# Numerical Methods in Civil Engineering

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## Lecture - 21
## Eigenfunction Solutions for the Wave Equation

Lecture 21 of our series on numerical methods of civil engineering.

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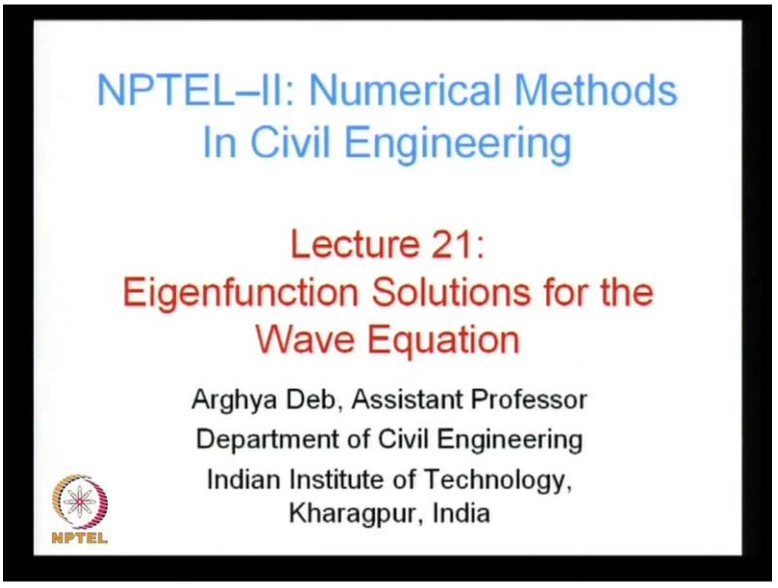


We will talk about eigenfunction solutions for the wave equation.

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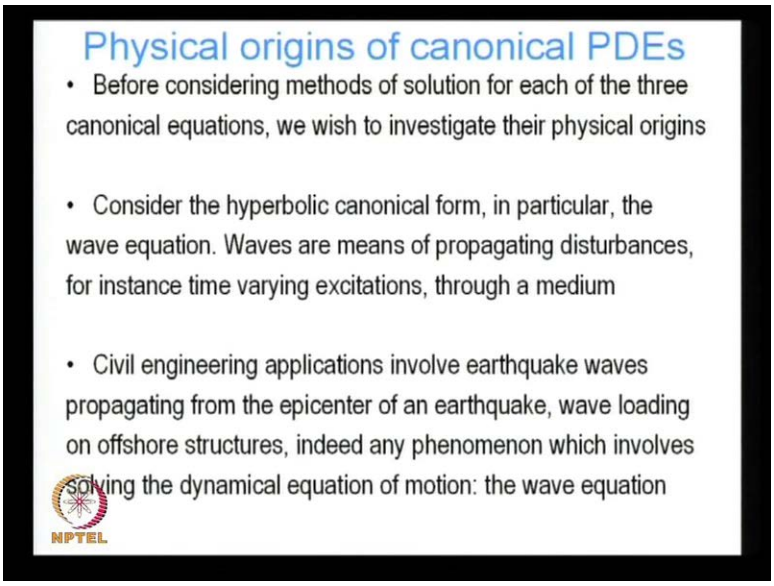


Before considering methods of solution for each of the three canonical equations recall last time. We found that the second order linear partial differential equation has three canonical forms, and we looked at examples of each of those canonical forms and we wish to consider methods of solution for each of the three canonical equations, but before we do that we wish to investigate their physical origins consider the hyperbolic canonical form in particular example of that hyperbolic canonical form the wave equation

Waves are means for propagating disturbances for instance time varying excitations through a medium through a physical medium civil engineering applications of the wave equation involve earthquake waves propagating from the epicenter of an earthquake wave loading an offshore structures indeed any phenomenon, which involves solving the dynamical equations of motion that is the wave equation any phenomenon that in any the equation of motion is basically the wave equation, if we consider the transient terms right

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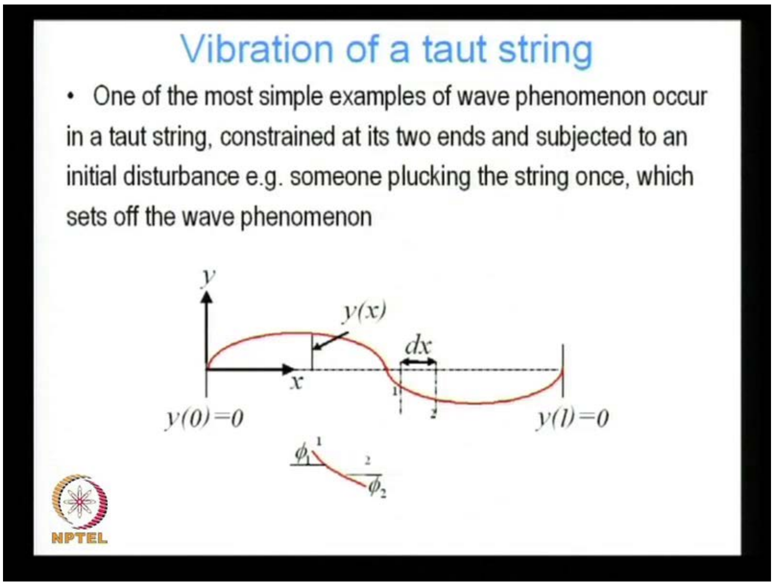


One of the most simple examples of a wave phenomenon occur in a taut string, when you hold a string at it is two end pull it, and then you apply a disturbance at any point. At any intermediate point you generate waves right and the waves you can see the waves actually traveling through the ends of the string that is what makes this example really nice you can see the waves traveling right. So, in this case we have a taut string we constrain it at its two ends basically, we hold let hold restrain the motion at its two ends and subject it to an initial disturbance. For example, we pluck the string once and, which sets off the wave phenomenon. So, in this case the x axis is the distance is the length of the string along the length of the string, while where the y axis is in the vertical direction right.

And we see a little wave there the amplitude of the wave is denoted by y(x), because the amplitude varies with distance from the end right, and we are looking at a little infinitesimal segment d x the little infinitesimal segment d x and may assuming that the ends of the segment lie on points one and two on the string right and.

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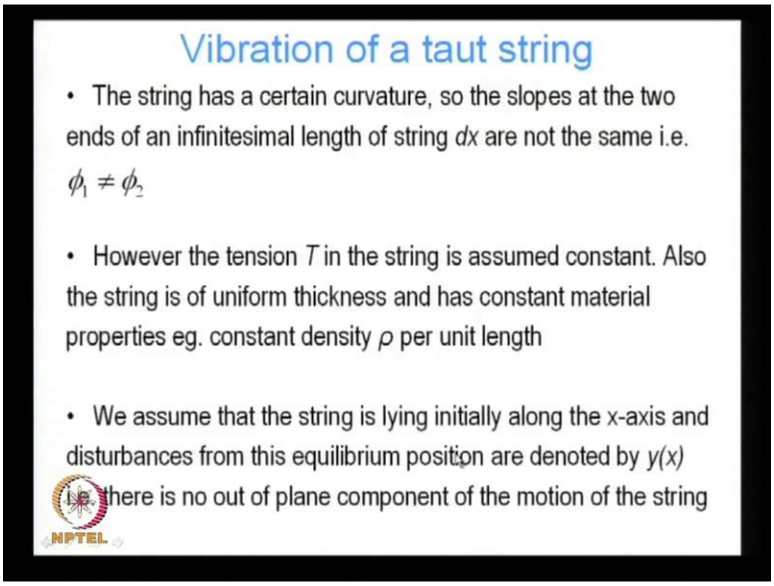


Since, the string has a certain curvature. So, that slopes at the two ends of the infinitesimals length of string d x are not the same that is this phi 1 is not equal to phi 2 the slopes are different right. However, that we assume that the tension in the string is constant, right the tension in the string is constant also we assume that this everything is homogeneous, the string is of uniform thickness the material of the string is the same throughout in pack in particular the density is the same throughout the density of the string is the same throughout.

And we assume that the string is lying initially along the x-axis and disturbances from this equilibrium position are denoted by y(x). That means, there is only one component right. There is no out of plane component right there is no component in this eight direction out of plane component of the motion of the string.

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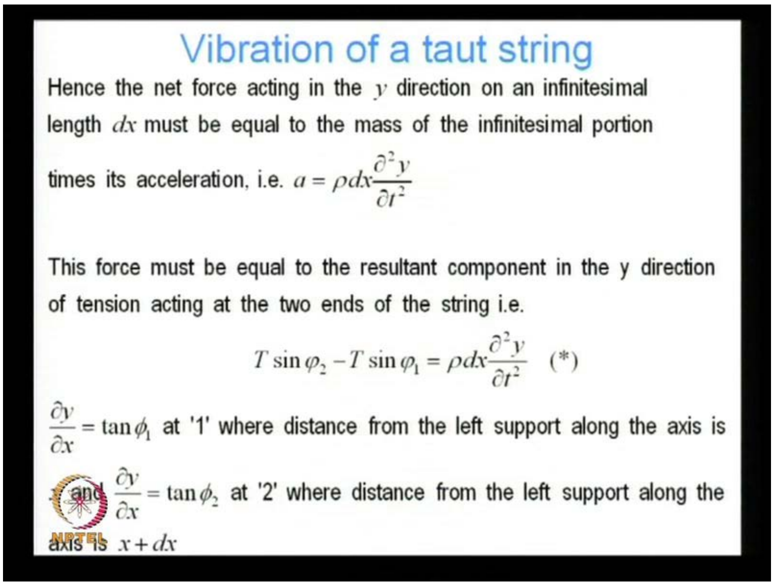


Hence, the net force acting in the y direction on an infinitesimal length d x must be equal to the mass of that infinitesimal portion times its acceleration. So, basically I am looking at this infinitesimal length of the string, and I am saying that whatever be the result of the of the tension acting on these on this little infinitesimal sample specimen that is got to result in the acceleration.

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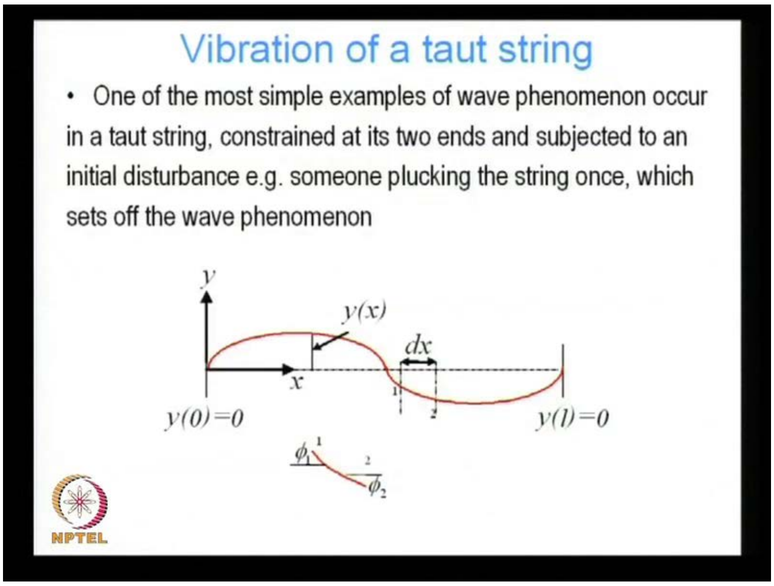

Of that specimen of that little infinitesimal length right. So, that is going to result in an acceleration and the acceleration is the rho times d x, rho times the infinitesimal length, which is the mass times the acceleration.

This force must be equal to the resultant component in the y direction of the tension acting at the two ends of the string, because that is the only force acting and.

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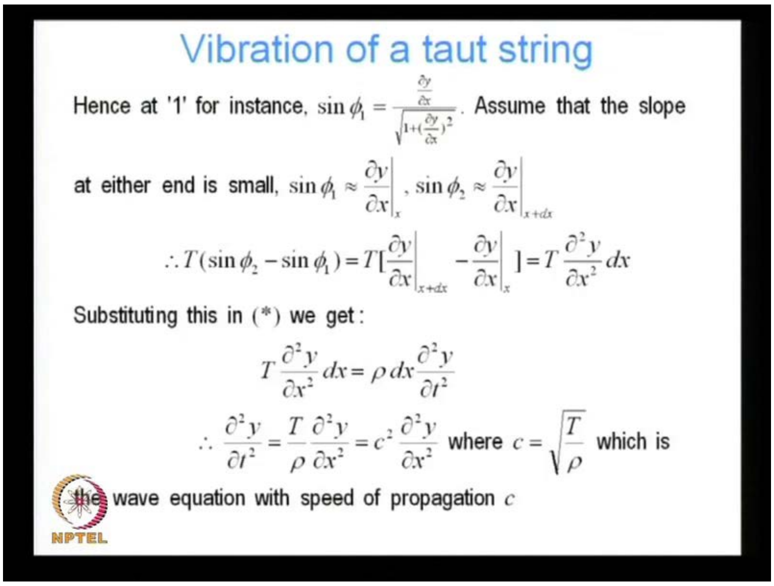


That is t sine phi 2 minus t sin phi 1. So, basically t is the tension acting in the string, t sin phi 2 is the force in the vertical direction at n 2 t, sin phi 1 is the force in the vertical direction n one. So, that difference in the force is what is resulting in the acceleration right. So, this is my equation then again I note that del y del x is equal to tan phi 1 at 1 right. So, del y del x is the slope. So, del y del x is the slope, here and that is given by tan tan phi 1 at one the distance from the left support along the axis is x at point, one it is x and del y del x is equal to tan phi 2 at 2, where distance from the left support along the axis is x plus d x,

hence at one for instance. So, we just since tan phi 1 is equal to del y del x sin phi 1 is given by that right del y del x by root over 1 plus del y del x square right basically. So, in and then we assume that the slope at either end is small right. So, the term in the in the denominator 1 plus del y del x square is approximately equal to 1. So, we have sin phi 1 is approximately equal to del y del x at x, and sin phi 2 similarly is approximately equal to del y del x at x plus d x right.

So, then what do we have t sin phi 1 phi 2 minus sin phi 1, becomes t times del y del x at x plus t x minus del y del x at x, which is again equal to this is the difference between the slopes at an infinitesimal distance of part right. So, that is got to be the secondary (( )) times that infinite distance at that infinitesimal distance right. So, that is equal to del 2 by del x square d x right. So, substituting this in this equation right in this equation we get our wave equation, which is like this t del 2 y del x square d x rho d x del 2 y del t square denoting t by rho as root of t by rho as c we get this is equal to c square del 2 y del x square one can ask why are you calling c the wave speed well. We will find out why we are doing that later on in the lecture right.

C the quantity c is actually the speed of propagation of the way right.

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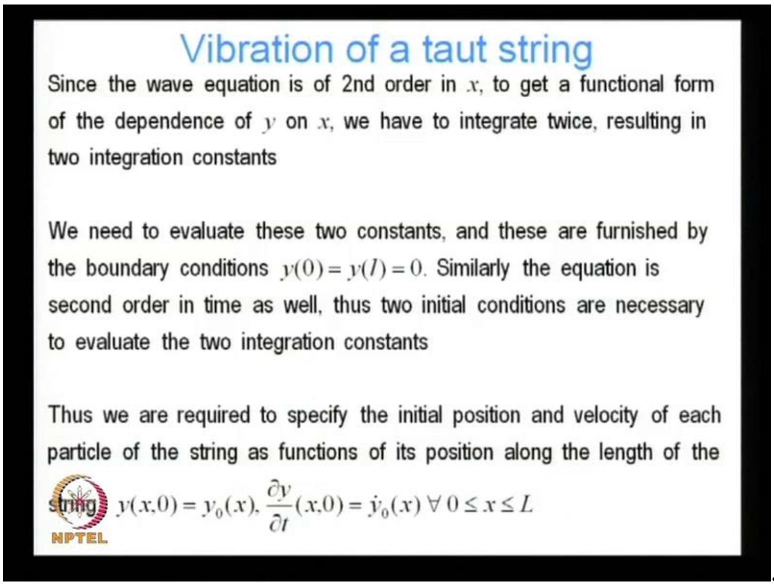


Since, the wave equation is of second order in x to get a functional form of the dependence of y on x. We have to integrate twice right resulting in two integration constants right, we need to evaluate these two constants and these are furnished and that is why we need boundary conditions right. We need the value of y at 0 and l. So, right. So, we need y 0 and y l and in this case we know that since the two ends are constrained y 0 is equal to y l is equal to 0 similarly, the equation is second order in time as well. So, again we need two conditions in time and those are my starting conditions. So, I need two initial conditions, which are necessary to evaluate the two integration constants in time due to integration in time.

Thus we are required to specify the initial position, and velocity of each particle of the string, as function of its position along the length of the string right. So, I must know the initial condition what does that mean along the string at each particle. I must know what is it is initial position right what is the initial value of y and also what is the initial value of y del y del t that is the initial velocity right. So, y x is equal to 0 must be a known function of x right, if at a fixed time del y del t again must be another known function of x at a fixed time t equal to 0.

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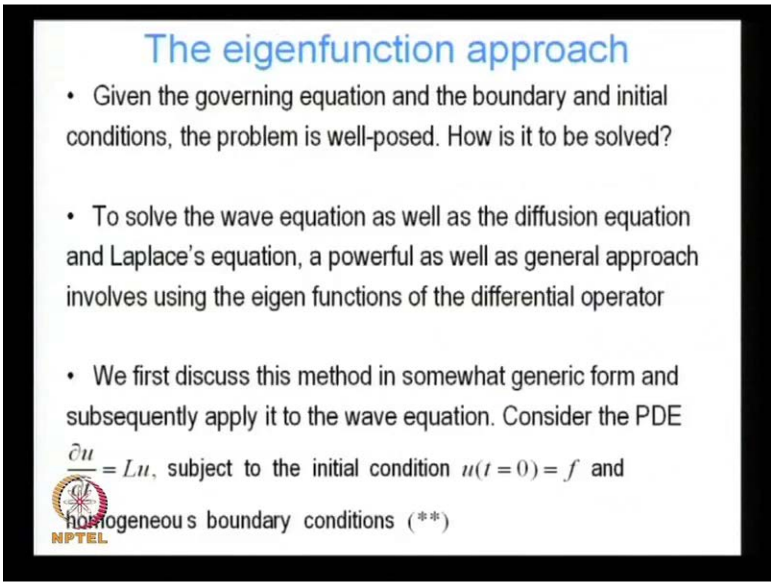


So, given the governing equation. So, now we have obtained my equation right we have looked at a physical example and we've seen that physical example they, when I enforce equilibrium, I generate the wave equation right, I generate the wave equation by enforcing equilibrium (( )). We just force equilibrium right just by enforcing force equilibrium. I got the wave equation, and then I looked at the boundary conditions. I looked at the initial conditions that are necessary for solving those that equation. Now, we have to figure out how we are going to solve it right well to solve there is it tends out that there is a veritably general approach for solving this equation to solve either the wave equation that is an example of the hyperbolic.

Second order equation or for instances the elliptic or parabolic equations examples being the diffusion equation, and laplace’s equation the approach that one approach that can be adopted is a general approach, and it is based on finding the eigenfunctions of the operator right. It is based on finding the eigenfunctions of the differential operator. We first discuss this method in somewhat generic form, and then we are going to apply to the wave equation right. So, let us consider the partial differential equation del u del t is equal to l of u, where l of u is an operator l is an operator acting on u right it his in our case it is a differential operator right. So, and it is subject to the initial. So, we want to solve this problem del u del t is equal to l u subject to the initial condition that at t is equal to 0. We know the value of u, which is equal to f and we also know that there are certain boundary conditions, but in our case we insist that these boundary conditions are homogenous boundary conditions.

So, my original problem this is very important to keep in mind the original problem. My original partial differential equation may have non-homogeneous boundary conditions, right boundary conditions may be non-homogeneous, when say it is not fixed at both ends. There is a string at that end right and the force and the displacement at that end is governed by the string force. So, it. So, that can be one or I prescribe a displacement. I prescribe a value of the displacement at one. So, the actual differential equation the actual p d e may have non-homogeneous boundary conditions, but when we are considering the eigenvalue problem, when we assume that the boundary conditions are homogeneous right boundary conditions are equal to 0 right whatever.

So, we consider this problem del u del t is equal to l u, where l is a linear operator. Since, we are concerned with linear partial differential equations and we are trying to solve this problem del u del t equal to l u subject to the initial condition, which is the same as my as my original problem right and spatial boundary conditions, but the spatial boundary conditions are not the same as in my original problem they are in the homogenous boundary conditions right.

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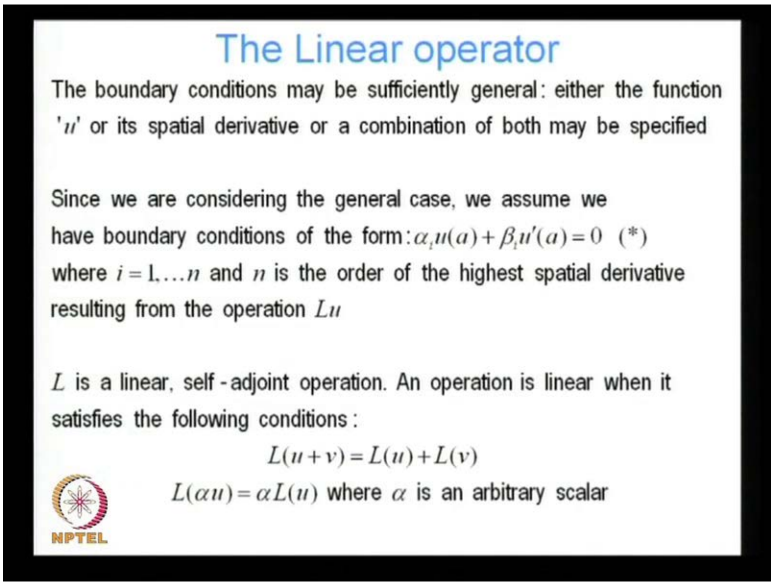


So, the boundary conditions may be sufficiently general what are those boundary conditions spatial boundary conditions well. They can be general either the function u or its spatial derivative or a combination of both may be specified something, what is called the robin boundary condition right either you specify you itself right, which is the essential boundary condition or you can specify the derivative of few right the dirichlet boundary condition or the neumann boundary condition or the combination of both (( )). Since, I have this like this alpha i u a plus beta i u prime a equal to 0. So, it is a combination of the function value, and it is derivative right this is the boundary condition. This is the most general form of the boundary condition.

So, I can require this to be satisfied at n points right and, what is n is the order of the highest spatial derivative resulting from the operation l u right l u. So, in our case l u is nothing, but for the wave equation l u is nothing, but del 2 del x square right del 2 del x squared u right and since, it is second order. I need two boundary conditions right two boundary conditions n becomes 2. So, n is the order of the highest spatial derivative that is 2 and. I need two boundary conditions of this form right of this form and the equation that we just looked at the boundary conditions. We just looked at, which was y 0 equal to 0 y l equal to 0 that is equivalent to alpha equal to 1, a equal to 0, for I equal to 1, alpha is equal to 1, a is equal to 0, and beta 1 is equal to 0 right, and the other 1 is alpha 2 is equal to 1, a u b right your u l is equal to 0. So, again beta 2 is equal to 0 right. So, it collapses to what we had earlier right.

So, but this is the most general form of the boundary condition. l has to be a linear self -adjoint operator right. So, it is has to be linear right we know that, because we are considering linear partial differential equations right. So, l is l had better be a linear operator right, but it is also it must also be something, which is known as self -adjoint right. what does self –adjoint mean, well we are going to look at that in next slide, but what does it mean, when an operator is linear well it just means that the operator acting on u plus v is equal to the operator acting on u plus the operator acting on v or the operator acting on alpha times u, where alpha is a scalar is equal to alpha times the operator acting on the argument u right. So, that is the linear operator but.

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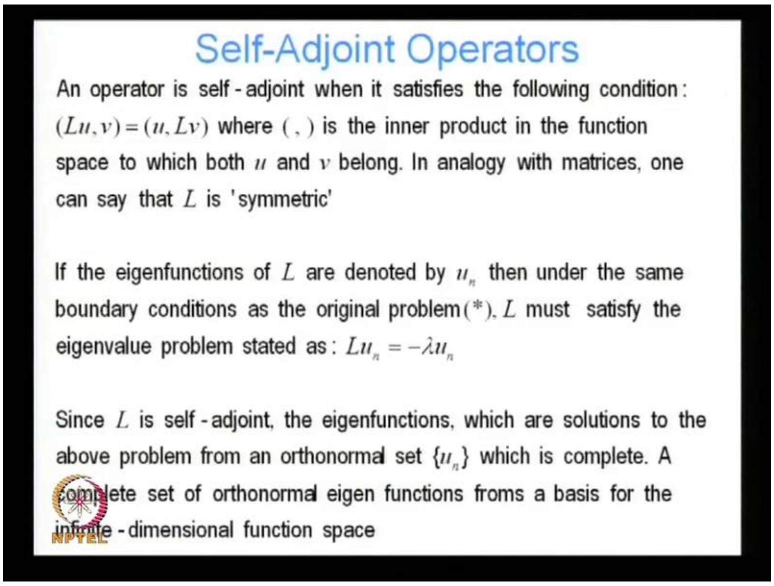


What is self –adjoint?

An operator is self –adjoint, when it satisfies the following condition what is that condition l u v is equal to u l v, where the operation within first brackets denotes the inner product inner product in the function space to, which both u and v belong. So, that we are considering this operator to be defined on certain functions u and v, which lie in a function space right and that function space is a normed space that is a norm there, and there is also a inner product there an inner product, which is related to the norm right there is an inner product in that function space any two members of that function space. I can use them to calculate the inner product right, and that inner product automatically gives rise to an norm in that function space to a measure of distance in the function space measure of magnitude in that function space right, and that inner product is denoted by this first bracket with a comma inside, and both sides of the comma are the two entities of, which you are taking the inner product right

and now, we are looking at inner products of l u some u belonging to that function space. So, I operate on that with l I get l u right, and then I take the inner product of l u with another member v of that function space and whatever, I get if my operator l is self –adjoint is equal to the inner product of if. I operate on v with l first right, and then I take the inner product with u right. So, that is what is meant? when an, when we say an operator is self minusadjoint right. So, an analogy with matrices we can say that the operator l is symmetric right, the operator l is symmetric. So, let us suppose the eigenfunctions of l are denoted by u n, then under the same boundary conditions. As, the original problem l must satisfy the eigenvalue problem same there is a little, when I say same it means it is similar right the form is similar, but it has to be homogeneous.

So, under those boundary conditions as the original problem l must satisfy the eigenvalue problem, which is stated as l operating on u n, where u n is an eigenfunction is equal to lambda, where lambda is a scalar times u n this is how, I define the eigenfunctions of l right the eigenfunctions of l the operator l as, such that if l operates on that eigenfunction the result is equivalent to multiplying that eigenfunction with the scalar minus lambda right. So, that is how we define the eigen very similar to work the to the eigen vectors and eigenvalues of a matrix, but now it is applied to an operator to a differential operator right. So, differential operator acting on a function gives me that original function times a scalar right, then that function is an eigenfunction of that differential operator and that scalar is an eigenvalue or a characteristic value of that differential operator right the same idea generalized to function spaces.

Since, l is self –adjoint the eigenfunctions, which are solutions to the above problem form an infinite an orthonormal set u n, which is complete well let us go back and take a step back recall that for symmetric matrices for symmetric matrices the eigen vectors right the eigen vectors are guaranteed to be orthonormal right. They are guaranteed to be orthonormal and they form a basis for that vector space, they form a basis for that vector space. Similarly, for a self –adjoint linear operator the eigenfunctions they form an orthonormal set right. So, if I take the inner product of two eigenfunctions to distinct eigenfunctions. I am going to end up with 0 right, if I take the inner product of the same eigenfunction with itself right, then I am going to end up with 1 right. So, they are they form an orthonormal set and moreover it is complete. So, remember in case of matrices, we said that the eigen vectors are form a basis right, what does it mean that any vector belonging to that space can be expressed as a linear combination of those eigen vectors right.

For set of functions, when we say that set of functions is complete. I mean there is a technical definition, I mean for completeness one can look at that, but if you think it think of it in a descriptive way right in a what it means, it means that I can express any function belonging to that space in terms of as a linear combination of my basis functions of my orthonormal basis functions. A complete set of orthonormal eigenfunctions form a basis for that infinite dimensional function space right.

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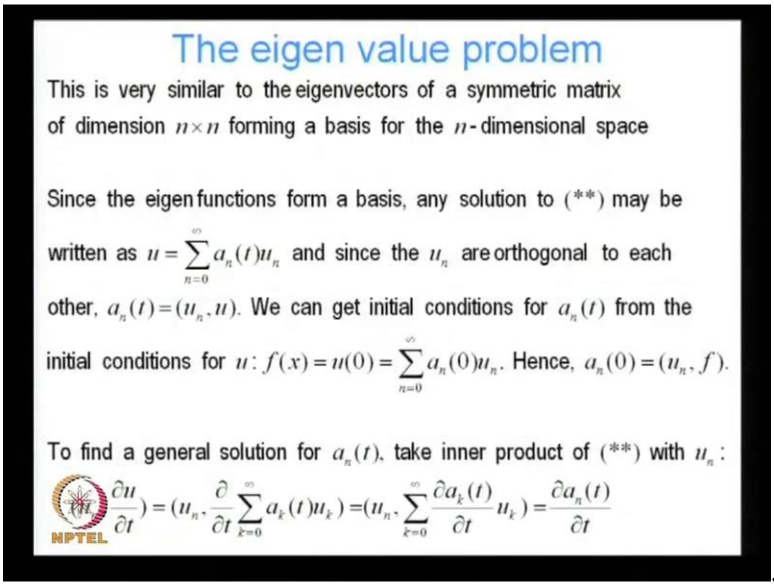


So, this is very similar to the eigen vectors of a symmetric matrix of dimension n by n forming a basis for the n dimensional vector space right.

Since, the eigenfunctions form a basis any solution to my equation star, which is this equation right, which is this equation del u del t is equal to l u subject to this initial and boundary conditions right the any solution to that equation can be expressed as a linear combination of my basis functions. So, suppose u is a solution of del u del t is equal to l u, then I can write u as a linear sum of as a sum of my basis functions right of my basis functions u n right. So, u is equal to sigma, n equal to 0 to infinity, a n t u n right why is there no t dependence on u n, because we know that this l this l involves only the spatial derivatives right this l involves only the this differential operator involves only the derivatives with respect to space right not time. So, any time dependence of the solution u to this problem has to come from my constants it is from my multiples sorry, I should not say constants not anymore my multiples a n right, which scale my basis functions right. So, the time dependence comes from this

from these coefficients a n right the coefficients a n contain the time dependence the eigenfunctions u n contain the space spatial dependence right, and that makes u dependent on space as well as time right and since, the u n are orthogonal to each other. We can write a n t is equal to u n u, we can if we take dot product of both sides of this expression u is equal to sigma, n equal to 0 a n t u n right on the left hand side. I have dot product with u n u n u right on the right hand side, I take I should not say dot product inner product right on the left hand side, I have inner product of u n with u on the right hand side, I have inner product of u n with a n t u n and again since, the basis functions are orthonormal. So, all the other terms are going to give me 0 and, I am going to left be left with a n t right, I am going to be left with a n t. So, I can get a n t is equal to u n comma u

Well that is interesting, but does that tell me anything it does not right where's the whole purpose is to find the solution u. So, if I do not know what is u right there is no question of taking the inner product of a of u n with u right. So, there is no way I can find the coefficients a n just by taking the inner product of u n with something, which I do not know right. So, that is not possible that is interesting to note that we can write a n as u n whatever. So, if I somehow manage to calculate u right and for some reason, I want to know what are my coefficients n I can do that right, but if I do not know u well, I got to find it right, how do we find? how do we find u well. We already know u n right we already know the functions eigenfunctions u n by solving my problem right by solving my problem I already know, but how are we going to find a n right, how are we going to find a n by solving a set of we will find out, that we solve we find a n by solving certain differential equations in time involving the constants n, let us see how we do that right.

So, we can get initial conditions for a n from the initial conditions for u right, why is that well suppose I know the initial conditions for u that is, I know that u at time t is equal to 0 is some known function of x f of x, such as f of x right then, I can write f of x is equal to sigma, n equal to 0 to infinity a n 0 u n right just from this equation just from this equation right. So, if I know u at 0, which is f of x that is equal to sigma, n equal to 0 to infinity a n 0 u n right and hence, I can get a n 0 is equal to u n f. I taking the inner product of u n and f right, I already know my u n's right, I already know my u n's right and suppose, I know my u n's, I know my f I can find out the value of a n at 0 right to find a general solution for a n at any time. We take the inner product of this equation with u n, which equation well that equation right that equation with u n.

So, if I take if I do that if I look at the left hand side I have inner product of u n with del u del t right, which is equal to u n inner product of del del t times u right u is equal to sigma, k equal to 0 to infinity a k t u k right, and then I take the partial derivative with respect to time inside the summation right. So, I have inner product of u n summation k equal to 0 to infinity, del a k t del t u k right and this is equal to del a n t del t wise that, because I have the inner product of u n u k only the term involving n is going to survive, and I am going to get del a n t del t right

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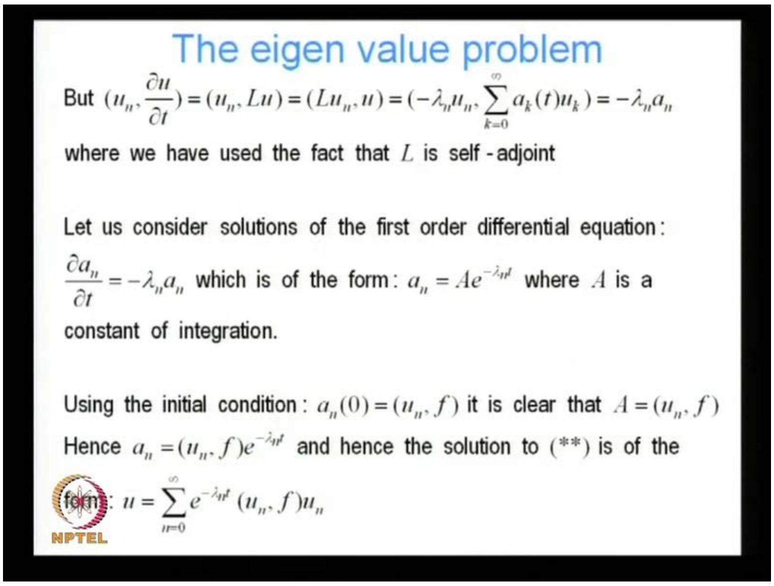


, but u n inner product of u n and del u del t is nothing, but u n comma l u wise that, because del u del t is equal to l u right del u del t is equal to l u right that was my original equation. So, u n with l u inner product of u n with l u, but since the operator is self –adjoint. I am now going to use the self –adjoint property and say that inner product of u n with l u is the same as the inner product of l u n with u right and l u n by definition the u n is then eigenfunction of the operator l. So, l u n must be equal to minus lambda n u n right l u n must be equal to minus lambda n u n. So, I've minus lambda n u n, and then I again replace u with my summation here sigma k equal to 0 to infinity, a k t u k again using the fact that the eigenfunctions form an orthonormal basis the only term that is going to survive is going to be that right

so, a n. So, only term that is going to survive is going to be minus lambda and u n right, because that is the only term that is going to survive. So, now, I have where we've used the fact that l is self –adjoint. So, we have this equation right u n comma del u del t is equal to minus lambda n a n a n, but just previously, if we found that u n comma del u del t is equal to del a n t del t. So, I have this equation this first order differential equation del a n del t is equal to 1 lambda n a n. So, I have a first order differential equation in time for my coefficients a n, and I can solve that if I do that for, if I solve that first order differential equation, I get n is equal to a to the power a into e to the power minus lambda n times t, where a is a constant of integration right and then to evaluate that constant of integration a, I use my initial condition right what was my initial condition on a n, which I calculated right here right I said a n 0 is equal to u n comma f right, I found that. So, now, I use that condition a n 0 is equal to u n comma f. So, if I do that it is clear that a must be equal to u n comma f right, because a n a n is equal to a e to the power minus lambda n t. So, n 0 is equal to a right it is obvious that a n 0 is equal to a right.

So that means, that a must be equal to inner product of u n with f right and therefore, a n must be equal to inner product of u n with f times e to the power minus lambda n t and therefore, the solution to my equation is my original equation is given by u is equal to sigma n equal to 0 to infinity a n t u n. So, u is equal to this thing u n comma f e to the power minus lambda n t times u n right summed from n is equal to 0 to infinity right. So, what does this finally, tell me that it tells me that if I know if I can find the eigenfunctions of my operator l and, if I can find the eigenvalues of my operator l and if I have initial conditions known initial conditions on the solution known right, then in that case I can solve the problem entirely in terms of the eigenfunctions the eigenvalues, and the initial conditions and the form of that solution is given by u is equal to sigma n equal to 0 to infinity e to the power minus lambda n t u n f inner product times u n.

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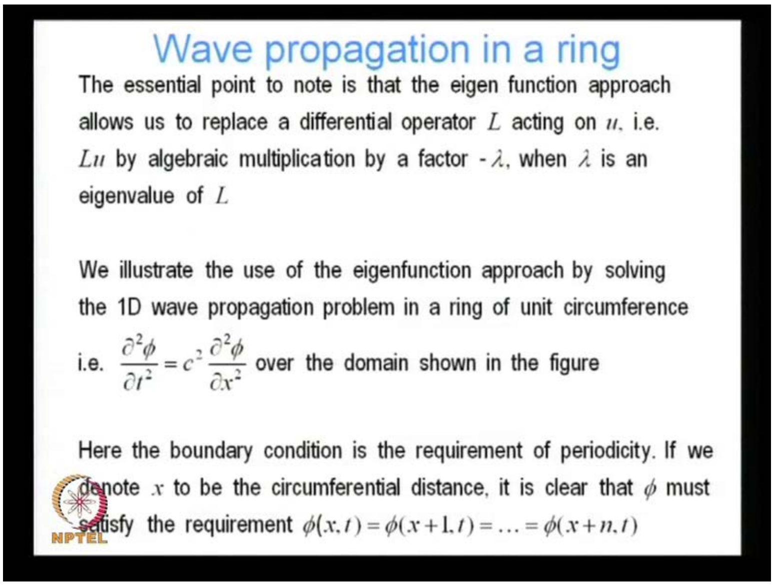


The essential point to note is that the eigenfunction approach allows us to replace the differential operator l acting on u that is l u by algebraic multiplication by a factor of minus lambda, when lambda is an eigenvalue of l right. So, because it is this, because this is an eigenfunction. I can replace l u by lambda operating on u right provided u is an eigenfunction right. So, we illustrate the use of the eigenfunction approach by solving. So, this is the general approach we looked at it will turn out, when we look other types of partial differential equations are like elliptic or parabolic, partial differential equations. You can see that the same approach can be used in that case as well right, but now we now we have that we have looked at the approach. In general, we want to see we want to apply directly to the wave equation and see how, we can solve that problem using this approach.

So, and the wave equation that I am going to solve for the first time is going to be a really simple one it is basically, I am going to solve the wave equation in a ring, in a ring of unit circumference that is a circular ring, and it is circumference is of length one right and I've waves propagating on that ring I saw the wave equation and I am going to find solutions to that wave equations. So, I am trying to find out what are the patterns what is the form of u as a function of x and time right there are waves, which are forming on that ring. So, I want to find out what is the displacement of a part of a point on that ring as a function of it is spatial position and time right.

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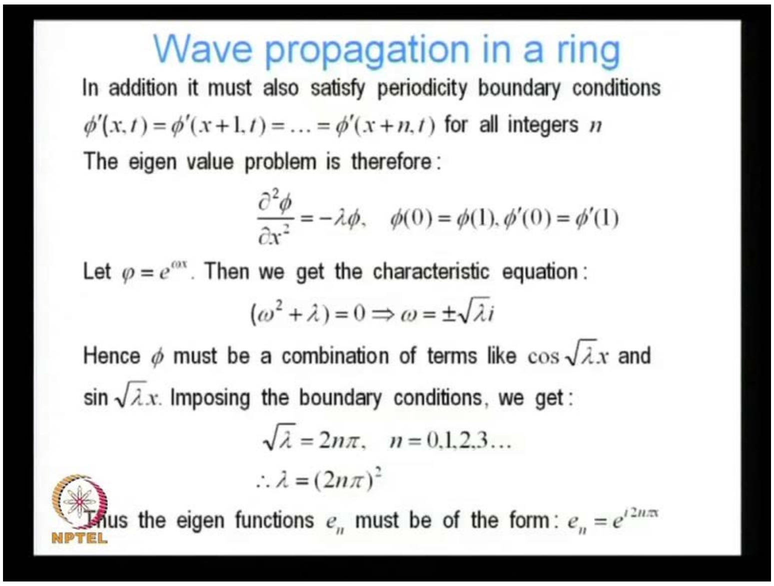


So, this is my domain of interest I do not know, where that is gone I seem to I've deleted that slide, but we have a ring right and I have these waves on that ring right. So, I am sorry the figure is missing, but you have to imagine right, I have a circular ring and I have waves on that ring and that circular ring is of circumference equal to one and x is the circumferential distance along the ring right. It is the distance along the circumference of the ring the boundary conditions, we impose is the requirement of periodicity, why is? why does it have to be periodic, because if I go around with one circumference is one. So, I start from x is equal to 0 right go around the ring I come back and since, it is a it is the ring has circumference one once I come back to that same position x is equal to 1 right and since, x is equal to 1 and x is equal to 0 at the same identical position the solution had also be identical right.

So, phi(x) at t had better be equal to phi(x) plus 1 at t must be equal to phi(x) plus 2 at t and must be equal to phi(x) plus n at t right. So, that is my requirement on phi similarly, I satisfy similar conditions on my derivative right not only must phi be the same the slope had also be the same. So, when I go back go around once and come back to the same location not only must my displacement be the same the slope of that the slope of the displacement also got to be the same right. So, phi prime of x at t x of t must be equal to phi prime of x plus 1 of t and. So, on and must be equal to phi prime of x plus n at t for all integers n right. So, these are my periodic boundary conditions right on the function phi on you can think of that at the displacement as well as on the derivative of the displacement right. So, that is my problem this is equal this I want to solve this subject to these boundary conditions right and some initial conditions right.

So, my eigenvalue problem is there for, what is my operator l my operator l is nothing, but del 2 del x square right partial derivative with second order partial derivative with respect to x right. So, my eigenvalue problem is del 2 l operating on phi must be equal to minus lambda phi right. So, first thing to if I want to solve a problem using the eigenfunction approach I have to find the eigenfunctions and eigenvalues of this operator. So, I have to solve this problem first right del 2 I have to solve del 2 phi del x square is equal to minus lambda phi, which is a purely spatial problem right it is a purely spatial problem, because l includes only the derivatives with respect to space. I solve that problem subject to these boundary conditions my periodicity boundary conditions right (( )) I going to solve that problem well it is of this form. So, I assume that phi is of the form e to the power omega x right.

Assume phi is equal to of the form e to the power omega x, then that differential equation let us convert it into a quadratic in omega right a quadratic in omega I have omega square plus lambda is equal to 0, which gives me omega is equal to plus minus square root of lambda times, I being the imaginary number right. So, so hence phi must be a combination, then the solution phi is of the is like e to the power root of lambda I x right e to the power plus minus root of lambda i x right and root of lambda i x that I can write using demoivre's theorem right, I can write it as cos x plus combination cos and sin right.

So, phi must be a combination of terms like cos of root of lambda x and sin of root of lambda x right, and then if we impose the boundary conditions on that if we impose the boundary conditions phi 0 equal to phi 1, phi prime 0 is equal to phi prime 1 and. So, so this phi must be like combination of cos of root lambda x and sin of root of lambda x imposing the boundary conditions. We can find out the value of lambda in terms out that we get root of lambda equal to 2 n pi must be equal to 2 n pi for n equal to 0 1 2 3, if I have to satisfy my periodicity boundary conditions right in that case lambda must satisfy that requirement and therefore, we get lambda is equal to 2 n pi whole square and therefore, the eigenfunctions e n must be of the form I know there are the form e to the power omega x omega, I know to be root of 2 n pi whole square that is 2 n pi. So, e n must be equal to e to the power i 2 n pi x right. So, now, I know my eigenfunction I know my eigenvalues right. So, I can move on to the next stage.

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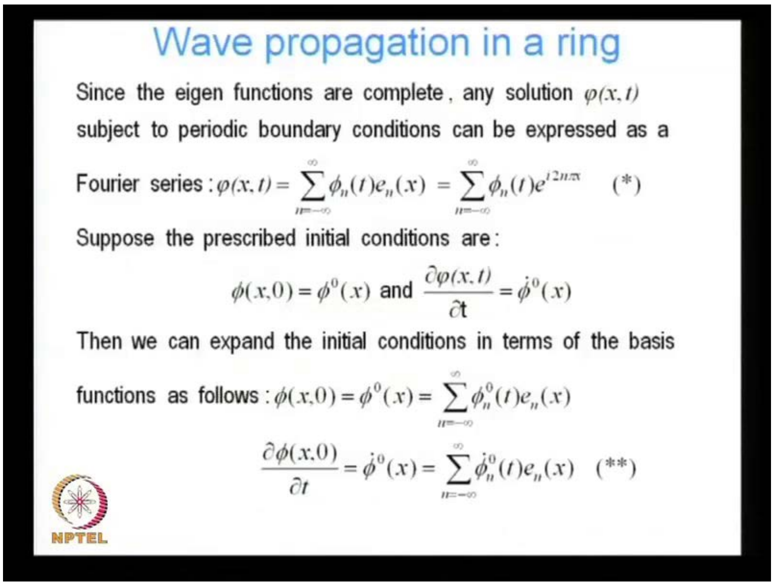


Since the eigenfunctions are complete any solution phi x t right now, I am the considering total solution phi x t of my original system. I can write it as a combination as a linear combination of my eigenfunctions e n x must scale with some constants phi n, which are functions of time right and if we do that we get an expression like this phi(x t) is equal to sigma, n equal to minus infinity to infinity phi n t e to the power i 2 n pi x right and this is like a this as the form of a fourier series right this is a form of a fourier series right. So, we still do not know what are my coefficients phi n t. So, I can describe my total solutions only, when I know phi n t I know I know eigenfunctions one part is known, but I still have to find phi n t right and how do I do that well I again I try to find a try to form a differential equation for the coefficients right and then, I solve that differential equation means that has got to be a differential equation in time, because my coefficients are only functions of time. So, I solve that differential equation in time subject to appropriate initial conditions, and find my coefficients phi n t once. I find my coefficients phi n t I get my total solution right.

So, how let us look at the procedure for doing that we suppose at the prescribed initial conditions, our phi x 0 is equal to phi 0 of x right suppose, I know that at time t is equal to 0. I know what is the solution phi 0 x and, I also know the velocity at time t is equal to 0. So, I know del phi del x. So, that is got be 0 right x at 0 is equal to phi dot 0 at x right. So, suppose I know those conditions then I expand the initial conditions right since, this is a function in x right and my eigenfunctions form a are complete and form a basis right. So, I can express this function of x in terms of my eigenfunctions right.

Similarly, I can express this function of x right in terms of the eigenfunctions right that is exactly what I do. So, I write phi 0 x is some constant, some coefficient phi 0 n t times e n x right phi 0 n now, that is a little doubt I have a something I have a mistake here there is no t here right there is no t here. So, phi 0 n e n x right and this phi dot 0 x I can write as some phi dot 0 n, where n is a constant right the summation constant of summation times e n x right. So, this please ignore this t that is that does not exist right. So, that is my initial conditions, and then I am expanding I am writing those initial conditions in terms of my basis, which are my eigenfunctions right initial conditions are functions of x, because they can vary along the circumference right well. I can initial condition of functions of x and since, the functions of x I can write them in terms of my eigenfunctions, which I am doing write there.

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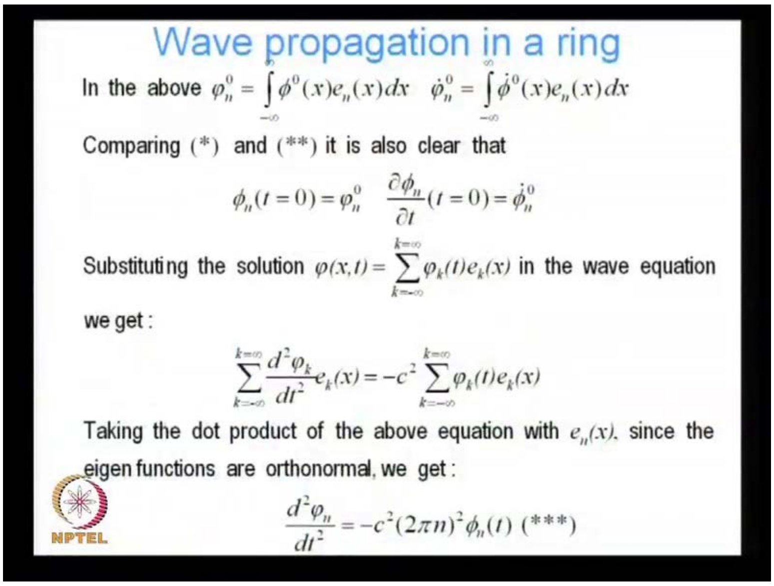


So, in the above how do I find these constants phi 0 n well, I just use the fact that my eigenfunctions orthonormal. So, I take inner product right on both sides with e k of x right and that is going to give me phi 0 k right similarly, I can get phi 0 dot k exactly the same fashion right. So, phi 0 n is equal to in a sum I taking the inner product is inner product in this case is the integral from minus infinity to infinity, that is the inner product for this function space right the inner product is defined in terms of an integral right like this is not an inner product for vectors an inner product is a dot product right, but for functions the inner product is in terms of an integral right integral over the domain of interest. So, in this case this inner product becomes phi 0 x e n x d x right, and this phi dot 0 n in inner product of phi dot 0 x with e n x right by taking those inner product, I neutralizing the fact that my basis functions are orthonormal. I can find out phi 0 n and phi dot 0 n right.

So, comparing these two equations this and this right it is also clear that phi n t is at t equal to 0 is equal to phi 0 n, and del phi n del t at t is equal to 0 is equal to phi dot 0 n look at this expression right. So, phi at t is equal to 0 is equal to phi n of 0 n x right. So, what does that tells me that tells me that phi n of zero is equal to phi zero n, I compare this with this right, I know that phi n of 0 must be equal to phi 0 n right is that clear similarly, phi phi n dot evaluated at 0 must be equal to phi dot 0 of n write, if I take the derivative write, if I take the derivative then again phi dot evaluated at t is equal to 0 and I compare that with this I see that that must be equal to phi dot 0 n right. So, that must be true right and substituting the solution phi x t is equal to sigma k equal to minus infinity to infinity phi k t e k x in the wave equation right. So, I know that phi must be of this form right, and then I substitute that in the wave equation right, I substitute that in the wave equation, and then I get d 2 phi k d t square e k that is the del 2 phi del t square that gives me that and on the right hand side of del 2 phi del x square times c square right, and then I get this right

is that clear. So, and then I take the dot product of the above equation with e n x and since, the eigenfunctions are orthonormal this part only gives me d 2 phi k d 2 d 2 phi n d t square. Since, I am taking dot product with n and this part gives me minus c square 2 pi n square phi n t right. So, now, I have finally, got and second order differential equation for my coefficients phi n right, and I need to solve them to find my coefficient phi n right.

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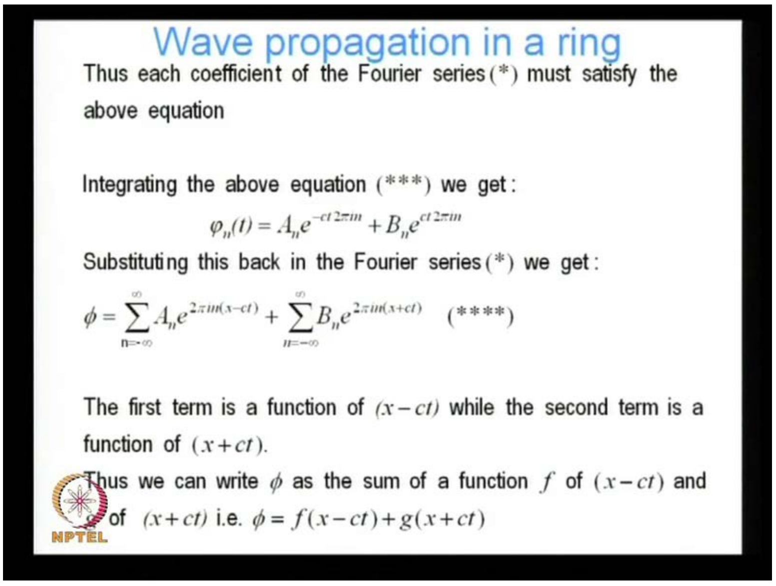


Thus each coefficient of the fourier series phi n t each coefficient, here must satisfy that equation right must satisfy that equation d 2 phi n d t square equal to minus c square 2 pi n square phi n t right. So, again to solve that equation we integrated twice right. We integrated twice and we get phi n t is equal to a n e to the power minus c t 2 pi i n plus b n e to the power c t 2 pi n right and this to integrating two times right, and I get that

and substituting this back in the fourier series. So, now, I know my phi n t, I know my eigenfunctions right. So, I can put this back together and I get my final solution phi in terms of x with depends on x as well as t right. So, the first equation is a function of x minus c t. You can see the first equation is a function of x minus c t the second equation is a function of x plus c t right, the second term sorry the first term is a function of x minus c t, the second term is a function of x plus c t. So, I can write my total solution phi as the some of two parts one part depends on x minus c t, the other part depends on x plus c t and this particular representation has a very important implication, and we are going to talk about that next class. So, again to continue to talking about this solution and we will see that by putting it in as splitting it into two parts as the some of a function of x f of x minus c t another function x plus c t that has a lot of very interesting physical implications thank you .