Numerical Methods in Civil Engineering
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Lecture - 24
Analytical Methods for Parabolic and Elliptic PDE's

In lecture 24 of our series on Numerical Methods in Civil Engineering, we will talk about Analytical Methods for Parabolic and Elliptic Partial Differential Equations.

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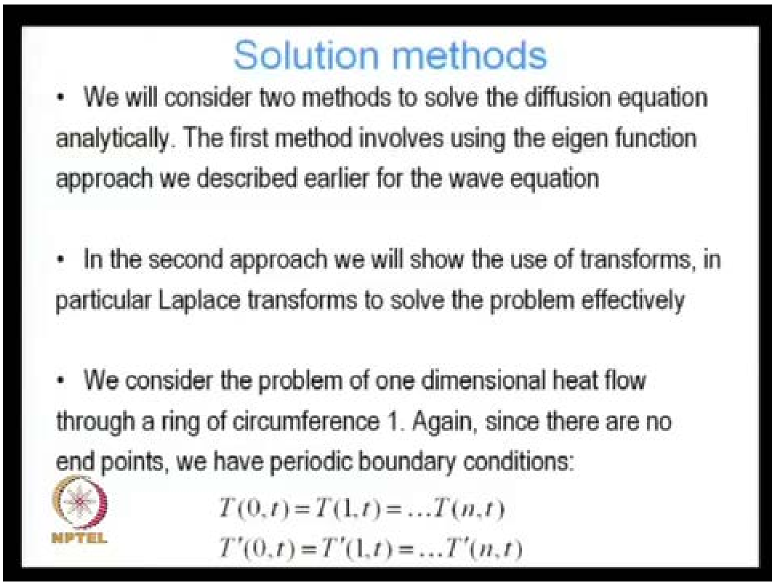

We briefly did start talking about analytical methods for parabolic partial differential equations in our last lecture. And we said that, we are going to talk about two solution methods, the first solution method will involve the use of Eigen functions, which I said is a sort of generic method, common to which can be used for all three canonical forms and then, we are going to use the method of Laplace transforms. So, we start midway, so I just briefly go over the parts that we have already covered.

So, we are considering the one dimensional heat flow through a ring of circumference 1 and we saw that, in that case we have periodic boundary conditions and the periodic boundary conditions are on the primary variable temperature and it is derivative with respect to space.

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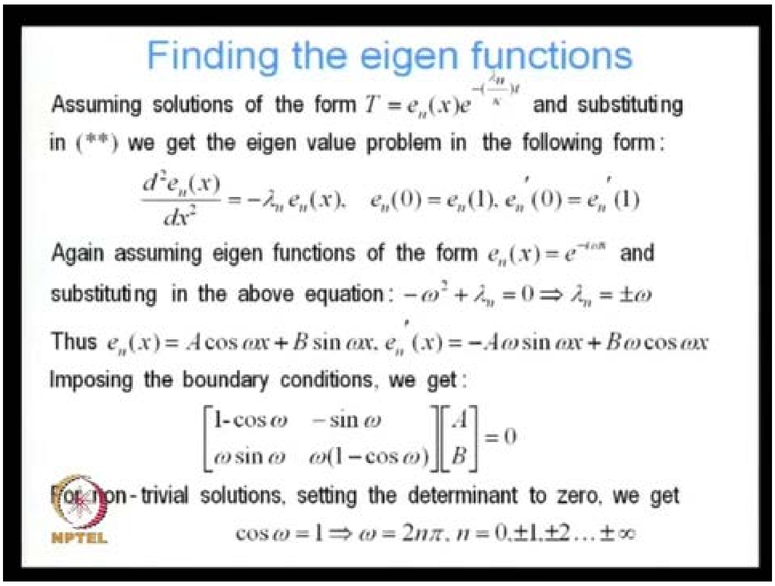

So, if we assume solutions of the form T is equal to e to the power n x, where e n x is nothing but the Eigen functions and e to the power minus lambda n k by k t is the time dependent part. If we assume solutions like that and we substitute it in my original differential equation then, I get the Eigen Value problem. Left hand side I have a linear operator and the right hand side I have this Eigen Value and this is my eigenvector and I solve the Eigen Value problem subject to homogeneous boundary conditions.

So, in this case, the periodic boundary conditions again, so this is my problem, this is my Eigen Value problem. So, how do I solve the Eigen Value problem, I have solve the Eigen Value problem assuming that, e n x has a certain form and that allows me to convert the differential equation into an algebraic equation. I solve the algebraic equation, that is a characteristic equation, the root of those characteristic equations are my Eigen values, I solve for the Eigen values.

And once I solve for the Eigen values, I get my Eigen functions, I know my Eigen functions upto certain undetermined constants. I know my Eigen functions upto certain undetermined constants and I evaluate those constants by imposing my boundary conditions. So, in this case, the boundary conditions are these ones, which we already talked about and then, on imposing those boundary conditions on these two, I get two equations, two unknowns and you can see this is a homogeneous equation.

So, for this system of homogeneous equations to have a solution, the determinant must vanish. Determinant vanishes gives me this condition, cos of omega equal to 1, which tells me omega equal to 2 n pi, n is equal to 0 plus minus 1 upto infinity.

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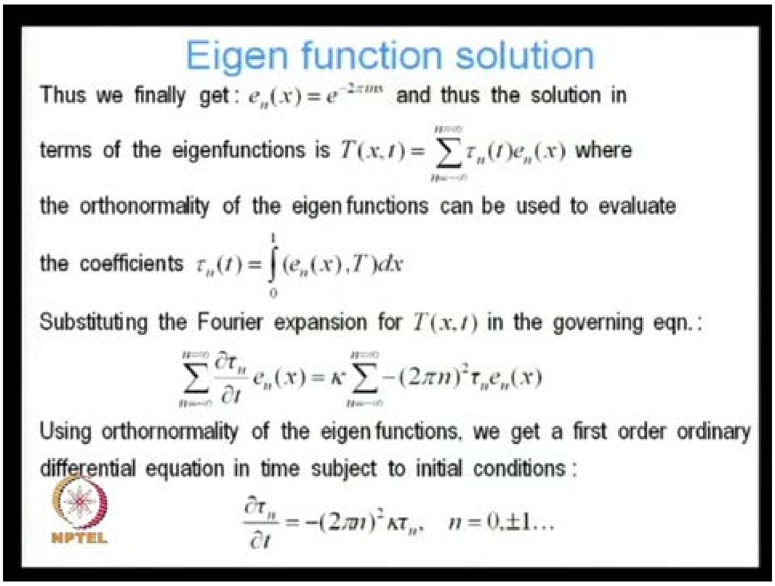

So now, I know my entire Eigen function, I know the constants also, so I know my constants, now I can write my solution in terms of the Eigen functions and some coefficients, which coefficients are functions of time. And then, I substitute, so this is my Fourier expansion, I substitute this Fourier expansion in the governing equation. So now, I get a function, a first order differential equation in time, so that gives me the undetermined constants tau n, which are functions of t. So, this is a ODE for tau n and then, I solve this ODE for tau n using my initial conditions, subject to initial conditions I solve for this, I get my tau n.

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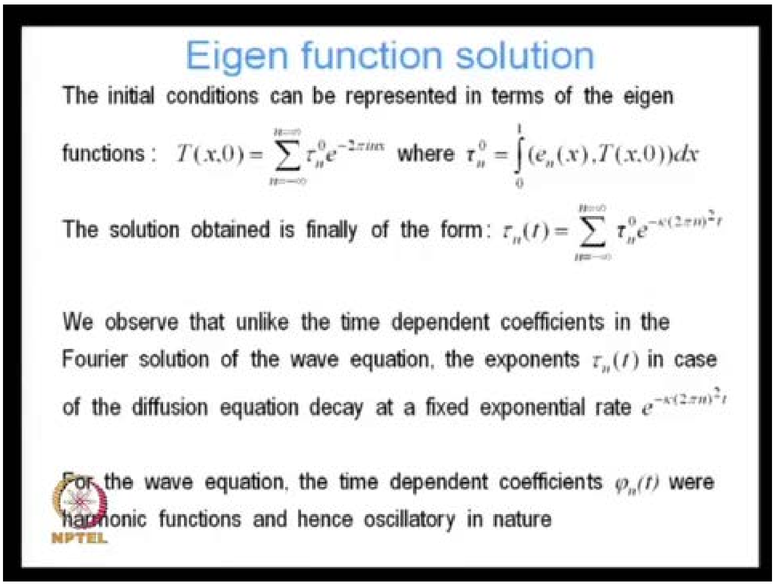

For doing that, I have to represent the initial condition in terms of the Eigen functions, with that system into immediate step, but once I do that, I finally solve for tau n and I get this. And once I get tau n, I have the whole Fourier expansion for my temperature, however one important thing to observe is that, the time dependent coefficients in the Fourier, unlike the time dependent coefficients of the Fourier solution for the wave equation, for the diffusion equation the time dependent coefficients decay exponentially.

You see this term minus k 2 pi n squared t, so it means that, as time increases, this becomes smaller and smaller. But, for the wave equation, if you go back to my to the notes to the previous lectures on the Eigen Value solution for the wave equation, you will find that, in that case the time dependent part was not one exponentially decay. It was not an exponential function at all, it was a harmonic functions, it was of an oscillatory in nature. Here, you can see it is very different, it decays with time as time increases, it goes down.

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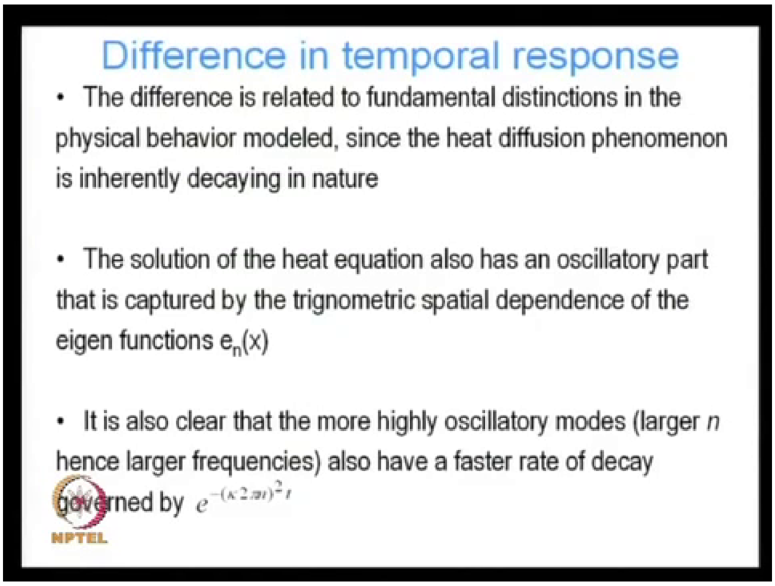

So, that is because of the fundamental nature of the problem, so we are modeling heat diffusion and heat diffusion inherently it is decaying in time, it is because of it is decaying nature, as if think of something like this, you have a boundary. You have a body, you apply heat temperature boundary conditions then, overtime you would expect all those peaks initially. Since you are applying the temperature at a boundary, you were expect the temperature to be high at a boundary.

And then, at the interior, the temperatures would be smaller as starting with, but overtime they are going to decay, those peaks are going to decay and you are going to reach some sort of steady state condition. So, that is exactly what the solution is telling us here, that solution is telling us the time dependent part of the solution is going to decay with time. So, the solution is eventually is going to the steady state solution is only going to have special dependence, it is not going to vary with time.

So, once it reaches steady state, things do not change with time, it is the equilibrium solution you can think of it like that, things do not change with time, they may vary over space, but they do not change with time anymore. So, that is what the solution is telling me, the solution of the heat equation also has an oscillatory part, but that is purely spatial, the time part does not have any oscillatory part. You can see i e to the power i, let us go back and take a look, again the time dependent part has no imaginary number, so it does not have any oscillatory part does not have any harmonic components.

However, the spatial part has an i, e to the power minus 2 n pi i n x, so there is an n x, so that is an oscillatory part. Using the Moivre's theorem, we can write it in terms of cosine's and sine's, so that has an oscillatory part, but the time dependent part has got no oscillatory contribution at all. So, that is going to decay if you look at it after sufficiently long time, so it has an oscillatory part that is captured by trigonometric spatial dependence of the Eigen functions e n x.

But, what is important is that, the higher oscillatory modes even in the space, there are oscillatory modes and those oscillatory modes will have different frequencies, some will have low frequencies, some will have high frequencies. And from the expression for those modes, you can see that, as n increases the frequency is also going to increase. For larger n’s those oscillations are going to have higher frequencies, while the smaller n’s they are going to have lower frequencies.

So, but you can also see that, those higher oscillatory modes are going to decay faster why, because n also appears in the coefficient of the time dependent part. So, as n increases, this thing is going to control the decay, so the higher modes are going to decay faster, higher spatial modes are going to decay faster in time.

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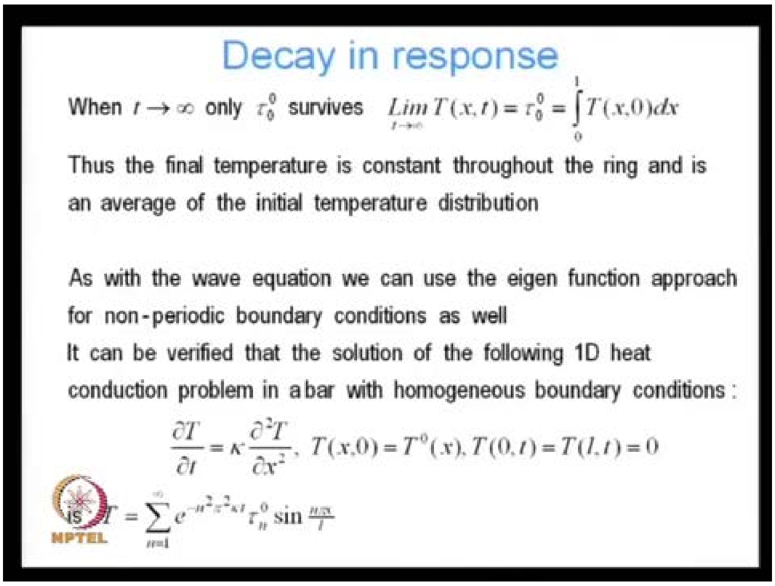

So, when actually t goes to infinity we can see that, if we look at this expression, ((Refer Time: 09:34)) tau n t is equal to sigma n equal to minus infinity to infinity. We can see when t goes to infinity, the only term that is going to survive is for n is equal to 0, all the other terms are going to go to 0, if alpha n is equal to 0, this term is going to go to 1. So, the only the term involving n equal to 0 is going to survive and because of that, in the limit when t goes to infinity, t x t is equal to tau 0 0 and which tau 0 0 we know from our Eigen expansion of the initial conditions, let us go and take a look at that again.

So, tau 0 0 is equal to e 0 x t x 0, e 0 x comes out to be 1, so it is basically integral from 0 to 1 of t x 0 d x, so that is going to be this. So, the final temperature at infinite time at a very long time after applying my initial conditions is telling me that, it is nothing but an average of the initial temperature, it is an average of that initial temperature distribution over my domain of interest. So, that is for the periodic problem with periodic boundary conditions, we can use the Eigen function approach for non periodic boundary conditions as well.

I am not going to solve that problem, because we have already done enough with Eigen functions that, you can be verified that the solution of the following 1 D heat conduction problem in a bar with homogeneous boundary conditions. So now, we can see the boundary conditions are no longer periodic, they are homogeneous, so t at 0 is equal to t at l is equal to 0. So, the above 0 at the two ends, the temperature at the two ends is fixed and it is at 0 for all times and I have an initial condition, which is t x 0 is equal to t 0 of x.

So, if I solve that using the method of Eigen functions, I am going to get the solution like this. Again you can see that, even in this case, even for the case of non periodic boundary conditions, the time dependent part decays exponentially, while this spatial part has a harmonic nature, this spatial part of the solution has a harmonic nature. So, the fact that, it is decaying exponentially, the time dependent part is decaying exponentially has got nothing to do with the boundary conditions, it has got absolutely everything to do with the fundamental nature of the diffusion problem.

And as time goes, it tries to reach steady state and steady state means that, all those transients, all those initial conditions, they decay and things with some sort of an average equally brighted state.

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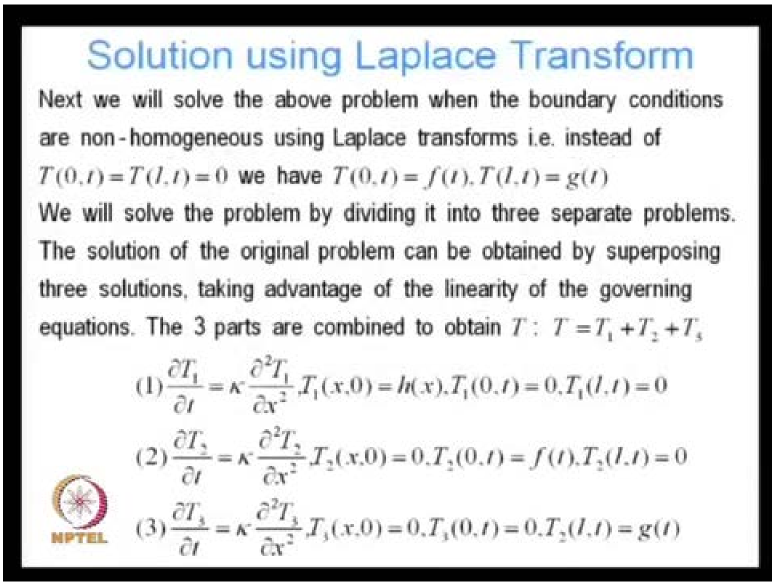

So, next we will solve the same problem, but now with boundary conditions, non homogeneous boundary, so I am going to solve the same problem. But now, I am going to assume that, I have non homogeneous boundary conditions and I am going to solve that using Laplace transforms. And the non homogeneous boundary conditions that I am going to assume is that, at the end x is equal is equal to 0, I have a functional dependence on time and known function of time, the temperature has a known function of time.

At the end l, I know the temperature as another known function of time given by g of t and we will divide, we will solve this problem by dividing it into three separate problems and we will use the principal of linearity, linear super position. So, the solution of the original problem can be obtained by superposing three problems, so one problem I am going to have initial conditions, the first one del T 1 by del t is equal to k del 2 T 1 del x squared.

So, in that one, I am going to have homogeneous boundary conditions, that I will have non homogeneous initial conditions. You can see this T 1 x 0 is equal to, h of x is my non homogeneous initial condition and then, I have homogeneous boundary conditions, which tell me that, at end 0 and at end l, the temperature is always 0 at all time, so that is that first one. Then, I have another one, del T 2 del t is equal to kappa del 2 squared T 2 del x squared and in this case, I have homogeneous initial conditions, but I have homogeneous boundary conditions at one end, but I have non homogeneous boundary conditions at the other end.

So, in this case, I have homogeneous boundary condition at the end l, but I have non homogeneous boundary condition at the end 0 and the third one is just the sort of the conjugate of that. So, in this case, I have homogeneous boundary conditions at end 0, but I have non homogeneous boundary conditions at end l and both 2 and 3 have homogeneous initial conditions. So, while superpose the solution of these three, I am going to get the solution of my original problem.

And you can see, if I add all those boundary conditions and initial conditions together, they are exactly the boundary conditions and initial conditions of my original problem. Now, the first problem I am not going to solve, because I have already solved this one, I know the solution, the second problem and third problem are near identical. So, if I can solve problem 2, I can solve problem 3, because they are just the mirror image of the other. So, if I can solve problem 2, I can solve problem 3, so I am going to focus on the solution of problem 2.

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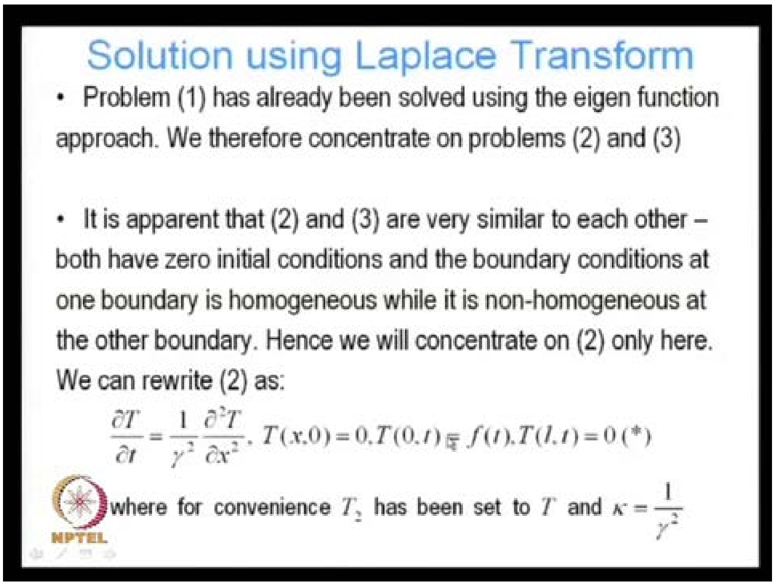

So, you will therefore, concentrate on problem 2 and we can rewrite problem 2, in order to do that, we are going to rewrite it to simplify it somewhat for the solution purposes. So, I am going to rewrite it, I am going to get rid of the two, because now I am only interested in problem 2, so there is no need on keeping that two, that subscript 2 carrying that around. So, I am going to remove that subscript 2 and I am going to replace kappa by 1 by gamma squared. And then, I have these boundary conditions and initial conditions, so T the homogeneous initial conditions. Non homogeneous boundary condition at l, 0 homogeneous boundary condition at end l, I am going to solve that problem.

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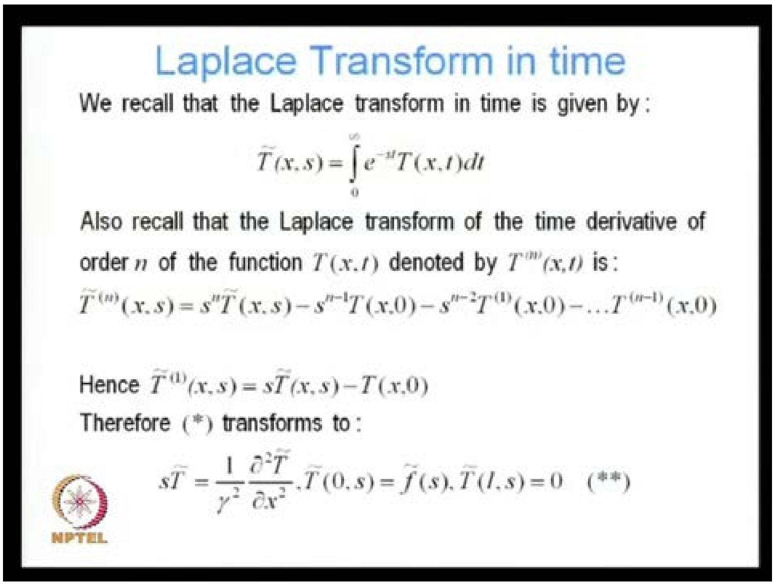

Before doing that, let us go back and recall that, what is the Laplace transform, so Laplace transform of a function T, which is a function of x and t. If I do Laplace transform in the time domain, I get something like this, T tilde of x s is equal to integral from 0 to infinity, e to the power minus s t T x t d t. So, you can see here, the variable of integration is t, so after integration t vanishes and you have a function in s and x nothing is changing, x just goes along for the ride, because you are not integrating with respect to x at all.

So, that is the definition of the Laplace transform, in case you do not remember, so with also we have expressions for the derivative in terms of Laplace transform. If I have d 2, if I have the second derivative of function with respect to time, I have the first derivative of function with respect to time or for that, might I have the n th derivate of a function with respect to time. I can do the Laplace transform of those derivatives and there is a simple rule, which tells me how I can find the Laplace transform of the n th derivative of a function.

So, if I have T n which means, d n d t n of t of the function t then, if I do the Laplace transform of that, I can write that in terms of the Laplace transform of my original function T tilde x s plus the initial condition. So, the Laplace transform of the derivative of T, the n th derivative of T can be written in terms as a polynomial in s and it also involves the Laplace transform of my original function, which is T tilde x s and these quantities T x 0, T 1 x 0, T n minus 1 x 0 and so on.

So, my derivatives, it involves my derivatives and those derivatives are evaluated at the at time t is equal to 0, so they all involved the initial conditions. Therefore, if we are interested in finding the Laplace transform of the first derivative then, n becomes 1. So, in that case, I get T tilde 1 x of s that is, the Laplace transform of d T d t, d capital T with respect to small t, derivative of capital T with respect to small t.

If I want to find the Laplace transform of that then, I get s to the power 1 T tilde of x s, which is nothing but the Laplace transform of T itself even that expression and the value of that function at the initial time at t is equal to 0. So, this is my Laplace transform of the first derivative of T capital T with respect to time and therefore, if we use this in that equation, so the left hand side involves the first derivative of T with respect to time. So, I know how that is going to transform as, so that is going to transform as this and so, if I substitute that, I get an equation like this.

So, the dependence with respect to x remains just the same, because there is no transformation in the spatial domain up till now at least, so nothing changes and then, I have the transforms of my boundary conditions. So, what were my boundary conditions, my boundary condition was T 0 t is equal to f of t and T l t is equal to 0, so these were my boundary conditions. So, I do that, I will have Laplace transform of the boundary condition and I have presented the Laplace transform of f of t as f tilde s and in this case of course, is 0, so it nothing changes.

You can see that, Laplace transform of 0 is not going to give you anything, that is still going to be 0, just from this expression you can see. So, this is now my transformed problem and you can see that, this is a second order ordinary differential equation. So, it is now no longer a partial differential equation, it is an ordinary differential equation in the spatial variable x, because we have got rid of all those derivatives with respect to time.

When we do the Laplace transform, we remove those derivatives and replace them by these algebraic expressions. So, we have got rid of the derivatives with respect to time, but since we have taken transform only in the time domain, so those derivatives with respect to x remained and I get an ordinary differential equation in x. The second order differential equation in x subject to two boundary conditions at 0 and l, if I want, I can integrate that twice, and find my T tilde in terms of x. I can do that and then, if I want, I can do an inverse Laplace transform and get back my time dependence, so that is one we have doing it.

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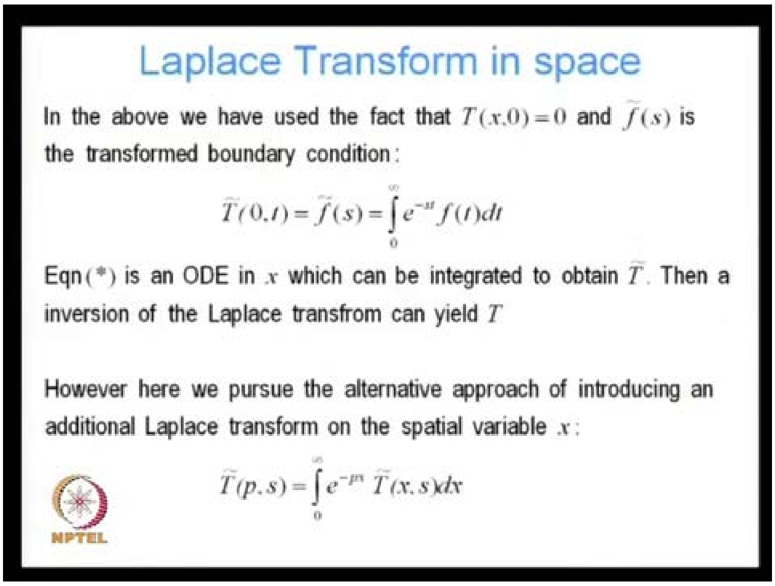

So, it is an ODE in x, which can be integrated to obtain T tilde then, I can do an inverse Laplace transform to get my final solution T. However, here we are going to have pursue an alternative approach, which will involve doing an additional Laplace transform, this time in the spatial domain, this time with respect to x. So, that is just identical to that previous expression that we saw, which is the Laplace transform, in the time domain, so Laplace transform in the spatial domain is very similar.

So, it is very similar except now the integration is performed with respect to this spatial variable x, instead of time. And once you perform the integration, the x goes away, so you have p left and you have s left, which was altered from the Laplace transform in the time domain.

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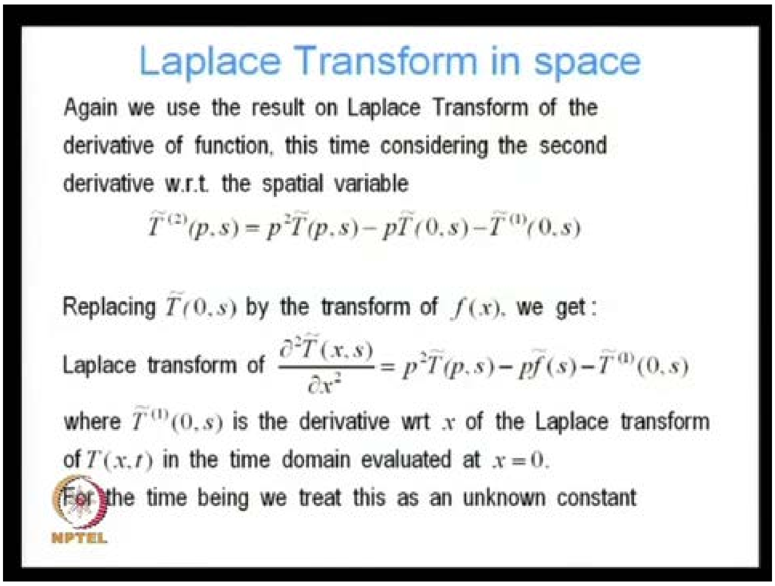

So, again we use the result on the derivatives, because we have now in this expression ((Refer Time: 24:57)) we have a second derivative in terms of x. So, if we want to find the Laplace transform of that second derivative, we again use this expression, but now our variable of interest is x, now t ((Refer Time: 25:15)). So, if we use exactly that same expression for the Laplace transform of the derivative, we can see that, the Laplace transform of the second derivative with respect to x.

If I do that using that expression, I get something like this p squared T tilde p s minus p T tilde 0 s minus T tilde 1 0 s. So, this T tilde 1 is now the derivative with respect to space, it is with respect to x, so and again we replace T tilde 0 s by the transform of f of x. We already got that before, T tilde 0 s is equal to f of s f tilde of s we already got that, so we do that and then, we get Laplace transform of del 2 del x squared T tilde at x s to be this p squared T tilde of p s minus p T tilde 0 s we have just replaced by that by the transform of the boundary condition minus T tilde of 1 0 s.

This is something which we do not know, because in my neither did it appear in my boundary conditions and in my boundary condition did not tell me anything about the first derivative. It just told me about the function value at 0 and l, did not tell me anything about first derivative, so I do not know that value. So, it is the derivative with respect to x of the Laplace transform of T x t in the time domain evaluated at x is equal to 0, so I do not know anything about that. For the time being, we will treat it as an unknown, let us let us carry it along and treat it as an unknown and then, using this expression and substituting it in my equation here, I get the following equation.

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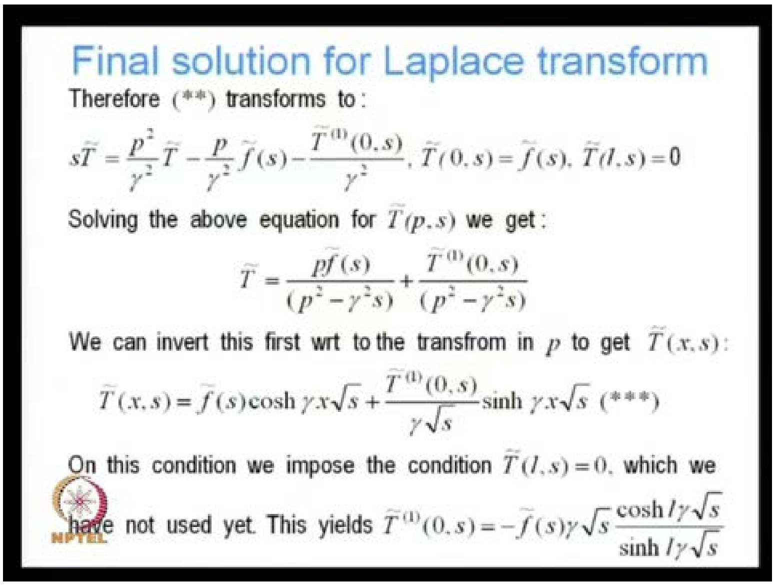

I get the following equation, s T tilde is equal to p squared pi gamma squared T tilde minus p by gamma squared f tilde s minus this, this I already know and T tilde l s is equal to 0. So now, again this is now a totally algebraic equation, so I have got rid of all my derivatives with respect to space, I have got rid of my derivatives with respect to time. So, this is a purely algebraic equation and I solve for T tilde appears in the transform domain, in the transform time and spatial domain.

I can solve for T tilde and get an expression for T tilde, however I still do not know this quantity. So, we can first, so once I know T tilde then, I have to do the inverse transform, in this case I have to do two inverse transforms with respect to space as well as with respect to time. So, in this case, I prove the inverse transform with respect space first and I get back an expression T tilde x of s, my p vanishes and I get back T tilde x of s, which comes out to be this.

I have not talk too much about, how to do the inverse transforms, that is going to take too long, but you can look up in any good book on engineering mathematics and the inverse Laplace transform, it is very similar to. So, this is the after doing, you can find the inverse transform, so on this after doing the inverse transform in the spatial domain, I get this and on this, I impose the condition that, T tilde l s is equal to 0, this condition, I impose that condition.

So, then using that, I find out this value, T tilde one 0 s, I find out that value and that value comes out to be something like this. So now, I know my T tilde x s totally, I know this value also, which I have evaluated by imposing that, that transform of the boundary condition of the spatial boundary condition at end l and so now, I know my T tilde entirely. So, all I need to find my T, is to do a reverse transform, is to do an inverse transform to get back my time dependence.

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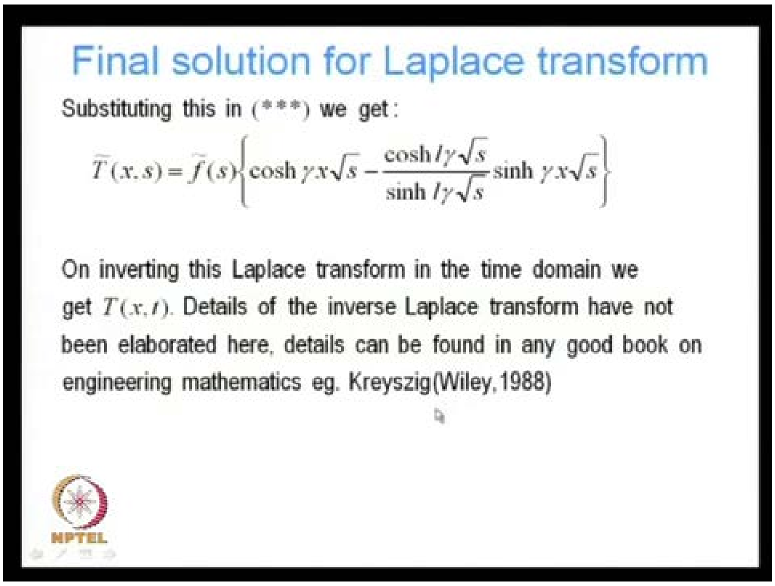

And if I do that, I finally get T tilde x s is equal to something like this, this is not I have not yet performed the transform, I just substituted that or I am just saying that, I have substituted that here. So, I get my expression for T tilde x s and that comes out as this, so and I do the inverse transform in the time domain to get T of x of t, T is a function of x and t. I have not talked about details of the inverse Laplace transform, here is a very good reference, where you can find all you need to know about it.

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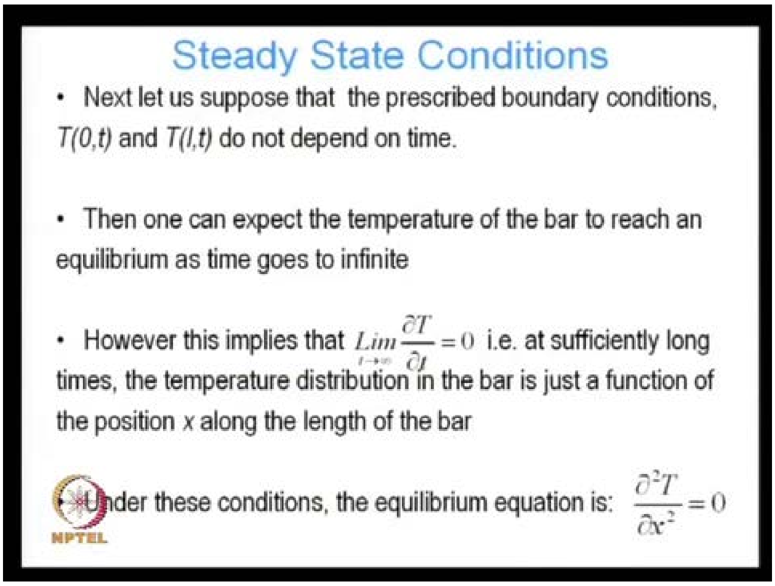

So, we briefly talked about the fact that, the solution to the diffusion equation has a time dependent part, which is decaying exponentially with time. So, if we look at the solution after sufficient till long time has elapsed then, we can expect the temperature of the bar to reach an equilibrium solution and that equilibrium solution is known as the steady state solution. That is the steady state solution and basically the steady state solution says that, the solution is no longer varying with time.

What does that mean, that means, the partial derivative of T, my temperature with respect to time in the limit t goes, small t meaning, time goes to infinity must be equal to 0. And in that case, my parabolic equation becomes an elliptic equation why is that, because the left had side become 0. Let us go back and look at my original equation, unfortunately I do not have that, but I must have it somewhere ((Refer Time: 32:26)), for instance let us look at this.

So, when this thing goes to 0, so I have only this, the left hand side left and that is like a Laplacian, Laplacian in one variable in this case, but that is like Laplacian. So, it is the elliptic equations, the third canonical form, which we have not yet talked about in detail, so that is the elliptic equation, Laplace’s equation. So, in this case, the equilibrium equation becomes this.

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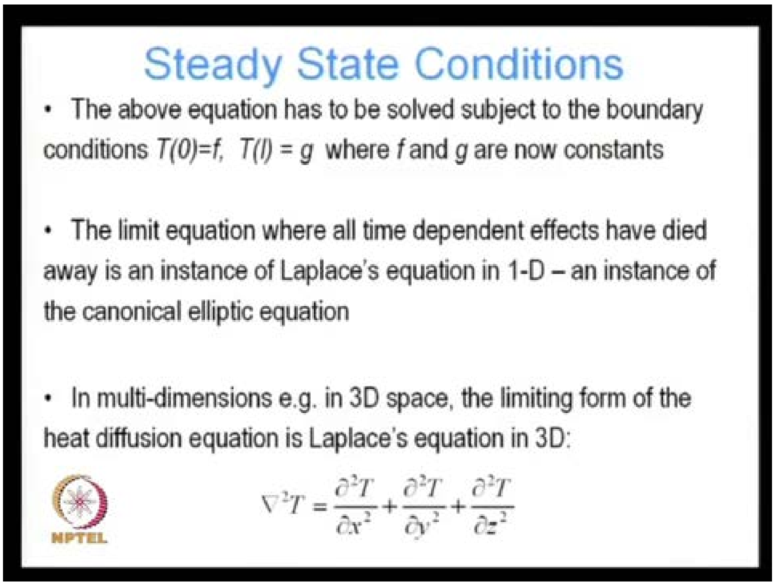

So, again this equation I can solve subject to the boundary condition, so I do not need any more initial conditions, because my time problem is looking at a solution after a long time. So, whatever effect the initial conditions had, had now totally died away, so now, I can solve that problem. So, that is often times like for instance, for the parabolic equation, if somebody is not interested in the transient, it often happens. If you are interested in the transient solution, we are just interest in the steady state solution of our problem.

We do not care how the temperatures are varying when just after I have applied the initial conditions. I am just interested in the temperature distribution of my domain after everything has reach steady state. So, in that case, I do not even solve my transient problem, I do not solve my parabolic equation, I straight away threw away my left hand side, I threw away my time dependence solution time, dependent part. So, make the derivative with respect to time equal to 0 and solve Laplace's equation after subject taken to those boundary conditions, so that is the solution of the steady state problem.

So, the limit equation where all time dependent effects have died away is an instance of Laplace's equation in 1 D and instance of the canonical elliptic equation in multiple dimensions functions in 3 D space. The limiting form of the heat diffusion equation is nothing but Laplace's equation 3 D, which is given by that.

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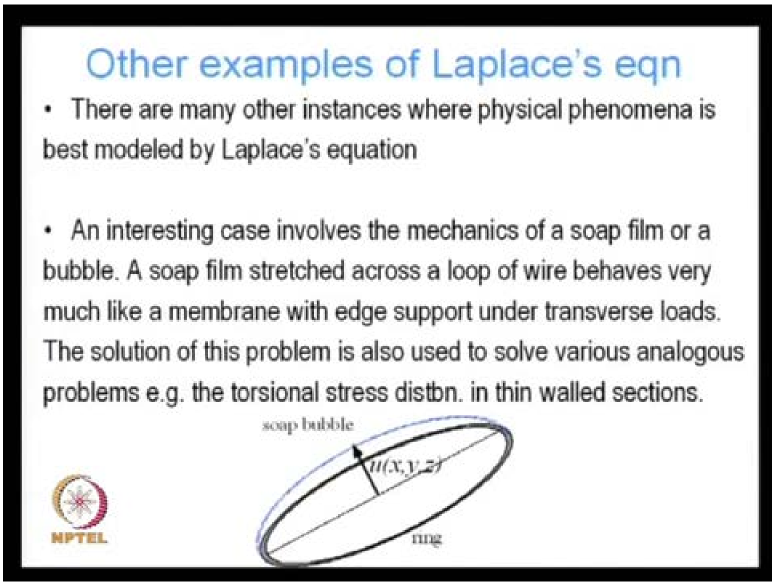

So, there are many other instances, so this is one instance, where we saw that, the parabolic equation becomes the elliptic equation after if you look at it at a sufficiently long time after I specified my initial conditions. But, there are the situations, where the Laplace's equation arises naturally from the physical nature of the problem, from the mechanics of the model, from the physics of the model. So, an interesting case involves the mechanics of a soap film or a bubble, you can think of a bubble, when somebody blows a bubble, there is a ring and somebody blows a soap bubble.

So, that thing expands and so, the mechanics of that is governed by Laplace's equation and actually this analogous problem is used in a number of places, basically it is a problem of membrane. The soap film or the bubble is basically a membrane problem, there is a very thin membrane subject to some edge conditions and then, I am applying transverse loads. So, here is my ring, here is my little ring right and then, initially if everything was flat, I have a really thin film connected to the edges of the ring.

And then, I applied a transverse load, so the little thin film bulges out, which is represented by this blue line out here. So, this blue line here, this blue dotted line represents the deformed shape of my soap bubble. But, the solution of this problem is useful in a number of places, it can be used to find the solution, it can be used to solve the torsional problem.

The thin walled sections if you remember, there are something called a candle wax equations and those equations can be solved using this analogous, this soap bubble or thin film analogy. So, if we can solve that problem, it can be solved, the solution can be used in a number of situations. So, for instance, the torsional stress distribution in thin wall sections.

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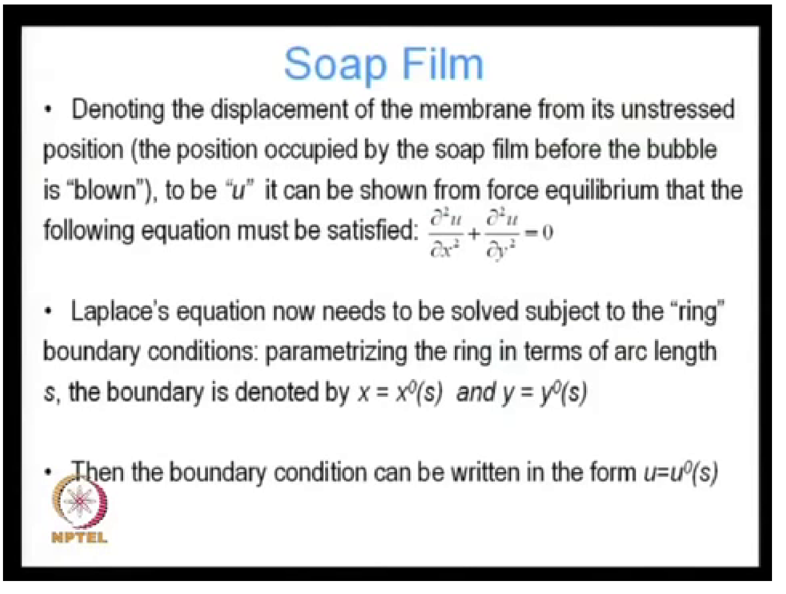

So, how do we solve that problem, we denote the displacement of the membrane from it is unstressed position as u. And then, we can show from force equilibrium, I have not gone into that, but if you look at any advance strength of materials book, you will find how they got that equation. So, the equation from the force equilibrium, we can show that, the displacement of my soap film or of my membrane must satisfy this condition, which is nothing but the Laplacian into D and which is equal to 0, so Laplace's equation into D.

So now, we need to solve Laplace's equation subject to the ring boundary conditions, which we can do like we parameterized the ring by the arc length along the ring. And then, we prescribe the displacement along the arc length, u is equal to u 0 of s and we solve that problem. So that, soap film problem can be solved using, that is nothing but the solution is nothing but the solution of Laplace equation subject to the appropriate boundary conditions.

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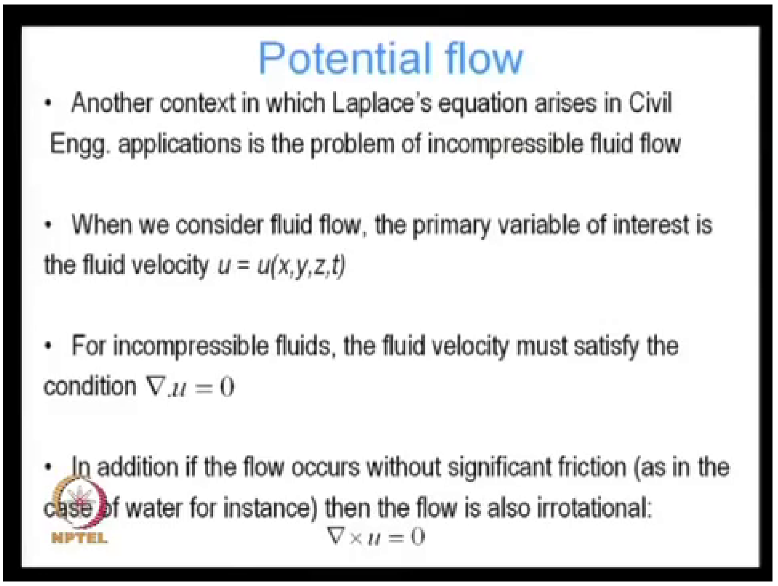

Another problem which arises in civil engineering applications is of potential flow, because suppose you consider the flow of an incompressible fluid, for instance what are people in hydraulics interested in fluid with the how water flows and solving the equations of motion. So, here the problem becomes complicated, because it is a constraint problem and what is the constraint, the constraint is that, the divergence of the fluid velocity.

Here the primary variable is the velocity of the fluid and the fluid is something like water, it is incompressible, that fluid is not compressible at all, water has a very, very huge bulk modulus, so you cannot compress water at all. So, on that condition, we can show I mean, some complicated mechanics, not really complicated, but some basic continuum mechanics one can show that, that incompressibility constraint can be represented, can be enforced by imposing this condition on the velocity of the fluid.

And what is that condition, the condition is that, the divergence of the velocity must be equal to 0. Now, in addition, if I assume that the fluid, suppose I am considering fluid through a pipe and suppose I assume that, there is not much friction, then, I can also assume that, the flow is irrotational. And what does that mean that means, that the curl of the velocity this equal to 0, this is the curl, is also the cross product of the gradient operator with the velocity.

So, the flow is irrotational which means that, the curl of the velocity must be equal to 0 and if the curl of a vector, the velocity is a vector, if the curl of a vector is 0 then, we know that, vector can be obtained by the gradient of a scalar function.

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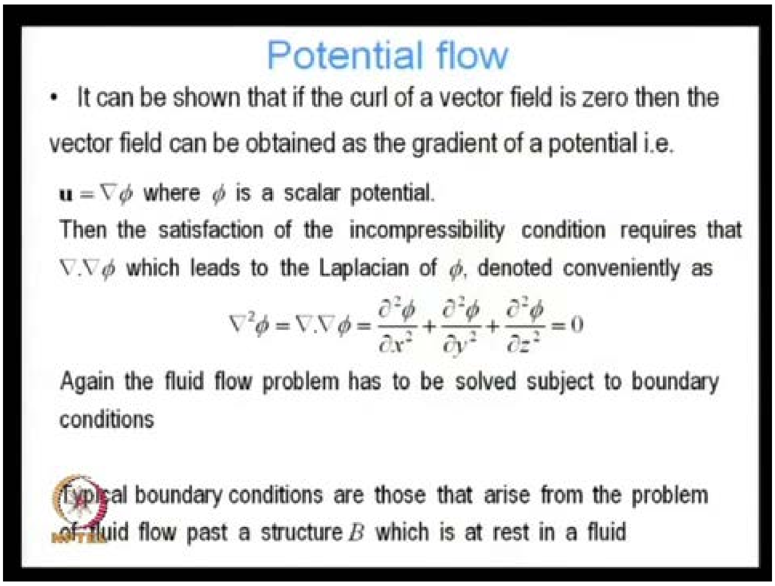

So, that is, it can be shown that, if the curl of a vector is 0 then, the vector field can be obtained as the gradient of a scalar potential. In that case, I can write, I can get my velocities, what I am trying to say is that, there is a potential, there is a scalar potential, whose gradient if I take, I am going to get the velocities. So, u is equal to grad of phi, where phi is a scalar potential then, the satisfaction of the incompressibility condition. What is the incompressibility condition, it is divergence of u must be equal to 0.

I am saying that, since u is irrotational, u has to be equal to the gradient of a potential, you can always be obtained as the gradient of a potential. So, what does that mean, divergence of u that means, divergence of the gradient of a potential and I am denoting the potential as phi, so divergence of gradient of phi got to be equal to 0. So, that is what my incompressibility to constrain translates to and divergence of gradient is nothing but the Laplacian.

So, divergence of gradient is nothing but the Laplacian, so what it tells me is the Laplacian of phi, where phi is my potential, is my scalar potential got to be equal to 0. So, instead of solving directly for the velocities in this case, if I can solve this equation for the potential, I can get my velocity just by taking the gradient of that potential. So, that is why is called potential flow, because ultimately I am solving for the potential phi. So, again we solve this subject to boundary conditions, Laplacian of phi equal to 0, we solve that subject to boundary conditions. And typical boundary conditions arise for instance, when you have fluid flow pass solid body, there is a solid body sitting there and suppose some sort of a rock sitting there and there is the fluid flowing pass there stationary rock, fluid flowing pass that stationary rock.

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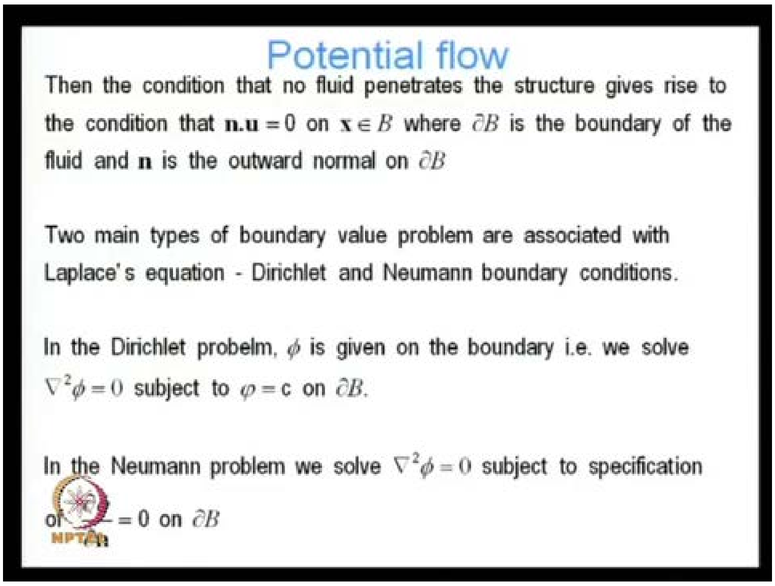

So, in that case, what are my boundary conditions, the boundary conditions is, it is that n dotted with u is equal to 0 for all x belonging to B, this should be x belonging to del B, del B is the boundary of the fluid and n is the outward normal on del B. So, have this little rock sitting there and my air fluid flowing pass that rock and I denote the surface of that rock as del B and on that rock, once the fluid velocities got to be 0, because there is no fluid penetrating that rock.

So, at the boundary, at the interface n dotted with u is equal to 0, so the normal velocity of the fluid is 0 at the interface. So, this is my boundary condition, this is the typical boundary condition, there can be all manner of boundary conditions, but this is typical boundary condition. So, you have that, so what this tells me is that, my fluid cannot penetrate the rock, but the fluid can flow pass the rock So, tangential velocity is not 0, but the normal component of the velocity is 0.

And we are going to look at two main types of boundary conditions for the elliptic equation for the Laplace’s equation and they are known as the, I might have mentioned them earlier in the course of this lecture, but here we are going to talk about them in slightly more detail with Dirichlet and the Neumann boundary condition. In the Dirichlet boundary condition, we specify that the boundary condition is specified directly in terms of the primary variable, in this case our primary variable is phi, so it is prescribed in terms of phi is phi, so phi is equal to some constant on some boundary.

So, somebody tells me solved Laplace’s equation given that, I tell you that, phi has this value on the certain part of the boundary, so that is the Dirichlet problem. The Neumann problem I solve Laplace’s equation subject to the condition that, somebody tells me, I do not know what the actual value of the primary variable is on the boundary, but I can tell you what the derivative is.

In this case of, if somebody tells me, since I am looking at this little problem of fluid flow pass the rock, somebody tells me that, I know the normal derivative of phi. I know del phi del n and it is equal to 0 on the boundary, if somebody tells mean that, that is a Neumann problem, both are well posed problems and I can solve them, I can solve both of them.

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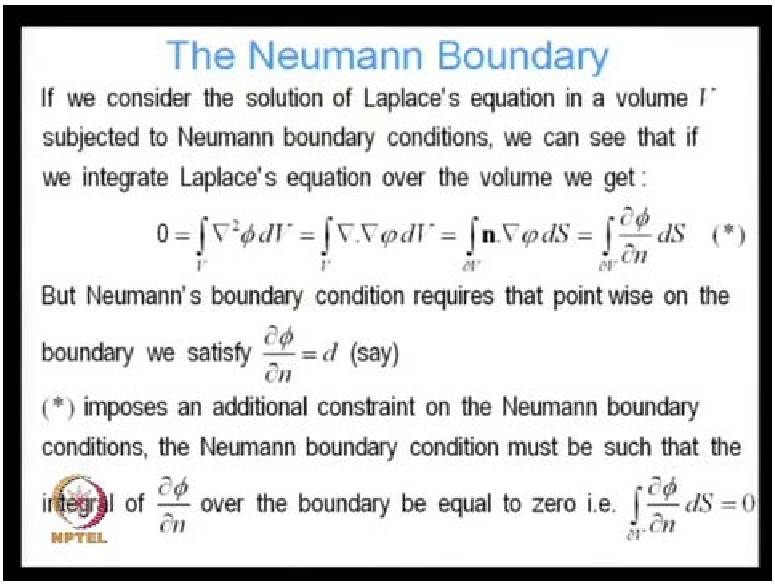

So, if I consider the solution of Laplace’s equation in a volume V, V subject to Neumann boundary conditions, we can see that if we integrate Laplace’s equation over the volume, what do I get. So, this is my equation, which must be satisfy that each point in the domain, in each point in the volume V, but suppose I integrate over the volume V, so again this equation has got to be true, because Laplacian of phi is 0 point wise. So, over if I integrate that over my volume V, that would also better b equal to 0.

So, in Laplacian of phi, I know that is equal to divergence of the gradient of phi and then, I use my divergence theorem. I use my divergence theorem to convert that volume integral to a surface integral, which involves the surface Neumann and the gradient of the phi at the surface. So, it this the dot product of n with the gradient of phi and this is nothing but the directional derivative of phi in the direction n. So, it is del phi del n d s, but suppose, I am solving Laplace’s equation subject to Neumann boundary conditions.

And what do my Neumann boundary conditions tell me, they tell me that I know what del phi del n is at the boundary. Suppose, somebody to has told me that, del phi del n is equal to d at the boundary, but then if I substitute del phi del n here, if I look at del phi del n here, see the del phi del n is appearing on the right hand side and it is appearing with in an integral over my boundary. So, this condition imposes an additional restriction on the Neumann boundary condition, what it tells me that, you cannot specify the Neumann boundary condition for Laplace’s equation in any arbitrary way you like.

The way the boundary condition you specify, the del phi del n value you specify must be such that, if I integrate that over the boundary, I get 0. So, you cannot specify any arbitrary dependence on the boundary, so del phi del n can be a function of s, because this is an integral over s, so del phi del n can vary over the boundary. You can specify that your normal grad, your derivative of find the normal directions is varying across the boundary, but it cannot verify in any arbitrary fashion.

It must satisfy the condition that, if I integrate that derivative over the boundary is got be equal to 0, so that is very important. So, because Laplace’s equation itself imposes this additional constraint on the Neumann boundary condition, the Neumann boundary condition must be such that, the integral of del phi del n over the boundary must be equal to 0, integral of del phi del n del V is equal to 0.

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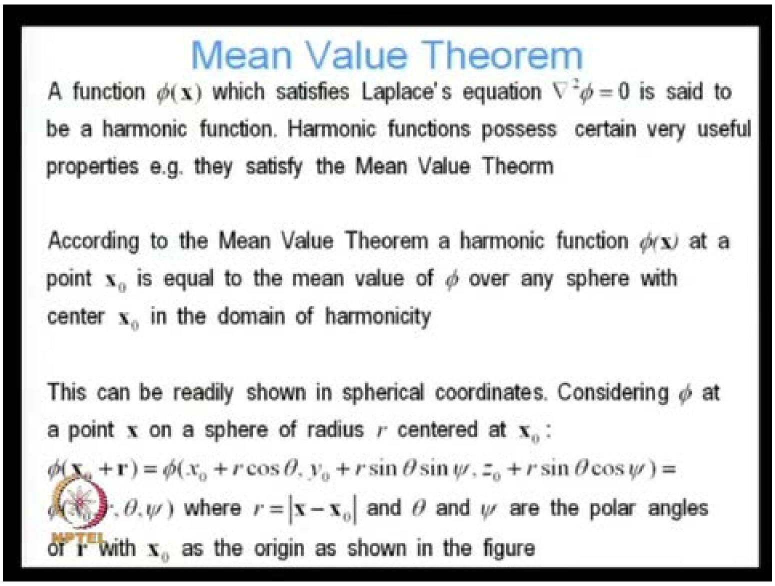

Now, function phi of x, which satisfies Laplace's equation is said to be a harmonic function and harmonic functions are very useful, because they posses certain very characteristic and important and useful properties. And one of those properties is that, they satisfy what is known as the mean value theorem, what is the mean value theorem, the mean value theorem tells me that, if I have a harmonic function at then, I can find the value of that harmonic function at any point x 0.

If I integrate that harmonic function over a sphere of any arbitrary radius centered around x 0. So, this is very important and it is it is very, very powerful, because what it is telling me is that, I want to find the value of that function at point. To find the value of that function at that point, all I need to do is to integrate, if I have the function specified on a boundary, I need to integrate that over the boundary and that boundary need not be at, it can be any sphere.

So, over any sphere with center x 0 in the domain of harmonicity, so find the value of phi at any point x 0, all I need to do is to integrate phi over a sphere with center at x 0 and that sphere can of any radius. So, if I integrate that function over that sphere, that is going to give me the solution over, I have to normalize of course. Because, this is the mean value, when I divided by my surface area of the sphere, so integrate it over the sphere, divided by the surface area of the sphere, I get the value of the function at the center of the sphere.

So, we are going to show why the mean value theorem works in next class and we are going to show that, in spherical coordinates, I just introduce spherical coordinates and then, end this lecture.

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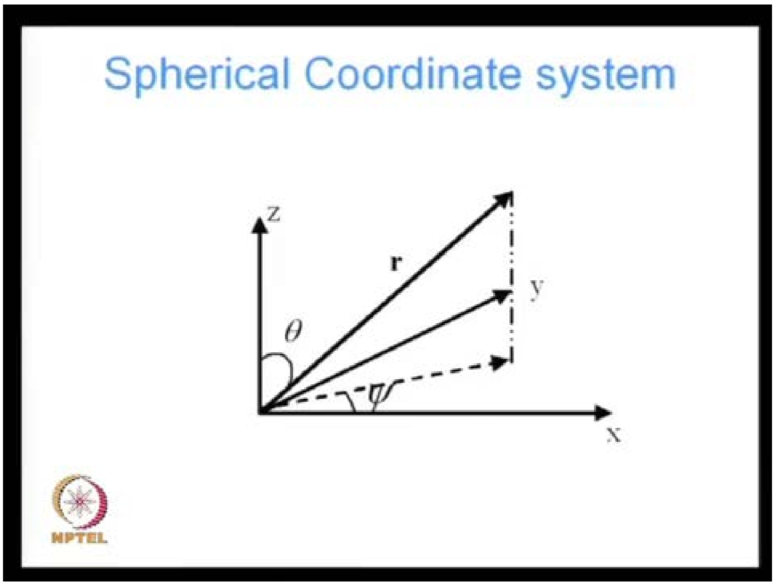

So, spherical coordinates, that is just a coordinates system which says that, if you have any point, so the spherical coordinates of a point are give in terms of the radial distance from the origin. And two angles, where one angle theta is the angle of this vector with respect to the Z axis and then, angle psi, which is the angle between the projection of this vector on the X Y plane and the X axis. So now, instead of X Y Z, we are going to look at r theta and psi, why we considering spherical equation, because we are going to look at the mean value theorem.

The mean value theorem tells me that, it tells me that, how can I find the value of a function at a point by evaluating the value of the function on the surface of any sphere of any arbitrary radius surrounding the point. Since this geometry is spherical, so I am going to use the spherical coordinate system.

Thank you.