Numerical Methods in Civil Engineering
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Lecture - 25
Analytical Methods for Elliptic PDE's

In lecture 25 of our series in Numerical Methods in Civil Engineering, we will wind up our discussion of analytical methods for partial differential equations by discussing analytical techniques resolving elliptic partial differential equations.

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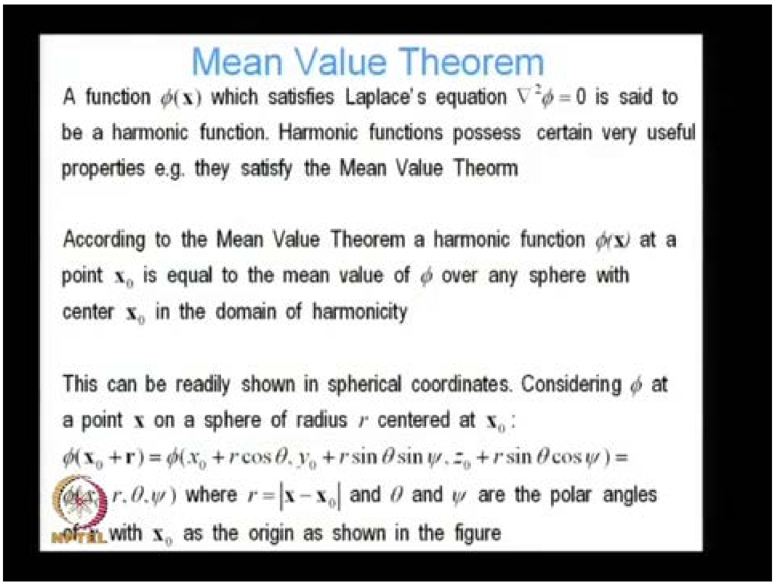

A function phi x which before doing, so I want to talk little bit about a very important result, which is known as the mean value theorem. And which starts with a function phi of x which satisfies Laplace's equation, and for any function phi of x which satisfies Laplace's equation is said to be a harmonic function. And all harmonic functions satisfy what is called the mean value theorem all harmonic functions satisfy the mean value theorem.

So, what does the mean value theorem say, it says that the value of a function at any point x 0 is equal to the integral the first you take the integral of that function value phi over any sphere with any radius centered at x 0, and you divided by the area of this sphere. So, basically to find it is very important because, what it says that if you want to find the function value at any point, what you need to do is you take the value of the function on any arbitrary sphere surrounding that point, you take the value of that function on that sphere.

And you take the integral of that value over that sphere, and then you take the average. So, it integrated and divided by the area of the sphere, and whatever value get is going to be the value of the function at the center of this sphere, so at any point x 0 is equal to the mean value of phi over any sphere with center x 0 in the domain of harmonicity. Domain of harmonicity meaning that we have to make sure that in that sphere laplacian of phi is equal to 0 everywhere.

Because, we are interested in finding phi such that it satisfies Laplace's equation, so this can be readily we want to show this, and we are going to show this in spherical coordinates, since we are looking at is sphere we are integrating over a sphere we need to use spherical coordinates.

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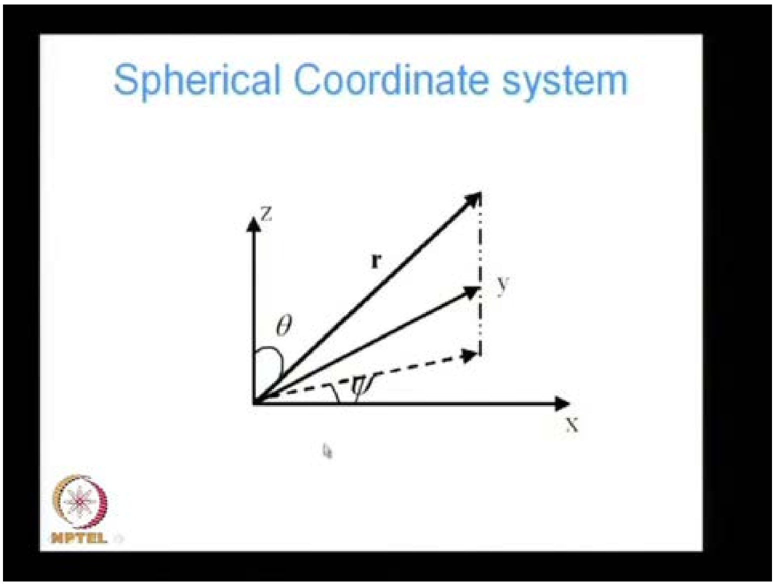

And spherical coordinates it is for just remind you is consist of coordinate system like this, if this is my Cartesian coordinate system x, y and z my any point I can describe the location of any point in the Cartesian system as x, y, z. In this spherical system as r theta and psi, where r is the distance radial distance of that point from the origin, theta is the angle that radial vector makes with the z axis. And psi is the angle which is the projection of the radial vector on the x, y plane makes with the x axis.

So, in this system theta goes from 0 to pi, psi can go from 0 to 2 pi, theta can go from as you can see, r can go from here to here. So, theta can go from 0 to pi where psi can be anywhere between 0 and 2 pi.

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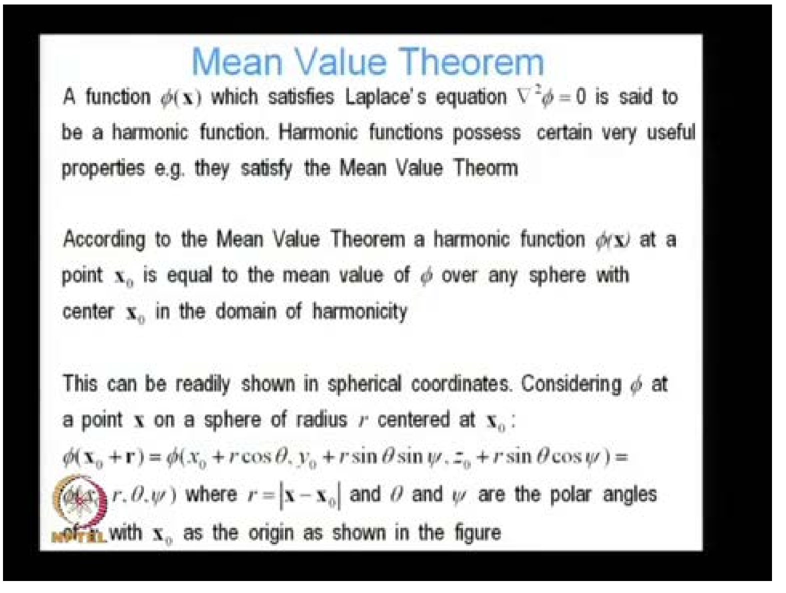

So, we are going to consider spherical coordinates and we are going to consider phi at a point x on a sphere of radius r center at x 0. So, phi at x 0 plus r is equal to it is spherical coordinates phi evaluated at x 0 plus r in turn this spherical coordinates is phi evaluated at x 0 plus r cos theta y 0 plus r sin theta sin psi z 0 plus r sin theta cos psi, where x 0 y 0 and z 0 is the origin of my spherical coordinate system. So, x 0, y 0, z 0 this is my point x 0, y 0, z 0 psi where r is equal to x minus x 0 and theta and psi are the polar angles of r with x 0 as the origin as show in the figure.

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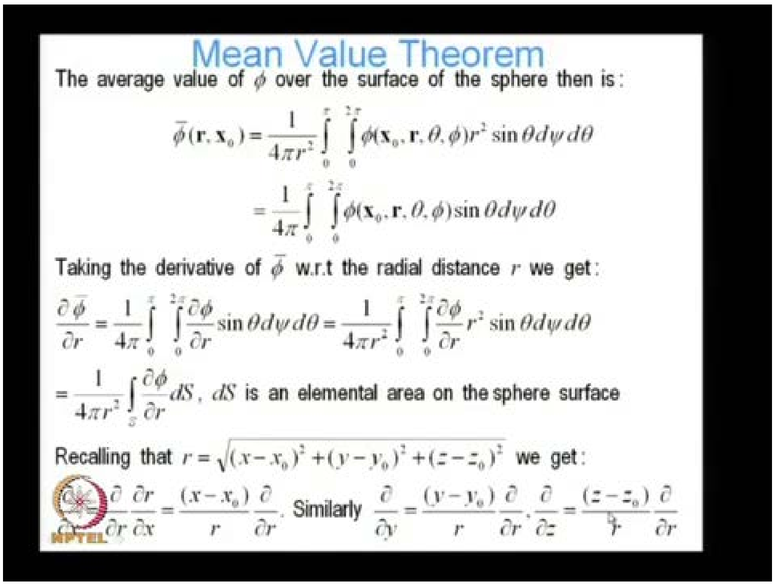

So, what does the mean value theorem say, it says that the average value we have to calculate the average value over that area of that sphere. So, how do we do that, well we take phi bar which is the average value at r x 0, r meaning the radius of the sphere. And evaluating it at a point on the surface of the sphere, so phi is the function of r, r the vector r it is not the spherical coordinate r, it is the vector r and the center of the of the system x 0.

And that is equal to 1 by 4 pi r squared, which is the total area of this sphere here r is now the radius of the sphere. And then I evaluate that function, and this is my infinitesimal area on the surface of the sphere r square sin theta d psi d theta, so this r square r square cancels out and I get something like this. When I take the derivate of phi bar with the radial distance, if I do that I get del phi bar del r which gives me del phi del r again I can take it inside the integral.

Because, the integral does not depend on r, so I take that partial derivative with respect to r inside the integral, I get del phi del r sin theta d psi d theta, which I can write as again I multiply the numerator and the denominator by r squared. So, I get 1 by 4 r squared del phi del r squared sin theta d psi d theta, and this is again my infinitesimal area d S. So, I get 1 by 4 pi r squared del phi del r d S, where d S is an elemental area on the sphere surface.

So, we have got that far, but before going any further let me go take a little side track, and try to talk about the gradient, the gradient operator on the surface of the sphere. So, on the surface of the sphere we know r is equal to x minus x 0 squared plus this, so x is a point on the surface of this sphere x 0, y 0, z 0 is the origin, so r can be written like that. So, I can partial with respect to x is equal to partial with respect to r and partial of r with respect to x.

So, that gives partial of r with respect to partial of with respect to x gives me x minus x 0 by r from this expression from r, if I take partial of r with respect of x I get x minus x 0 by r and I have partial with respect to r. Similarly, I get del del y is equal to y minus y 0 by r del del r del del z is equal to z minus z 0 by r del del r, now what is this x minus x 0 by r it turns out this x minus x 0 by r is the component of the normal at the point x.

So, if I consider the point x on the surface of the sphere, and I consider the normal to that to the sphere at the point x, the components of that normal in the x, y and z directions are given by x minus x 0 by r, y minus y 0 by r, z minus z 0 by r well how is that let us take a quick look.

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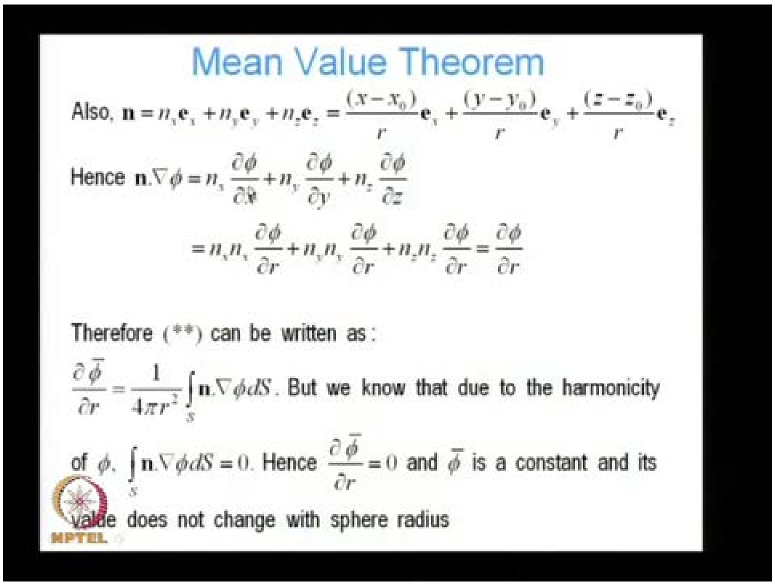

So, suppose my normal is n x e x n y e y n z e z, and I am evaluating the normal at a point x on the surface of this sphere. So, what is the x component of that normal, it is x minus x 0 by r, so the vector is x minus x 0 e x y minus y 0 e y z minus z 0 e z, so the normal is just the x component of the normal is just x minus x 0 divided by the normal that vector r. So, n x is x minus x 0 by r n y is a y minus y 0 by r n z is equal to z minus z 0 by r z 0 by r.

So, n dotted with gradient of phi suppose I have a function phi defined on the surface, and I evaluating it is gradient. The gradient by definition grad phi by definition is del phi del x e x plus del phi del y e y plus del phi del z e z, so n dotted with grad phi is equal to n x del phi del x plus n y del phi del y plus n z del phi del z. But, now what about this, this is del phi del x, we have seen that we can write del del x as x minus x 0 by r del del r.

So, what is this, this we already known is n x, we saw that this is n x, so del del x is n x del del r. So, we have n x del phi del x is equal to n x n x del phi del r plus n y del phi del y is n y n y del phi del r plus n z del phi del z is equal to n z n z del phi del r. So, we take out del phi del r outside, and we have n x n x plus n y n y plus n z n z which gives me 1 and I get del phi del r.

So, what this shows is that if I evaluate the gradient on the surface of the sphere, and take the dot product of the gradient with the normal to the surface of the sphere that gives me how phi is varying with the radial distance it gives me del phi del r. Therefore we can our equation which was I do not know where that is I suspect it is this we I am talking about this. So, del phi bar del r I can write this as del phi bar del r is equal to 1 by 4 pi r squared, go back let us go back again.

So, del phi bar del r equal to 1 by 4 pi r squared del phi del r and that is nothing, but n dotted with grad phi. So, that is equal to n dotted with grad phi d S, let recall in our previous lecture we showed that if function is satisfies if the function phi satisfies Laplace's equation. Then that is very nature the integral of n dotted with grad phi over the surface must be equal to 0, we showed that last time.

So, what does this mean this means that del phi bar del r is must always be equal to 0, and what does that mean. That means, phi bar is a constant, and its value does not change with this sphere radius. So, I am taking radius I am taking I want to evaluate the function at a certain point, so I am evaluating the function on the surface of this sphere, and I am dividing it by the area of this sphere.

So, what I have shown here is that irrespective of the size of the this, this will I mean whatever be the sphere may be of infinite size, sphere may be of finite size does not matter phi bar is always going to be the same.

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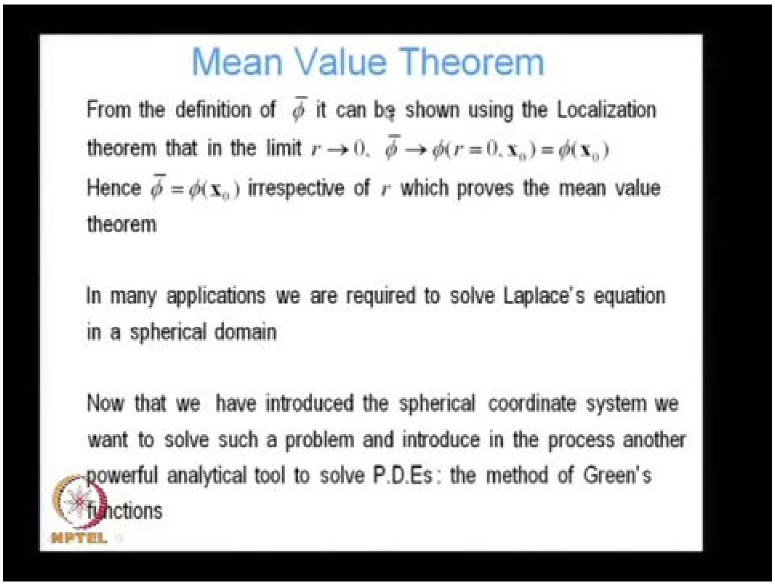

So, from the definition of phi bar it can be shown using something known as the localization theorem, basically it is intuitive that if you want to show it rigorously, we have to use something called the localization theorem. But, since it is intuitive I did not go into that, but what it basically says is that in the limit as r goes to 0 phi bar must be equal to phi of x 0, why is that well the size of this whatever be the size of this sphere my phi bar is not changing.

So, I make my sphere smaller and smaller, smaller and smaller and the limit that r goes to 0 phi bar it has to be equal to the value of the function at x 0. So, therefore, phi bar is equal to phi of x 0 irrespective of r which proves the mean value theorem, so that is important, in many this why is it important well because, in many applications we are required to solve Laplace's equation, in a spherical domain particularly in fluid mechanics, in a acoustics in many, many applications even in heat transfer.

Suppose we people are interested in solving in a acoustic problem, if they interested mid problems of scatter of scatter scattering problems. So, they have or for instance problems of bubble formation, when you have bubble forming in a infinite fluid and the bubble is going rising up found somewhere in the bottom, and if the body of water and it rises up.

So, how is that how what is the mechanics behind that when is the bubble going to burst things like that.

So, in all those situations it is necessary to describe the physics using spherical coordinates, and we need to solve Laplace's equation well why would you need to solve Laplace's equation well again because, we are considering in most cases, we are considering irrotational in viscid flow, so potential flow that is why we need to solve Laplace's equation. So now that we have introduced this spherical coordinate system we want to solve such a problem.

And introduce in the process another powerful tool to solve partial differential equations, the method of green's functions. So, up till now we have looked at two methods, we have looked at the method of Eigen functions, we have looked at the method of using transforms using Laplace's transform, now we want to talk about the method of green's functions.

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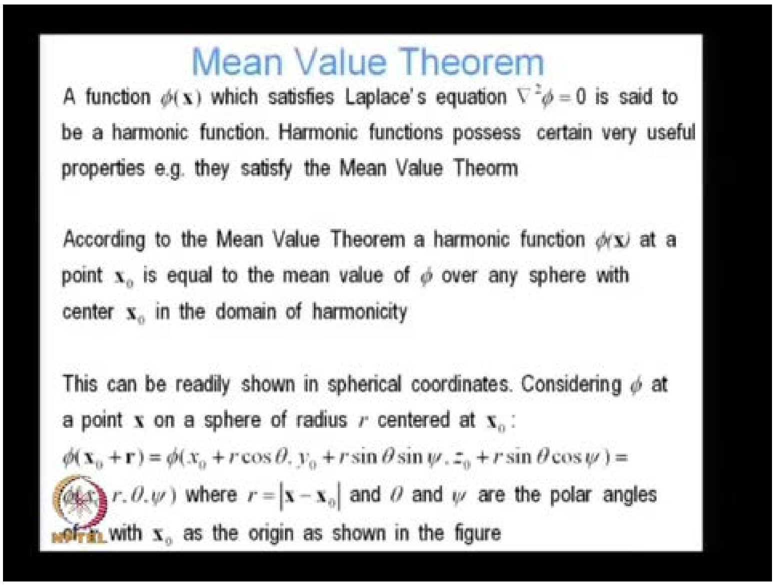

So, the laplacian operate in spherical coordinates is expressed in terms of variables r theta and psi as this 1 by r square del del r this. So, in terms of partials with respect to psi, and partials with respect to theta, now consider a function phi hat which I defined to be equal to minus 1 by 4 pi r. So, r being the radial distance of that point from the origin, so I am defining a function phi hat, which is at any point I am going to define that function, and the value of that function is going to be 1 by 4 pi times the radial distance of that point from the origin.

So, and it is clear that if r is not equal to 0 laplacian of phi hat is always going to be 0, so phi hat by definition is always going to satisfy Laplace's equation unless r is equal to 0. You can see that these are partials with respect to psi and theta there is no dependence of phi hat and psi and theta, so the only part that want to be is this del del r of minus 1 by 4 pi r, r which going to give me 1 by r squared term r squared r squared cancels out I have a constant I have del del r of that, that is going to give me 0.

So, by construction this function is going to always satisfy Laplace's equation unless we are at the origin. And this solution of Laplace's equation in spherical coordinates is known as the fundamental solution, next we consider a function f of x, where f of x is a smooth, but otherwise arbitrary function that is derivatives are continuous. But, it is value is totally arbitrary, and it is defined in a domain omega centered about the origin, such that mod of x less than epsilon for all x belong to omega. So, it is centered around the origin, which has a size which is bounded by epsilon it is radius bounded by epsilon.

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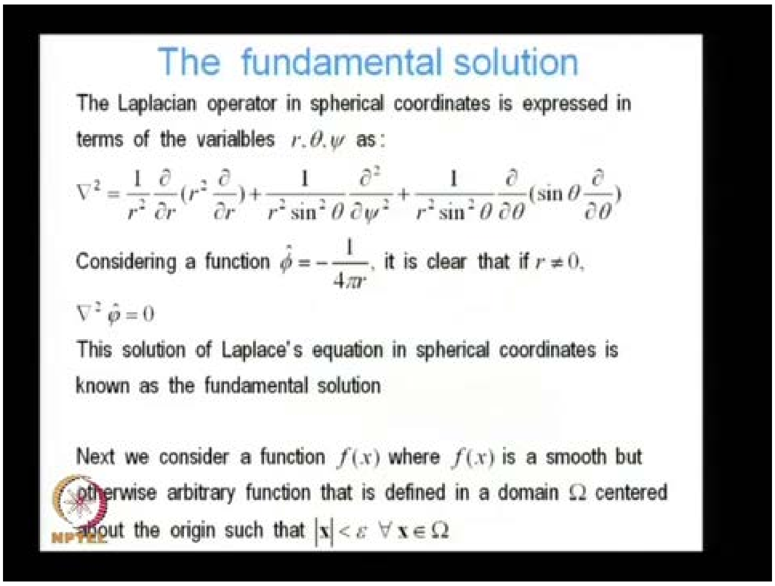

So, let us take the integral of f of x with laplacian of phi hat, so we know that laplacian of phi hat is always equal to 0 except at the origin. So, let us rewrite this f of x as f of x minus f of 0 laplacian of phi hat plus f of 0 laplacian of phi hat just I have added and subtracted f of 0.

Then we look at this part of this equation, and if we look at this part and then we use divergence theorem I know that the laplacian is nothing, but the divergence of the gradient. So, f of 0 is integral of omega, divergence of gradient of phi hat, so this can be written as divergence of gradient of minus 1 by 4 pi r because, that is the form of phi hat, we look just said that phi hat has the form minus 4 1 by 4 pi r, and that is equal to f 0 n dotted. So, then I use the divergence theorem I bring this, this is divergence of this vector which is gradient of the scalar.

So, if I use the divergence theorem I can write it as n dotted with gradient of minus 1 by 4 pi r d S. Now, recall that we showed just little while earlier that n dotted with the gradient of any function is nothing, but del del r of that function on the surface of the sphere, we just showed that n dotted with the gradient is the rate of change of that function with respect to r. So, we can write f 0 divergence of gradient of minus 1 by 4 pi r d omega as f 0 integral of over del omega that is on the boundary.

And now we are replacing n dotted with grad of this by del del r of minus 1 by 4 pi r, so that is going to give me 1 by 4 pi r squared. And then again I replaced d S by r squared sin theta d theta d psi, the incremental in infinitesimal area I replaced in terms of r theta and psi r squared r squared cancels out, I pull out my 4 pi r outside. And then I have this integral, and if we integrate this, this exactly comes out to be equal to 4 pi, so that 4 pi 4 pi cancels out and I am left with f of 0. So, this term is actually f of 0 this actually contributes just one, so this term is f of 0.

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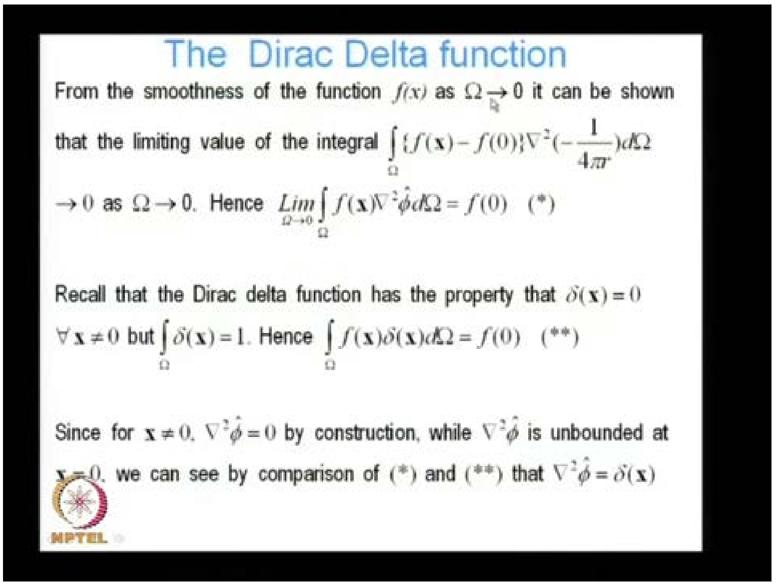

And we can show that as omega tends to 0 in the limiting value, since this function is smooth as omega tends to 0, this function goes to 0. You can see why that is happening because, as omega becomes smaller and smaller f x becomes closer and closer to f of 0, and it go and since that is smooth it does, so in a continuous fashion. So, in as omega goes to 0, this integral also goes to 0, so we can write in the limit that omega tends to 0 this integral actually goes to f 0.

So, this integral goes to f 0, so in the limit omega goes to 0, integral of f of x laplacian of pi hat d omega is equal to f 0. Now, let us take a step back, and recall that there is something called the Dirac delta function, and the Dirac delta function has the property that it is 0 everywhere in my domain except at x is equal to 0. And moreover if I integrate del x Dirac the Dirac delta function over my domain I am always going to get 1.

So, this is different from the chronicle delta, which this is the Dirac delta function, so in this case delta x is equal to 0 everywhere except at x is equal to 0. But, the integral of delta x over my domain is always equal to 1, hence if I have a function f of x and I have delta x, if I take the product and I integrate it over my domain, this is going to give me this product is going to be 0 everywhere, except at the origin except at the origin it is value is going to be undefined the value of delta x is undefined.

The only thing that I know is this if I integrate delta x over the domain I am going to get 1, so this is, but since delta x is 0 everywhere, the value of x does at any other than 0 is not going to count because, this is going to give me 0. But, over that in, but at x is equal to 0 it has the value f 0, and then it is to going to skill the value of the Dirac delta function, and what we are going to get is something called something like f 0.

So, if I compare these two equations, if I compare this equation, and compare this equation. We see that laplacian of phi hat by construction, since laplacian of phi hat is equal to 0 everywhere by construction except at the origin it is very similar to the Dirac delta function. And in fact, it is exactly equal to the Dirac delta function because, it acts in exactly the same way I know that laplacian of phi hat is equal to 0 everywhere in my domain, except at the origin, and in the origin it is undefined because, as you can see there is a r at the bottom, so r as r goes to 0 that becomes undefined.

So, it is very similar to the Dirac delta function, now I see that if I take any arbitrary function, and scale it with laplacian of phi and I integrate it over omega then I get the value of the function at 0, at the origin, which is exactly what the Dirac delta function does. So, at the origin it behaves exactly like the Dirac delta function, outside far away from the origin it behaves exactly like the Dirac delta function, so it is got to be equal to the Dirac delta function. So, laplacian of phi hat must be equal to delta x laplacian of phi hat must be equal to delta x, where delta x is the Dirac delta function.

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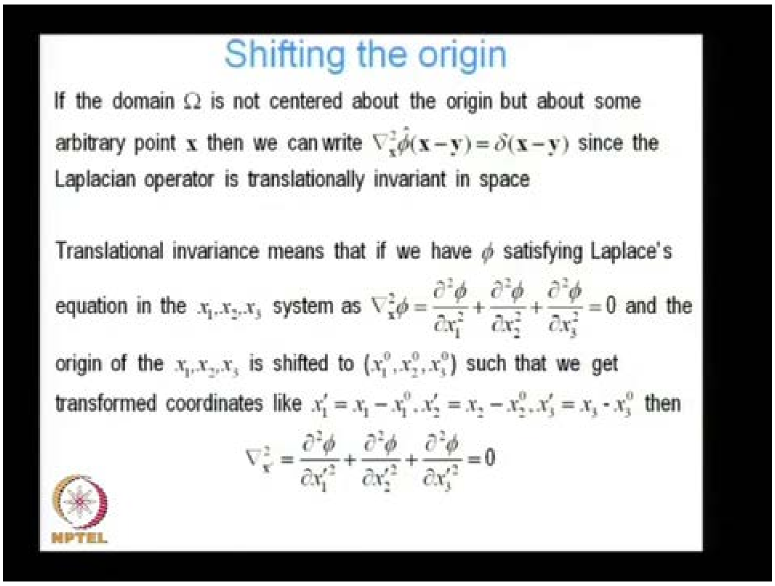

In case my domain is not centered about the origin, but about some arbitrary point x then we can write laplacian that should be why I am sorry, but about some arbitrary point y then we can write laplacian of phi hat x minus y is equal to del x minus y. Why can we write that well we can write that because, the laplacian operator is translation ally invariant in space, what does that mean well if I have my if write my laplacian like this, and in the origin at 0 I have something like this.

So, del del 2 phi del x 1 squared plus del 2 phi del x 2 squared was del 2 phi del x 3 squared, and then I shift the origin. So, I change my origin from 0 to x 1 0 x 2 0 x 3 0, and such that we get a transformed coordinates like x 1 prime, x 2 prime, x 3 prime where x 1 prime is equal to x 1 minus x 0, x 2 prime equal to x 2 minus x 0 x 3 prime equal to x 3 minus x 0, then laplacian with respect to x prime is going to be exactly equal to 0 can be shown, so it is translation ally invariant in space.

So, I can write laplacian of phi hat x minus y is going to be exactly equal to delta of x minus y. Translation ally invariance means the property is remain exactly the same, it is just that the origin has shifted, so let us go back to the equation laplacian of phi hat x minus y is equal to del x minus y, and we know that phi hat was of the form 1 by 4 pi r when the origin was at 0.

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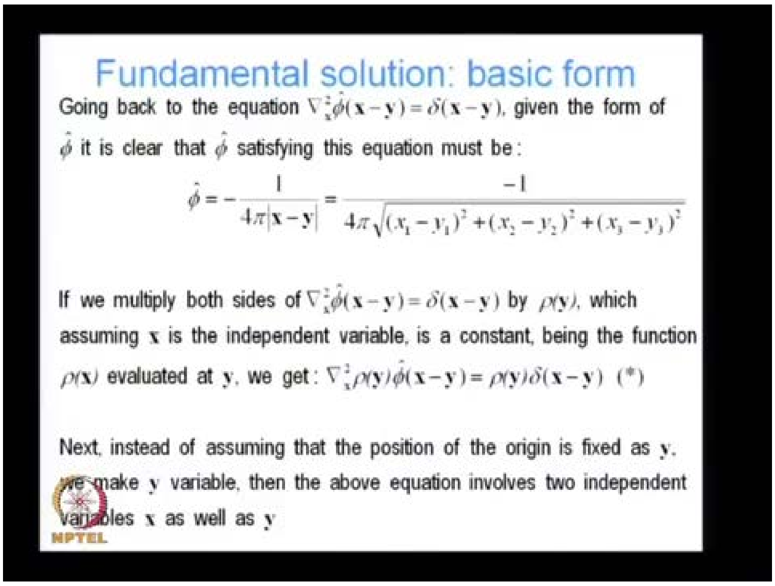

So, if my origin has shifted to y bar my phi hat is going to be slightly different, it is going to be 1 by 4 pi mod of x minus y instead of r I have mod of x minus y. And that is going to be given by this 4 pi root of x 1 minus y 1 squared plus x 2 minus y 2 squared plus x 3 minus y 3 squared. So, we look at this function and let us take multiply both sides of that equation laplacian of x square phi hat x minus y is equal to del x minus y by rho y, which assuming that x is my independent variable is a constant.

So, well as for the time being rho of y is nothing, but the value of rho of x evaluated at y, so x is still my only independent variable, and rho of y let me think of rho of y as a constant, it is the value of rho of x evaluated at y. And in that case I get Laplace's since it is a constant, I can pull it inside my laplacian operator and I get laplacian squared rho of y phi hat x minus y is equal to rho y del x minus y. Next, and assume that the instead of assuming that the position of the origin is fixed at y I make y variable.

So, my origin now becomes variable instead of thinking of y as a point in space, as a fixed point in space, now I assume that y is a variable. So, my origin can be anywhere in this space, then this equation involves two independent variables, at not only involves independent variable x, and it is components and it also involves the independent variable y when if we integrate this equation. So, now, there are two independent variables x and y and I am going to integrate this equation with respect to y, I am going to integrate that equation with respect to y throughout my volume v.

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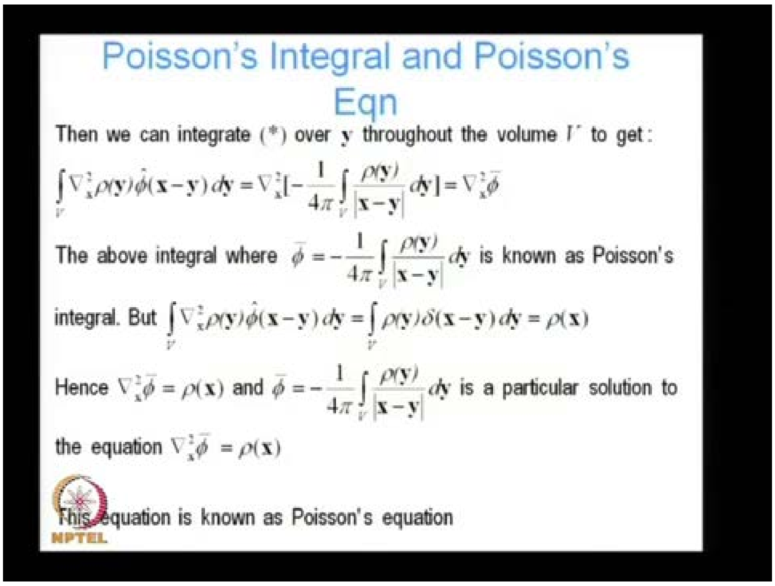

So, I take the integral of laplacian of rho y phi hat x minus y divide over the volume v, so this I know is 1 by 4 pi x minus y. So, I have laplacian of x square minus 1 by 4 pi integral of v rho y mod of x minus y d y, and if I denote this integral 1 by 4 pi integral rho y by mod of x minus y d y as another variable phi bar, where phi bar is now a function of x only, in that case I have because, I am integrating it out with respect to y.

So, it is now a function of x only, so phi bar, so this is equal to laplacian of squared phi bar because, I am denoting this as phi bar, and this integral is known as Poisson's integral. But, let us recall laplacian of x squared rho of y phi hat x minus y d y is equal to rho of y del x minus y because, I know that laplacian of x squared phi hat x minus y is just like del x minus y, we have seen that before exactly here.

So, that is del x minus y, so then I have rho y del x minus y I am integrating over d y, so again I am going to get rho of x, it is going to give me rho of x. So, what do we have, we have this thing is equal to rho of x, and this thing is equal to laplacian of phi bar, so I have laplacian of phi bar where the laplacian is now with respect to x, x is my independent variable.

Laplacian of phi bar is equal to rho of x and phi, so this gives me a non-homogeneous form of the elliptical equation. So, Laplace's equation has rho laplacian of x equal to laplacian of that function equal to 0, this is laplacian of phi bar is equal to rho of x, and this equation is known as Poisson's equation, this is Poisson's equation. And you can see that for the Poisson's equation I already have a solution, if laplacian of phi bar is equal to rho of x, then to find phi bar all I need to do is to evaluate this integral what is that integral I take rho of y divided by mod of x minus y.

Suppose I want to find phi bar at some point x I want to find phi bar at some point x in my domain, and I know that phi bar satisfies Poisson's equation. So, all I need to do is to evaluate this integral rho of y divided by mod of x minus y, where x is the point where I want to find my want to solve Poisson's equation. And evaluate that integral and that is my solution, and that is actually a particular solution to this equation, it is a particular solution to that equation.

And this see is up till now we have not talked anything about boundary conditions, so this Poisson's equation may have certain boundary conditions as well. So, we have this solution does not include the effect of all those boundary conditions, so it is a particular solution, so there is going to be another part which includes the effect of the boundary conditions.

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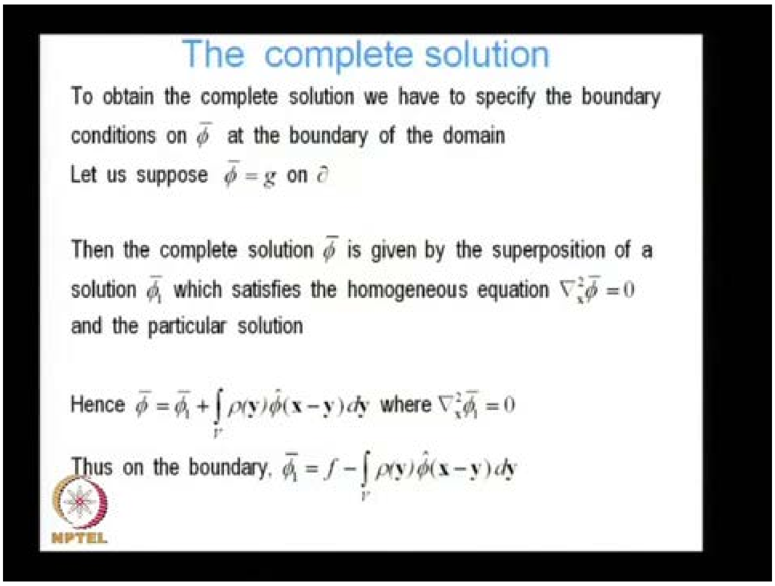

Since obtained the complete solution we have to specify the boundary conditions on phi bar at the boundary of the domain. So, in let us suppose phi bar is equal to g on my domain which is del v, so I am missing a v here, so it is phi bar equal to g on del v, then the complete solution phi bar is given by the super position of the solution, which satisfies the homogeneous equation, laplacian of phi bar equal to 0 and the particular solution.

The homogeneous solution the solution to the homogeneous equation must satisfy the boundary conditions, actually both of them together must satisfy the boundary conditions. So, what is phi bar, phi bar is equal to phi 1 bar plus integral of rho y phi hat x minus y d y, this is my particular solution, this is the solution to the homogeneous equation. Therefore, on the boundary phi 1 bar must be equal to f minus this value.

Because, this value this is what I get, so phi for phi bar to be equal to g on del omega, so this must be g I apologies again. So, phi bar to be equal to g on del v then phi 1 bar must be equal to f minus this on the boundary g minus that on the boundary.

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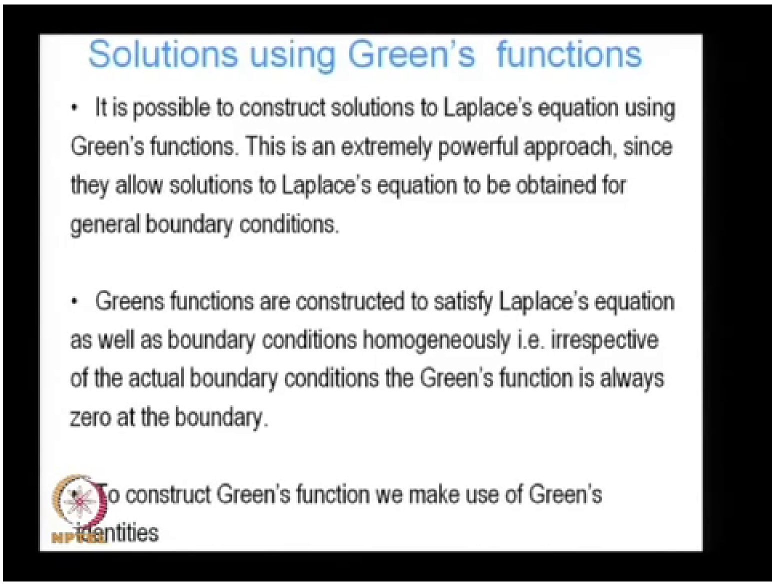

So, we have seen we have found a way to solve Poisson's equation, in if using the fundamental solution to the Laplace's equation. So, we have first looked at Laplace's equation, and we found the fundamental solution of Laplace's equation, which is phi hat. And then we looked at Poisson's equation, which is basically the elliptical operator equal to a non zero value, and we said that if we know the fundamental solution to Laplace's equation.

We can also construct the solution for Poisson's equation using that fundamental operator phi hat and the value on the hand side of Poisson's equation which is rho, look let us go back and take a look at Poisson's that was my Poisson's equation. So, the solution of the Poisson's equation I have been able to construct in terms of the fundamental solution of Laplace's equation. And my operator plus there is the this term which is the homogeneous solution, which is nothing, but the solution of Laplace's equation subject to those boundary conditions.

So, that is one approach to solving Laplace's equation using the fundamental solution in spherical coordinates, we want to talk about another approach which is a very powerful approach. Because, it allows us to solve Laplace's equation for very general boundary conditions, so whatever be my domain, whatever be my boundary conditions I can solve Laplace's equation using this green's function approach.

So, the green’s functions are constructed to satisfy Laplace's equation as well as boundary conditions homogeneously that is irrespective of the actual boundary conditions, the green’s function is always 0 at the boundary it is very analogous to your Eigen function approach. Remember that what does the Eigen function do, what do we do in the Eigen function approach, we try to find the solution of my operator of my linear operator which satisfies the homogeneous boundary conditions.

So, Eigen value problem always has homogeneous boundary conditions, and using those Eigen functions solution to that problem which are the Eigen functions, I can construct any solution to my partial differential equation. So, they form a basis for my solution space, similarly, green’s functions also, green’s functions you can think of them again as solutions to Laplace's equations satisfy the boundary conditions homogeneously.

But, once we know the green’s function we can construct the solution for any arbitrary boundary condition. So, green’s functions satisfy Laplace's boundary condition for Laplace's equation for homogeneous boundary conditions, it satisfies Laplace's equation for homogeneous boundary conditions. But, using the green’s function I can solve the non-homogeneous problem, and we will see how we can do that, but to in order to construct green’s functions, we make use of something known as green’s identities which are basically rather simple.

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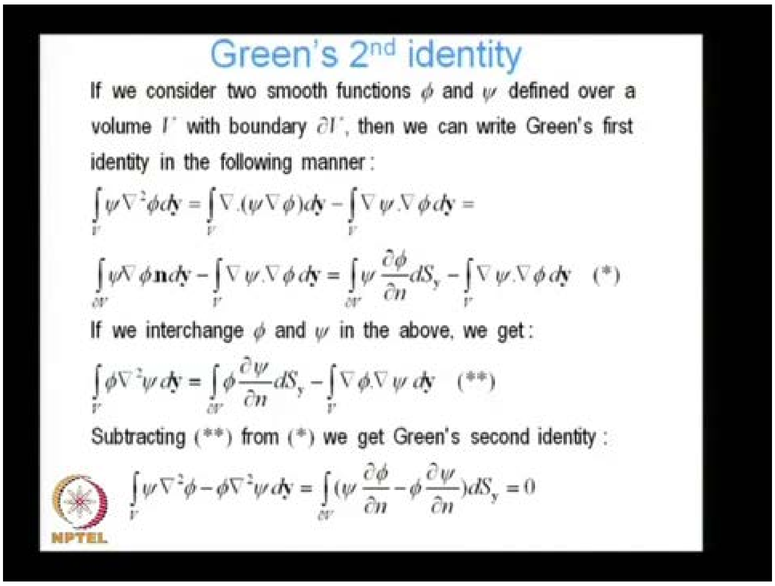

Basically, we consider two smooth functions, phi and psi defined over a volume v with boundary del v, then we write green’s first identity in the following manner, which is just really, really straight forward. Because, it is just writing the laplacian operator in terms of it is definition, in terms of it is definition, in terms of the divergence and the gradient.

So, let psi if I take the integral of psi times the laplacian of phi, I can write it as divergence of psi times the gradient of phi minus gradient of psi dotted with gradient of phi. This is a very simple identity from vector calculus, it can be proved almost trivially using there are lots of ways of proving it, but psi laplacian of phi is equal to divergence of psi gradient of phi minus gradient of psi dotted with gradient of phi.

And then we say we use the divergence theorem on this part, so divergence of psi grad phi is nothing, but psi grad phi dotted with n over my surface minus this part remains the same, the gradient of psi dotted with gradient of phi. And this gradient psi dotted with n is nothing, but del phi del n, we have seen that before that that is nothing, but del phi del n, where psi del phi del n d s y. So, again I have made a mistake here, this should be d s y because, this is the integral over the surface.

So, psi del phi del n d s y minus gradient of psi dotted with gradient of phi d y, so that is my first identity green’s first identity, to get green’s second identity we change interchange phi and psi. So, now, we have interchange phi and psi, we have phi laplacian of psi d y is equal to phi del psi del n minus grad phi grad psi d y, so all you have done is we have interchanged the phi and psi, in that expression.

And then we subtract this from this, if we subtract this from this, this part you can see vanishes because, this is identical to that. So, that part vanishes and we have psi laplacian of phi minus phi laplacian of psi integrated over the volume must be equal to psi del phi del n minus phi del psi del n integrated over the area, so that is my green’s second identity.

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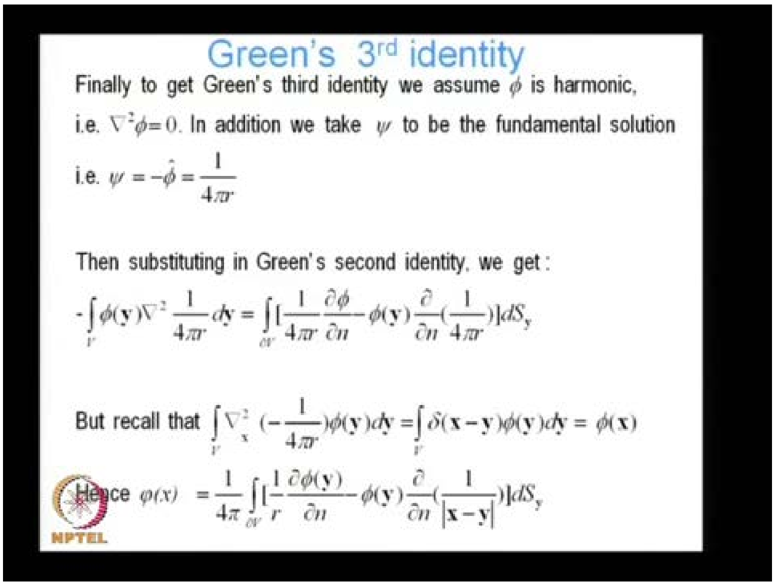

And to finally, to get green's third identity we make a further assumption up till now, the only thing that we have assumed over phi and psi about phi and psi is that they are smooth functions, they are defined over the volume v with boundary del v we have not assumed anything else. So, now, we make the additional assumption that laplacian of phi is equal to 0, if we assume that laplacian of phi is equal to 0, and we assume that psi is my fundamental solution to Laplace's equation that is psi is equal to phi hat, which is equal to minus 1 by 4 pi r.

And then I substitute these values, these conditions in green's second identity, if I do that what do I get well I have psi laplacian of phi I have assume that laplacian of phi is equal to 0. So, this first term is going to give me 0, and then I have minus phi laplacian of psi d y, so that gives me minus phi laplacian of 1 by 4 pi r d y, and that is equal to the integral over the surface that is equal to the integral over the surface, which is psi del phi del n psi being equal to 1 by 4 pi r.

So, I have 1 by 4 pi r del phi del n minus phi del psi del n, so that is equal to minus phi del del n of 1 by 4 pi r d s y. So, substituting these in green's second identity I get that, but let us recall laplacian of minus 1 by 4 pi r laplacian of my fundamental solution is nothing, but my Dirac delta function. So, I have delta x minus y phi y and that is going to give me phi of x.

So, what do I have, I have on the left hand side I have phi of x and that is equal to 1 by 4 pi 1 by r del phi del n minus phi y del del n 1 by mod of x minus y, where I have replaced r by mod of x minus y here, and I have pulled out my 4 pi outside the integral. So, this what do you see well you see that using green's third identity I am if I want to find if I know that phi satisfies Laplace's equation, I already have my solution phi in terms of using green's third identity, provided I know the value of the gradient and the value of the function on the boundary.

So, you can see it is automatic, so I want to I have only thing I have assumed is that laplacian of phi equal to 0 that is phi is the solution of Laplace's equation well, what is the solution, well I can always find the solution if I know the value of the solution at the boundary, and the gradient at the boundary. Green's third identity allows me to find the value find the function at any point in the domain, if I know the value of the function at the boundary and the value of the and the normal derivative of the function.

You can see the power of this approach, so it can find the value of the function at any point I want to find a function which satisfies Laplace's equation. And somebody has told me that the value of I know the value of the function at the boundary, and the gradient at the boundary if I know that I can solve I know the function straight away. So, given the boundary conditions I know the function that is why green's functions are, so powerful.

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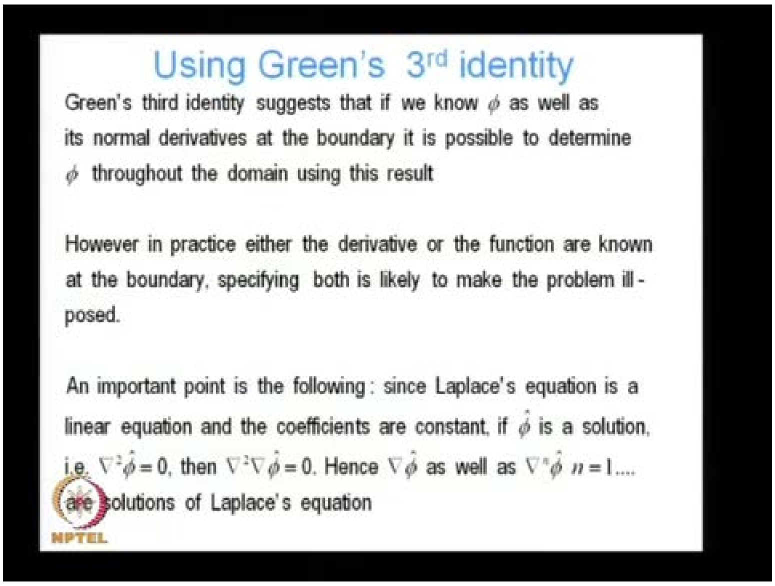

So, green’s third identity suggests that if we know phi, as well as its normal derivatives at the boundary, it is possible to determine phi throughout the domain using this result, it is possible to know the function anywhere in the domain. However, in practice we do not know at any point on the boundary, we usually do not know both the function as well as it is derivative.

If we both the things are known, the problem is not usually will-posed we go back to our mechanics. We go back our structural mechanics at a boundary we cannot specify both the displacement as well as the traction, you can specify either, if you specify both then the problem is will-posed. So, usually we do not know the function value as well as it is derivative at a point, we either know the function value or we know it is derivative.

So, an important point is the following, since before coming back again I have taken a little side track, but it is worth emphasizing here. Since, Laplace's equation is a linear equation, and the coefficients are constant if our phi hat is a solution that is laplacian of phi hat is equal to 0, then laplacian of gradient of phi hat is always going to be 0, why is that going to be true well think of it like this. Laplacian of phi hat what does that mean del del x 1 square phi hat plus del 2 del x 2 squared phi hat plus del 2 del x 3 squared phi hat.

And then, suppose I want to take the gradient, so if I take gradient I take a again del del x, del del y, del del z or del del x 1, del del x 2, del del x 3. So, since those coefficients are constant I can pull in my gradient operator inside my laplacian, so the order of the differentiation does not make a difference. Because, my coefficients are I cannot say they do not depend or there is no x dependence of the coefficients of Laplace's equation.

So, since that is true what that tells me is that laplacian of grad of phi hat had of better be 0 too, if laplacian of phi hat is equal to 0 and I take gradient of both sides, I can pull the gradient in because, of the nature of this operator. So, if phi hat satisfies Laplace's equation the gradient of phi hat must also satisfy Laplace's equation, and if gradient of phi hat satisfies Laplace's equation, the second gradient the second operate on that with another gradient operator that is got to satisfy Laplace's equation as well.

So, if I have grad to the power n of phi hat all of them are going to be solutions of Laplace's equation. We can see that this gives rise to something very important known as spherical harmonics, we are going to talk briefly about them if we get time, but this property is very, very important.

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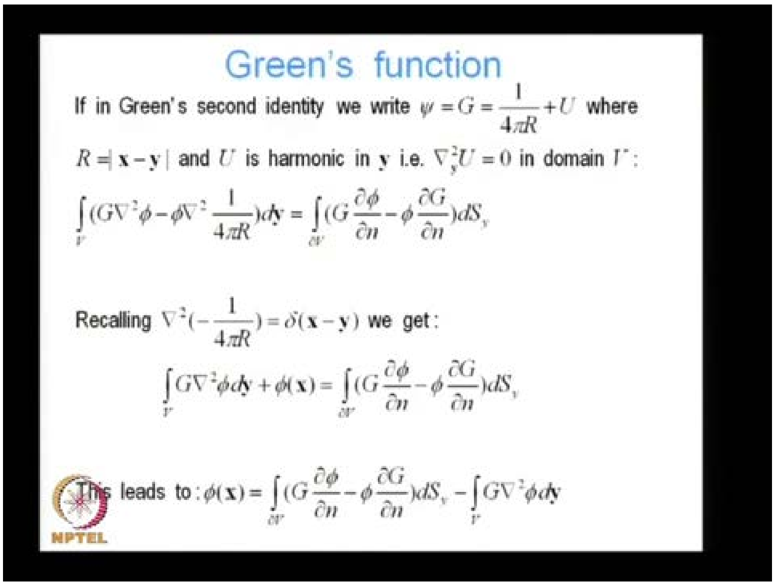

So, we look we saw green's third identity, and we saw that using green's third identity I always know that I can always find a solution of Laplace's equation, provided I know the function as well as it is derivative on the boundary. But, then we said that well it is not usually possible to know both the function as well as it is derivative on the boundary, so what do we do, well we do this. In green's second identity we write psi is equal to g is equal to 1 by 4 pi r plus u we define side to be 1 by 4 pi r by u r is mod of 1 by 1 by r is equal to mod of x minus y, so that is my fundamental solution.

So, here before that what did we do we assume that psi is nothing, but my fundamental solution, in green's second identity we assume that psi is my fundamental solution. Now, I am saying that well do not assume psi is my fundamental solution, we assume psi is equal to my fundamental solution plus some another term, you assume that psi is my fundamental solution 1 by 4 pi r plus another term. So, we had changing tracks like this, so first we assume size equal to 1 by 4 pi r, we obtain the solution for phi, but we said that that solution it is not very useful.

Because, it tells me that I must be not only my function value, I must need my derivative also at every point in the boundary. So, that is not very useful, so now, I see that well then let us instead of assuming psi equal to 1 by 4 pi r let us assume psi is equal to 1 by 4 pi r plus u, where u is harmonic in y, where u satisfies the solution of Laplace's equation in that domain. And then I substitute that again back in my green's second identity, so which was this I substitute that back again in green's second identity, and if I do that I get now g of laplacian of phi because, laplacian you go back and take a look let us go back and take a look.

So, we have psi laplacian of phi minus phi laplacian of psi, so g laplacian of phi minus phi laplacian of 1 by 4 pi r, if we do that it comes to be that. And then on the hand side we have g del phi del n minus phi del g del n, g del phi del n minus phi del g del n psi is equal to g. So, g del phi del n minus phi del g del n, so psi laplacian of phi minus phi laplacian of 1 by 4 pi r because, laplacian of u is always 0, so g psi laplacian of phi minus phi laplacian of psi laplacian of u is going to give me 0, I have laplacian of 1 by 4 pi r.

And then we recall that laplacian of 1 minus 1 by 4 pi r is equal to my Dirac dalta function with argument x minus y. Then what do we get, we get g laplacian of phi minus phi del x minus y integrated over d y, so that is going to give me phi of x that must be equal to integral with over the surface of g del phi del n minus phi del g del n. So, now this gives me phi of x is equal to g del phi del n minus phi del g del n minus g laplacian of phi, I have brought this laplacian of phi to the hand side.

So, I have brought this to the right hand side I have phi of x here, and I have this thing, now if I want to find out phi that satisfies Laplace's equation, if I want to find put phi that satisfies Laplace's equation, then this term is automatically 0 because, phi satisfies Laplace's equation. So, this term is automatically 0, and then in that case phi is going to be given by g del phi del n minus phi del g del n S y.

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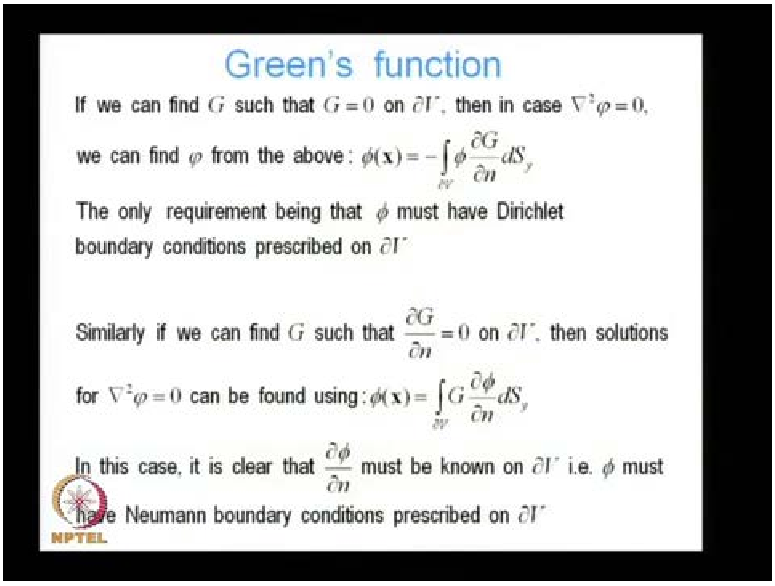

If we can find green's function g such that g is equal to 0 on my boundary, then I have a solution. So, if I find g such that g is 0 on the boundary, then I am getting phi of x is equal to minus phi del g del n d S y. So, you can see now I do not need the function value as well as it is derivative, I only need the value of the function phi on the boundary.

So, if I know if g satisfies g is going to be 0 everywhere on the boundary, then I can find out phi at any point in my domain, only if I know the value of phi on the boundary I do not need to know both phi, and it is derivative at the boundary which makes the problem well-posed. So, if g is equal to 0 on del v then I can find phi x is equal to minus phi del g del n d S y, so if I know phi on my boundary I know g then I can find out phi x, so the idea is this, if I know my green's function.

And the green's function satisfies the boundary conditions homogeneously, which was another very important criteria for green's function. Then I can find my solution to Laplace's equation, if I know my value of that function on the boundary, if I know my phi on the boundary. So, the only requirement is that phi must be prescribed on the boundary, so phi must have Dirichlet type boundary conditions.

Similarly, on the other hand if we can find g such that the gradient of g is equal to 0 on the boundary, then what happens, then this term is going to be 0. And in that case I can always find phi if I know my gradient of phi on the boundary, so they are complementary if I know if g is equal to 0 on the boundary then I must know phi on the boundary.

If del g del n is equal to 0 on the boundary, then I must know del phi del n on the boundary. So, if I know that I can find out my phi of x, so this is the basically the fundamentals of the green's function approach for solving both the dirichlet as well as the Neumann's boundary condition.

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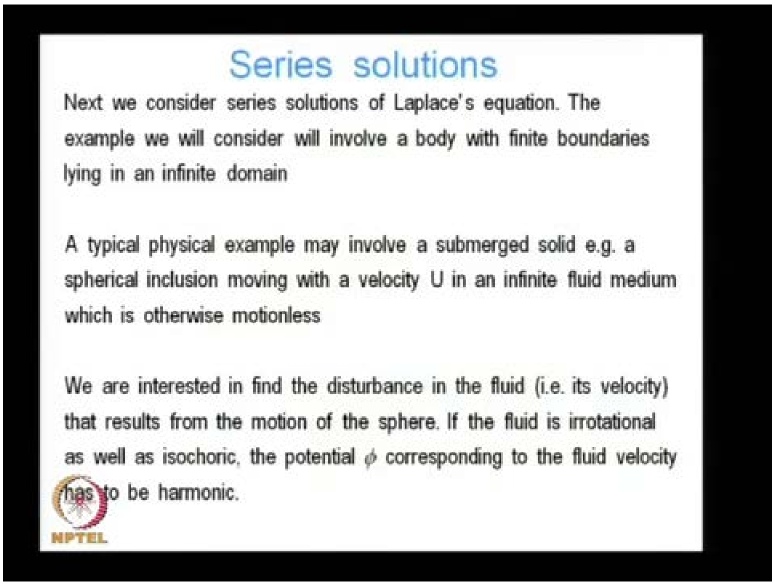

So, I thought I would finish elliptic equations this time, but I am still left with the series solutions, I am not sure if we have time to cover that. So, we have to take a decision on that whether we want to continue with series solutions or move on to numerical techniques for partial differential equations.

Thank you.