpha, in the right hand side and when x becomes equal to alpha that term is going to go to 0.

So, f j alpha is going to be 0, for all j less than q then we can write f of x again doing a Taylor series expansion of f of x about alpha. Then f of x can be written like this, because again f of alpha is going to be 0, f prime of alpha is the second term in the Taylor series f prime of alpha is going to be 0, f double prime of alpha is going to be 0, up to order j where j is less than q.

So, all those terms in the Taylor series are going to go to 0 and the only term and the term which is going, which is the first term which is not going to go to 0, must have must have derivative of order q, because for all j less than q f j of alpha is equal to. So, in the Taylor series we are going to evaluate this derivative at alpha that is why expanding this Taylor series about alpha. So, all those j terms j less than q are going to go to 0, so the first term that is going to be survive is going to be this term.

And if you evaluate repeat this term as the remainder term, we can write it as x minus alpha to the power q times of f qth derivative of f evaluated at psi where psi belongs to the interval x alpha divided by factorial of q. So, that becomes my f of x, and what is my f prime of x f prime of x I just take the derivative of both sides x minus alpha q minus 1 f q of psi q q minus 1 factorial where psi belongs to alpha, so now, what do we have?

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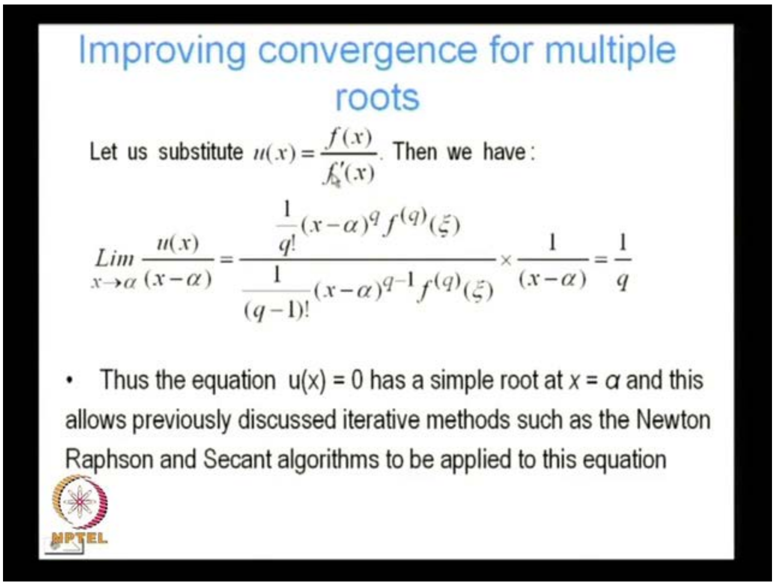

Let us write u of x is equal to f of x by f prime of x, then if we write u of x by divided by x minus alpha, and then if I take the limit as x goes to alpha what is u of x u of x is f of x by f prime of x.

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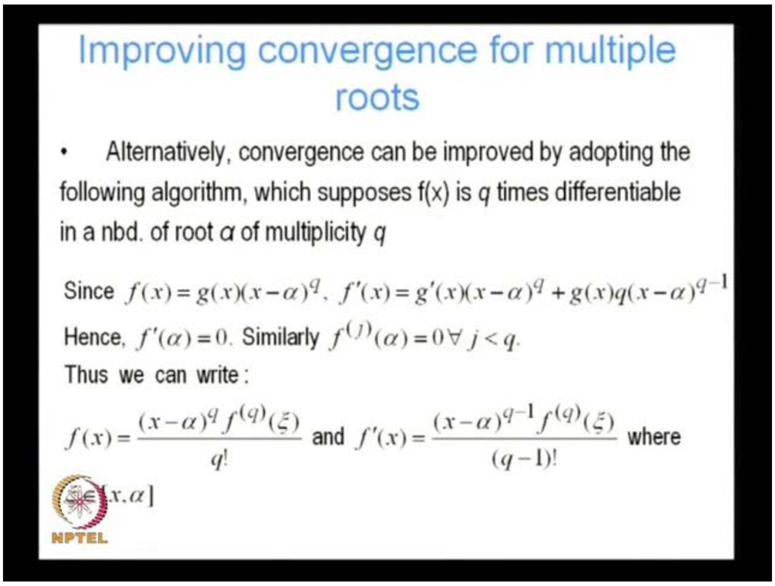

So, f of x is this f prime of x is this, so if I divide this by this.

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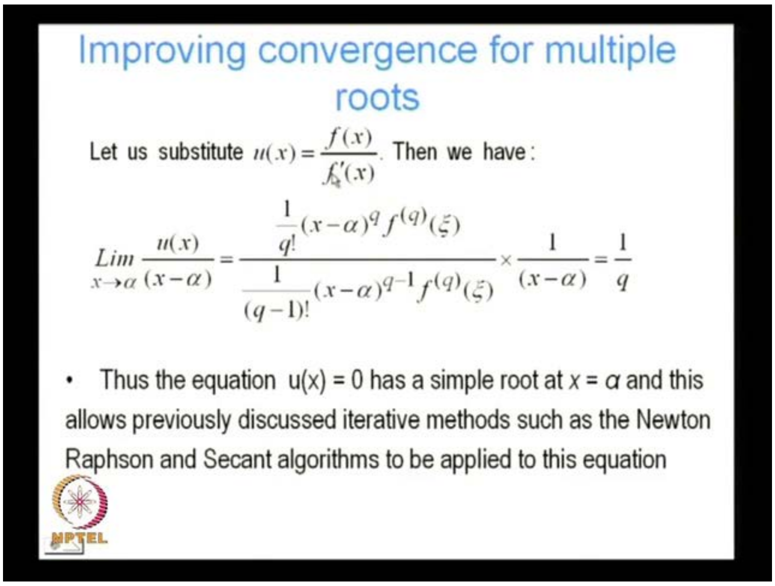

And I divide that by x minus alpha I am going to get one by q I am going to get one by q, so what does that mean? That means, that the equation u of x has a simple root at x is equal to alpha. Why? Because, if I take u of x and divide it by x minus alpha then in the limit as x goes to alpha, I am getting a constant. If this was not a simple root, then there would have been terms like, x to the power minus alpha on this side, I have to got u of x I am dividing it by x minus alpha.

If u of x had more than one root at alpha, then I would have had x minus alpha on the I would have had some x minus alpha surviving, this quotient would have had x minus alpha. And then when x went to alpha that would have gone to 0, but we are saying that that is not going to 0, that is actually going to 1 by q; that means, a u x must have only 1 root at alpha, because after I divided by x minus alpha and I take the limit x goes to alpha it goes to a constant.

It does not go to 0; that means, u of x has a simple root at x is equal to alpha and this allows since it has a simple root; that means, I can use my Newton Raphson, Secant algorithms, and this equation and recover my original order of convergence, which is for the Newton Raphson I am going to get quadratic convergence. So, if we define a new function, u of x which is f of x by f prime of x, this was my original function and then if an u of x I do my new Newton iteration, if I do my newton iteration in u of x I am going to get quadratic convergence. Like for f of x I did not have quadratic convergence, so that that is a very easy way of converting, u f of x to a function, such that I have quadratic convergence.

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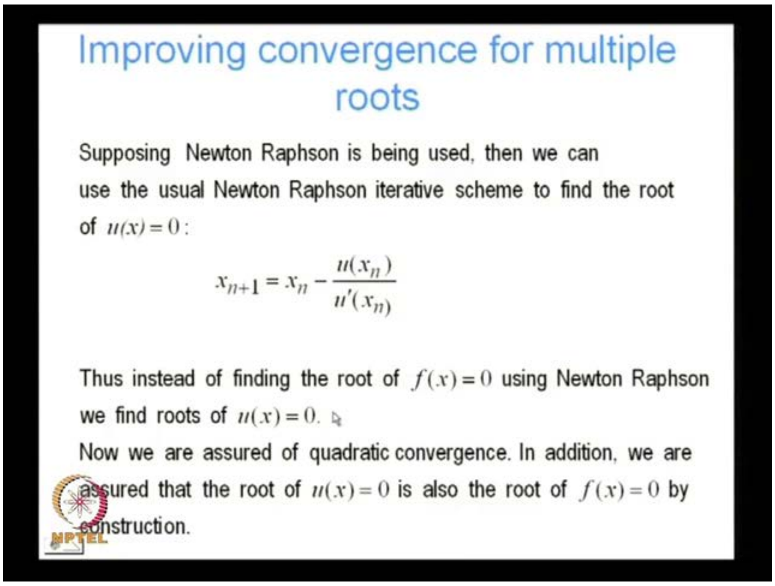

Suppose, Newton Raphson is being used, then we can use the usual Newton Raphson scheme to find the root of u of x is equal to 0, so this is my usual Newton Raphson scheme, but now I am using a term u. Then instead of finding the root of f of x is equal to u 0, by using Newton Raphson to find the roots of u of x is equal to 0. So, now, we are assured of quadratic convergence in addition, we are assured that the root of u of x is also root of f of x, because that we saw by construction. So, even though we have multiple roots, we have a easy way of converting it into improving the order of convergence by writing an another function, which is just a the original function divided by it is derivative.

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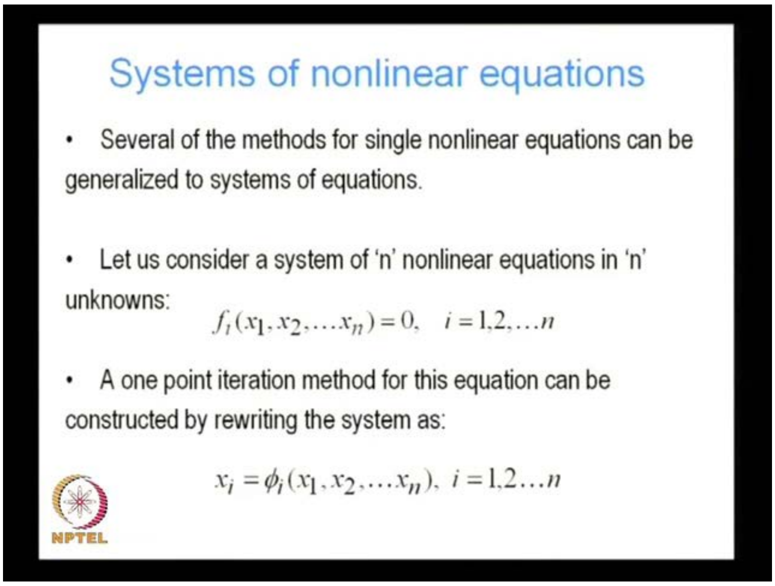

So, I thought we will go we will have a system of that will continue with that in the next class, where we are going to talk about systems of non-linear equations.

Thank you.