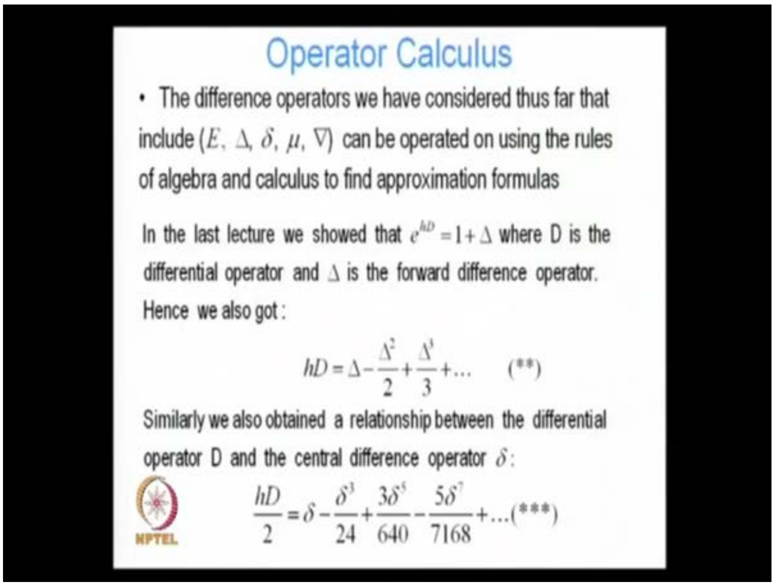
Numerical Methods in Civil Engineering
Prof. Arghya Deb
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 29
Differential Operators – III

In lecture 29 of our series on numerical methods in civil engineering we will continue with our discussion on differential operators.

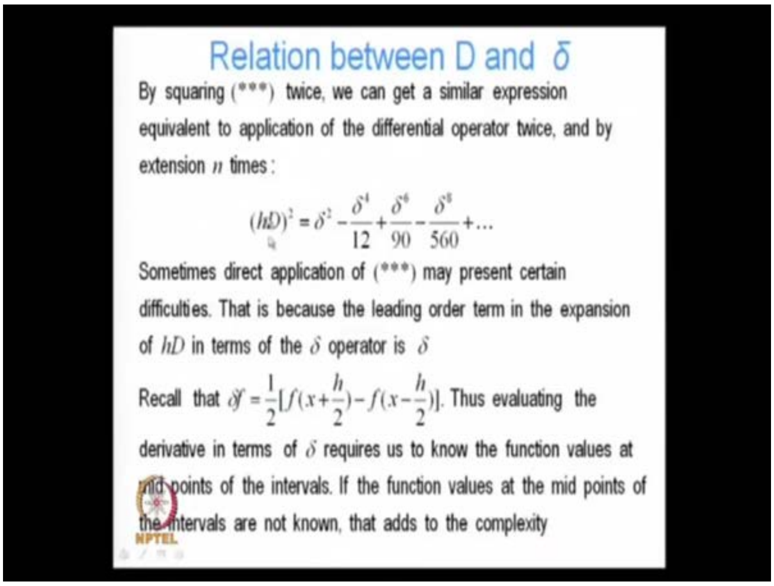
(Refer Slide Time: 00:29)



The differential operators, we have consider thus for include the shift operator. The forward difference operator, the central difference operator, the average difference operator and the backward operator and we have seen that, they can be operated on using the rules of algebra and calculus to find appropriate formulas in the last lecture. We showed, we dealt with operator calculus and during that we showed relations between various operators and various operators for instance between the difference operators. The forward difference operator and the differential operator and we showed that, they are related to through the exponential function.

So, e to the power h D is equal to 1 plus delta where, D is the differential operator and delta is the forward difference operator using this. We also showed that h D is given by this relation in terms of delta. We also obtained a relation between the difference operators in this case the central difference operator and the differential operator and we showed and we showed that relation is something like this.

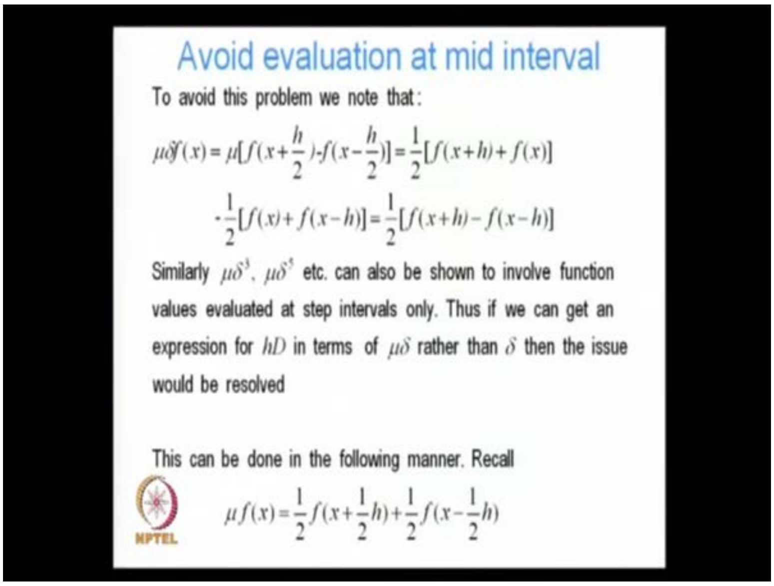
(Refer Slide Time: 01:45)



By squaring this last relation twice we can get a relation between D Square, that is the differential operator applied twice and various powers of the central difference operator. However, sometimes if you want to apply this relation directly, it creates problems well why is that well, let us recall that the central difference operator is given in terms of function values at the mid step at the mid step. For instance delta f is equal to half of f of x plus h by 2 minus f or x minus h by 2. So, we need the function values at the middle of the step at x plus h by 2 and x minus h by 2.

If we do not have function values at those steps at the mid steps then, it is hard to use the central difference operator evaluating the derivative in terms of delta therefore, derivative D in terms of delta requires us to know the function values at mid points of the intervals, if the function values at the mid points of the intervals are not known that adds to the complexity meaning the cost you have to evaluate the. So, it is always better if we can get the get the difference operators in terms of function values at the grid points at x plus h x plus 2 h x minus h and so on.

(Refer Slide Time: 03:13)

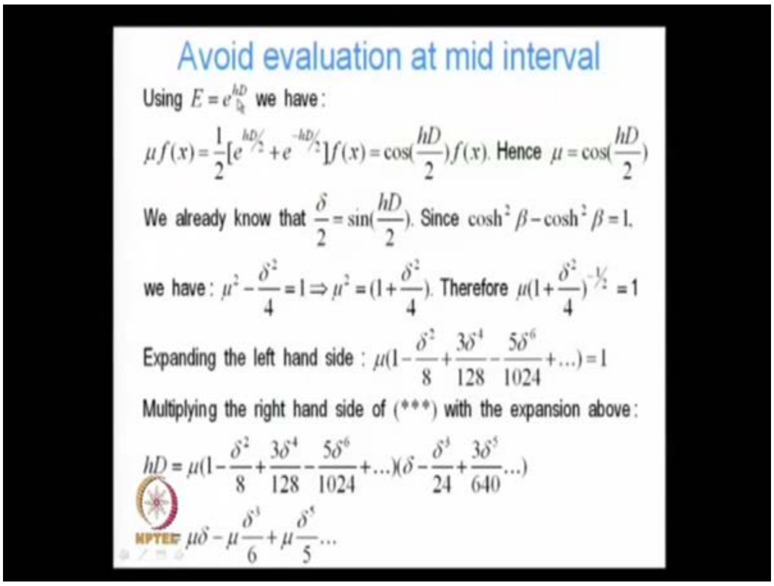


How do we avoid this problem? to avoid this problem, we note that mu del f x that is if we apply the central difference operator first on a function and then we apply the average operator the mu then, what do we have. So, if we apply delta on f we get f of x plus h by 2 minus f of x h minus 2 on top of that if we apply mu then, what do we get half of. So, mu of f of x plus h by 2 is equal to half of f of x plus h plus f of x minus half of f of x plus f of x minus h. So, mu operating of f of 1 of x minus h by 2 is equal to f of x plus f of x minus h it is the average difference operator. So, x minus h by 2 it takes it goes 1half steps back and it goes half step forward and adds them together and takes the half of that. So that is my average operator, that operating on this part gives me this part that operating on this part gives me that. So, again it goes half step forward, half step back and adds it together. So that is given by that.

So, and then if we add these 2 together, we get half of f of x plus h minus f of x minus h. So, what do we see, if instead of directly applying the central difference operator if we apply the central difference operator first and then use the averaging operator then we can get it in terms of the grid point value. So, f of x plus h and f of x minus h similarly, we can show that, mu delta cube that is mu operating on delta operated three times or mu delta to the power 5 mu operating on delta after it has been operated 5 times can also be shown to involve function values evaluated at step intervals only but, step intervals I mean grid points at grid points only.

Thus, if we can get an expression for h D h D instead of mu delta cube, mu delta 5 instead of delta cube, delta 5 and delta. Then the issue would be resolved. We would be able to evaluate the functions at the grid points. This can be done in the following manner, let us recall that mu of f of x is equal to half x plus half h plus half x minus half h, that is we just talked about that is just little while ago.

(Refer Slide Time: 06:07)



Then using e is equal to h D, we have f of x plus half h. I can write this as h D by 2, we can think of it as this shift operator. x plus half h, I can think of that as the shift operator operating over half a step. So, e f of x plus half h is equal to e operating on f for half a step and then, I know there is a relation between e and the difference operator that involves the exponential. So, I can write this as half of e h to the power D by 2. F of x plus h is equal to e to the power h D. So, f of x plus h by 2, I am writing as e to the power h D by 2.

Similarly, I can, I am writing f of x minus half h as e to the power minus h D by 2. So, I write this, like that and that we recognize is the cos h is the cos hyperbolic of h D by 2. It is the cos hyperbolic of h D by 2 operating on f of x and again, we said at the beginning of our discussion and operate a calculus, that we will say that 2 operators are the same, if there action on any arbitrary function is the same. So, in this case f is an arbitrary function. So, we can write mu is equal to cos of h D by 2.

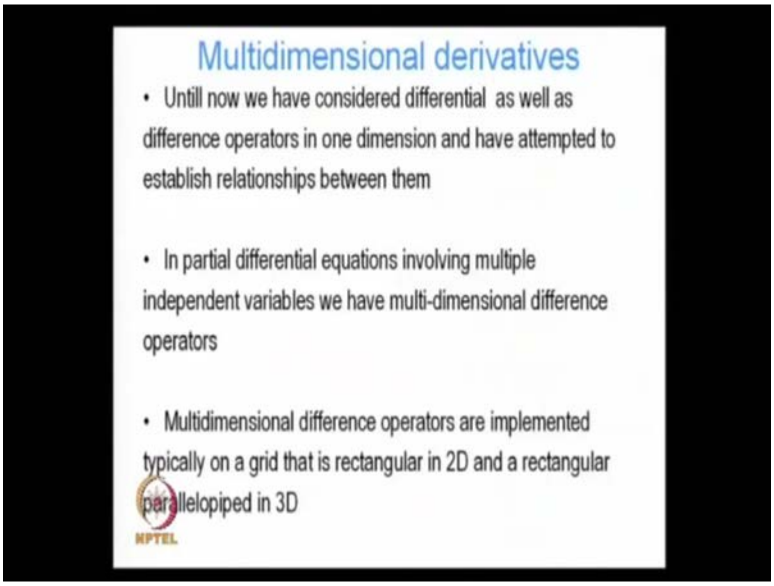
Now, we already saw in the last class, that delta by 2 is equal to sin h D by 2 if we look back we discuss that. So, since delta by 2 is equal to sin h by 2 mu is equal to cos h by 2. So, I can write cos hyperbolic square and. So, I know that cos hyperbolic square anything minus. Now, what is that sin? There is a sin, I am sorry. So, that is a type. So, that is actually cos hyperbolic square beta minus sin hyperbolic square beta is equal to 1 that is an identity. So, from that, we get mu square minus delta square by 4 is equal to 1 mu square minus delta square by 4 is equal to 1 from which, I get mu square is equal to 1 plus delta square by 4 therefore, I can write mu 1 plus delta square by 4 to the power minus half is equal to 1. So, I just take the square root of both sides and then bring it to this side and I get 1. So, if I expand this on the series expand the series by on the left hand side then, I get this expression. So, mu 1 minus delta square by 8 plus 3 delta 4 by binomial expansion. So, I get that is equal to 1 and then. So, this is equal to 1. Now, let me go back to my expansion here. Which was this?

Now, I am going to multiply this hand side with 1. So, I will get the same thing. So, if I multiply this by 1. Or that is equivalent to multiplying it by this whole thing, because this whole thing is equal to 1. So, I can get h D is equal to mu times, that was already there. So, I get this, I might have a factor of 2 missing, I am not so much sure about that, let us check. So, this mu 1 minus delta square by 8 times that delta minus delta cube by 24. So, I have a factor of 2 missing. So, but, then I have adjusted that. So, here, it is just h D by 2. So, that gives me mu D minus mu C mu delta cube square plus mu delta 5.

So that involves terms like mu delta a cube, mu delta to the power 5, which I know can be written in terms of grid point value. So, which I know, I can be written in terms of grid point values. So, I get an expression for the differential operator in terms of mu delta, which involves evaluating the function only at the grid point values. So that is enough about 1 dimensional differential operators and 1 dimensional difference operators and the relation between them, basically the relation between differential operators and difference operators, how we can write the differential operator in terms of various difference operators and most importantly recognizing, what is the error? What is the order of the error? When, we write those difference operators in terms of the differential operators.

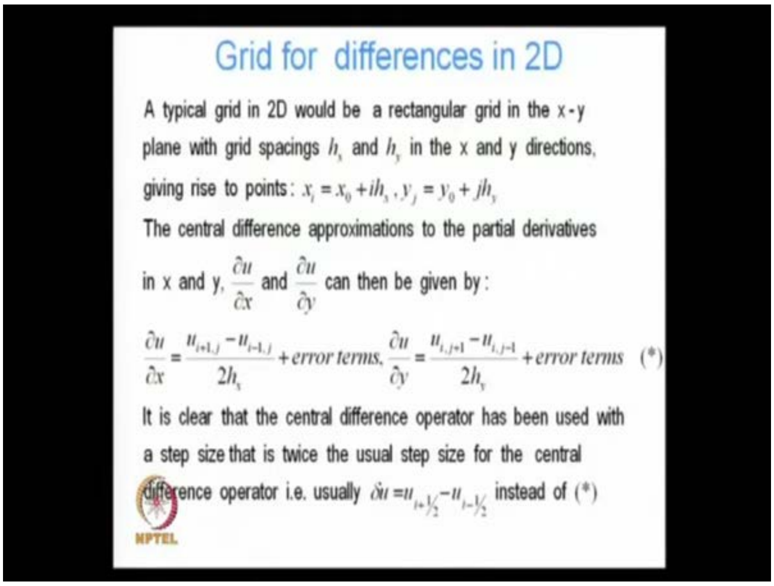
Recall that, we found that for the forward difference operator our error is linear. We have a linear error but, when we write the differential operator in terms of the forward difference operator. We have a linear error term the error is of order h while, when we wrote the central difference operator, when we wrote the differential operator in terms of the central difference operator and we expanded it in series. We found that the error was of second order, that was the error the leading term in the error was of h square was of order h square. So that is very important to recognize. When we use different difference operator we can use different difference operators to approximate our differential operators and we can get different orders of error where until.

(Refer Slide Time: 12:43)



Now, we have considered differential as well as difference operators in 1 dimension and have attempted to establish relations between them however, we looked before we started looking at difference operators. We looked at partial differential equations and we saw that most physical problems like they are multi-dimensional and they are represented by multi-dimensional partial differential equations. So, the differential operators that we have to deal with are going to be multi-dimensional. So, we do need multi dimensional approximations multi-dimensional difference approximations to multi-dimensional differential operators. So, typically multi-dimensional difference operators are implemented on a grid in a grid in 2 D. It is usually a rectangular grid and in 3 D. It is 3 dimensional ordered grids. So, it is like a grid rectangular parallelepiped in 3 D, the grid is like a rectangular parallelepiped in 3 D.

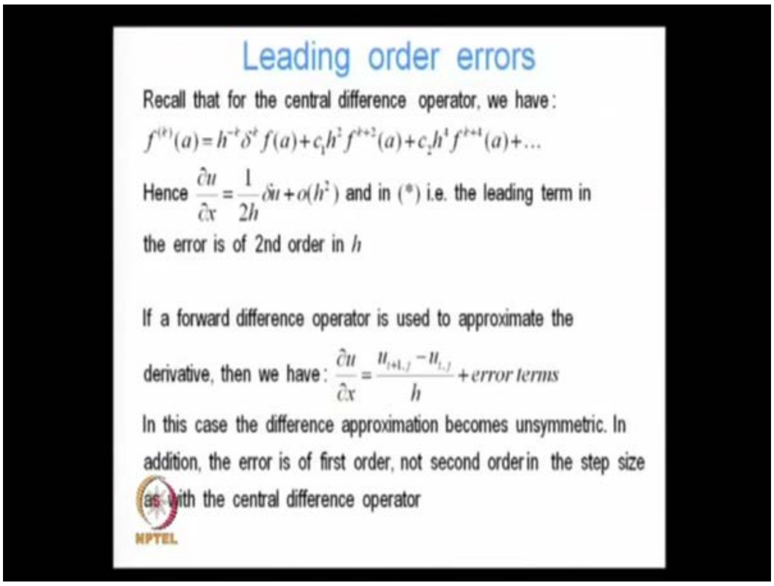
(Refer Slide Time: 13:50)



A typical grid in 2 D would be a rectangular grid in the x and y plane with grid spacing's h x and h y in the x and y directions, my grid spacing need not be the same in the x and y directions. I can have different grid in the x direction and different grid in the y direction and this gives rise to points x i is equal to x 0 plus I h x y j is equal to y 0 plus j h y. So, it is term from its location on the grid. The i'th location on the grid is given by x 0 as i times h x where, h x is the grid size in the x direction while, the j'th location in the y direction is given by y 0 plus j times h y.

The central difference approximations to the partial derivatives in x and y del u x and del u y can be given by del u by del x is equal to u i plus 1 j minus u i minus 1 j by 2 h x. Now, note several things, this involves only I am taking in the x direction. So, I plus 1 i minus 1, the j the j'th index remains fixed and similarly, when we look at del u del u del y we have u I j plus 1 minus u i j minus 1 by twice h y plus of course, the error terms now we have used it is obvious that we have used the central difference operator but, note that we have used it with the step size that is twice the actual step size the step size that we talked about. So, we have used twice, which is perfectly fine. There is no problem with that we have used this with twice. The step size for this, the central difference operator, because when we define the central difference operator, we said that is equal to u i plus half minus u i minus half. So, at a grid point I am taking half a step on the left half a step to the right that instead of here, I am taking a full step to the left full step to the that is perfectly fine if i divide by 2 h x if i divide by 2 h x.

(Refer Slide Time: 16:19)

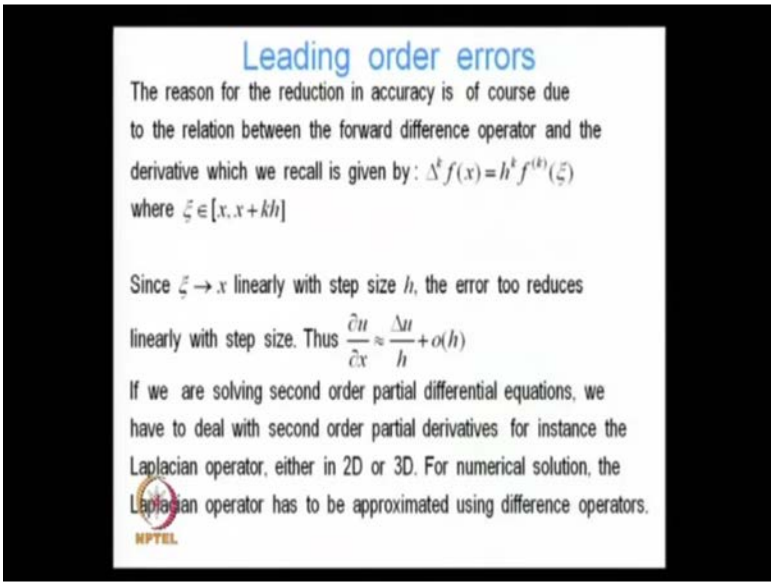


Also, let us recall, we have looked at this expression before, the relationship between the k'th order derivative, the k'th order differential operator and the k'th order central difference operator. We looked at this relationship before we found, that it was like this hence, del u del x. So, from here we can directly get the order of the error. So, del u del x is equal to 1 by 2 h times del u 1 by 2 h times, del u plus order h square plus a term of order, because if we put k is equal to 1 write k is equal to 1. That is telling me that, the derivative is equal to h to the power minus 1 h to the power minus 1. So, there is 1 by h del u there again, 1 plus order h square plus the term is of order h square is that clear, we have this 2 again, because of the we have taken twice the step.

So, is it clear in this expression? Therefore in this expression, the error terms of are order h square. So, this is second order accurate, this is second order accurate if a forward difference operator is used to approximate the derivative then, we have del u del x is equal to u i plus 1 j minus u i j. The forward differences you look go forward and subtract. So, u i plus 1 j minus u i j by h plus error terms, again these error terms are going to be linear they are going to be of order h.

So, in this case the difference approximation becomes unsymmetric, you can see it is unsymmetric. We are looking at something at i and then, it is unsymmetric about i. So, i plus 1 minus i since, it become that is why it becomes unsymmetric in addition, the error is first order not second order in the step size not second order in the step size like here, it is first order in the step size as with as is always true in case of the central difference operator.

(Refer Slide Time: 18:46)



the reason again, let us go back, the reason for the reduction in accuracy we have talked about this earlier just recapitulate is of course, due to the relation between the forward difference operator and the derivative unlike, this relation between the central difference operator and the derivative, which involves this h square term.

In case of the forward difference operator, that was given by this and xi del del to the power k f x was equal to h to the power k f k xi and we said that, x i belongs to x to x plus k h. So, x i will become closer and closer to x as and it is dependent on h. So, as h becomes smaller x i is going to be closer and closer to h but, that relation is linear.

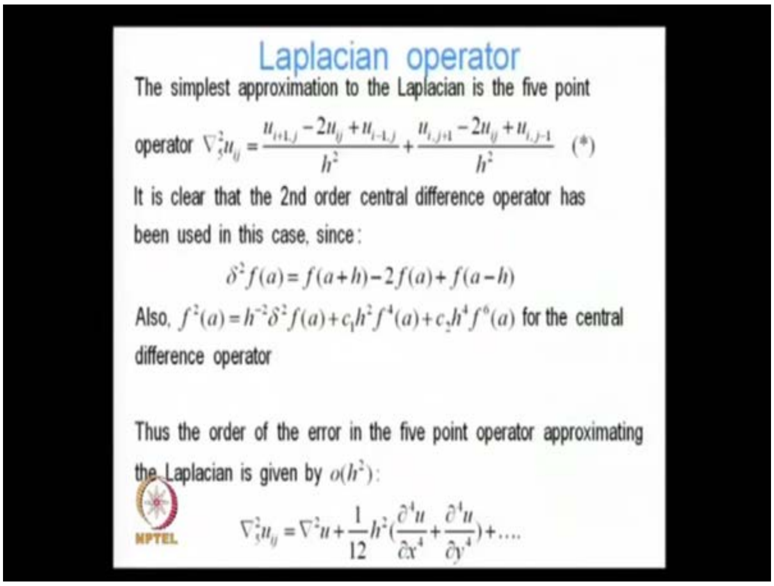
Since, x i goes to x linearly with step size h as you reduce h xi becomes closer to x but, it goes, it becomes closer to x that relation. It goes decreases linearly with h as h decreases that relation, decrease xi becomes closer to x in a linear fashion while, in the other case as h reduce, it as h became smaller. The error became smaller like h square. So, in 1 case if the error was 0.1 in the next if I reduced h, it would become 0.01. It would become 0.01, it would go as h square, if my h is of order is that clear? Is h is of order point 1 then the error is of order 0.01 and similarly, if it is 0.01. It would be 0.001. So, it goes it becomes. So, if I increase the order by 1 10'Th that is reduced by square of that.

So, the error too reduces linearly with step size in this case therefore, this is of order h the error is of order h. So, if you have. So that is something very important again, we recognize that depending on a choice of the difference operator, we can encounter different error terms. So, typically in most numerical algorithms, particularly most commercial codes, they will insist on a difference operators or integration algorithms, which are at least second order accurate, which are at least second order accurate if we are solving second order partial differential equations.

We have to deal with second order. So, we have just looked at first order partial, first partial derivatives of order 1 but, when you look at second order partial differential equations, we have to look at second order partial derivatives for instance we looked at the wave equation, we looked at our diffusion equation, we looked at our elliptic equation laplace's equation. In all those equations, we encountered the laplacian, that was common, the laplacian which present everywhere.

So, in order to solve those equations, we have to be able to represent, we have to a difference equation for the laplacian on a grid. So, we have to represent the laplacian operator in terms of difference operations either in 2 D or in 3 D space, how do we do that?

(Refer Slide Time: 22:34)



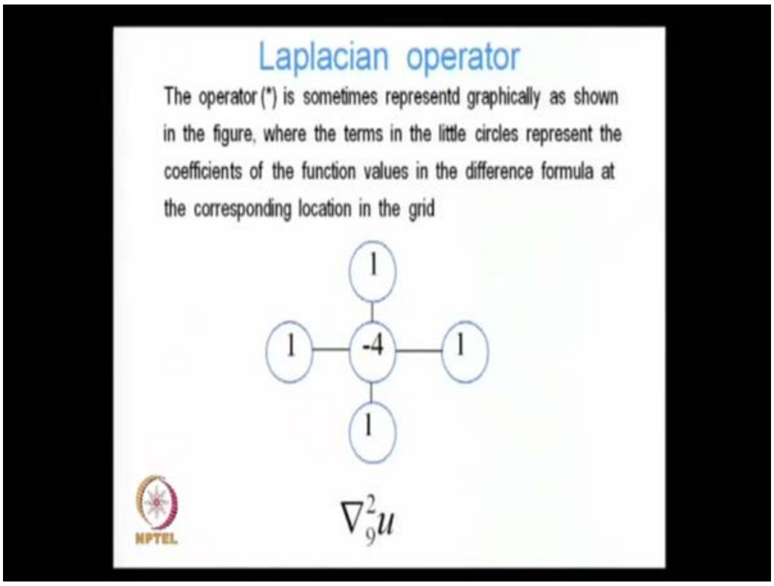
the simplest approximation to the laplacian is the 5 point operator, it is called a 5 point operator, because we will come to that is why, it is called a 5 point operator but, this is the laplacian, this is an approximation to the laplacian in 1 direction, this is the approximation to the laplacian in the other direction. So, you can see, it involves u i plus 1, j minus u i j u i minus 1 j divided by h square. Now, there is h square u see can see that consider of h, because this is a second order difference operator and similarly, this is the. So, if you think of i as representing the index in the x direction. So, the first part of this representing del square u del x square. The first term, the second term represents del square u del y square. So that is the simplest approximation to the laplacian again. It is called a 5 point operator, because u can see it involves the evaluation of the function at 5 points at i plus 1 i minus 1 j plus 1 j minus 1 and of course, at i j itself.

So, it involves evaluating the function at 5 points that is why, it is called the 5 point operator and it is also clear, that we have used the second order central difference operator in this case since, del square f a is equal to f a plus h minus twice f a plus f a minus h, we have seen that earlier on we showed that, may be in the first lecture on differential operators, that del square f a is equal to f a plus h minus twice f a plus f a minus h, you can see that we have used that here, because if u i plus 1 is actually 1 step forward. So, that corresponds to plus h u i j is of course, the function evaluated at the point that we want to evaluate the laplacian. So, that is f a, that corresponds to f a and a minus h, that corresponds to u i minus 1 that corresponds to u i minus 1.

So, this is very similar to that you can see and again this is just the same thing except that is in the y direction also, Let us recall from our previous relationship that we just saw here. f squares a means, the differential of f double derivative of f that can be written as h minus square central. The central difference operator, operated twice plus the error term, which is order of h square thus, the error the order of the error in the 5 point operator approximating the laplacian is of order h square is of order h square and therefore, we can write del 2 5 u i j is equal to the laplacian operator plus. This error term and it can be shown that, this error term is of this form you can see by comparing with this. We have f to the power 4 which basically involves the 4th derivative my u of my function u.

So, the error is something, some constant times h square times, the 4th terms involving the 4th derivative of u. So, this by approximating the laplacian with this 5 point difference operator, we are going to get an error with like this, which is of order h square as you reduce the as you change the size h. The error is going to reduce as h square, it is going to reduce by h square. So, and it is magnitude is given by the actually the magnitude of my 4th derivative of my function, which you can expect to be small as you take higher and higher derivatives, you expect those derivatives to become smaller for a well behaved function for a well behaved function.

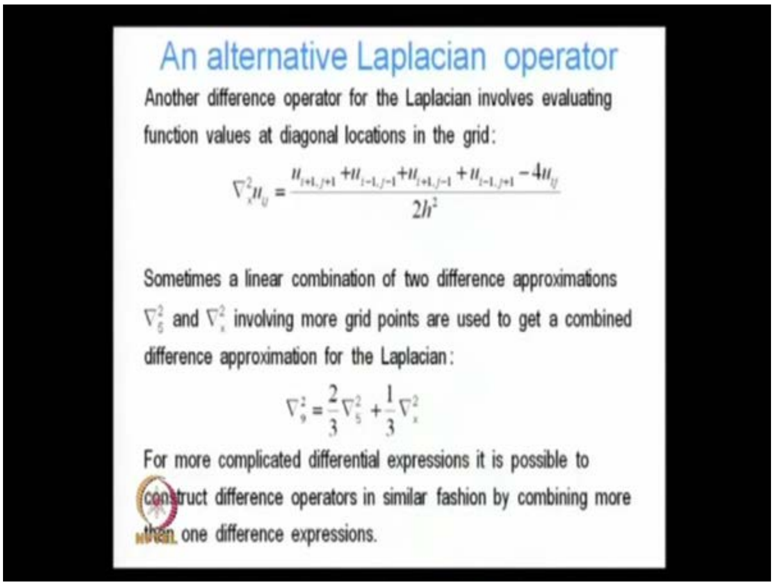
(Refer Slide Time: 27:14)



So, this operator is sometimes this 5 point difference operator is sometimes represented graphically as shown in the figure where, the terms in those little circles represent the coefficients of the function values in the coefficients of the of the function values in the difference formula.

So, the function values had 1 minus 2, so 1 minus 21 in 1 direction again 1 minus 21 in the other direction. So, at the center it becomes minus 4 and 1111. So that is represented like this, sometimes represented like this.

(Refer Slide Time: 27:59)

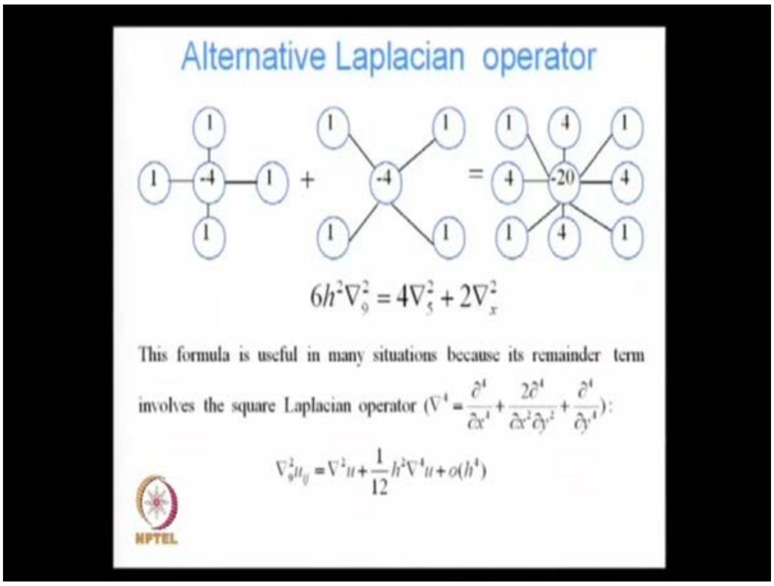


Now, it turns out that there is another approximation to the laplacian operator, not just this 5 point operator and this involves evaluating the function values at diagonal locations of the grid. So, here we evaluated the function values at the original location. The point at which at we want to find the laplacian plus h in the x direction minus h in the x direction plus h in the y direction minus h in the y direction instead of this, we can evaluate the function values at the diagonal points on the grid and this cross del square cross u i j cross mean, the diagonal laplacian operator it involves evaluating the function value at i plus 1 j plus 1 which, if we look at the previous picture that is basically, evaluating the function value here. i minus 1 j minus one, which involves evaluating the function value here, i minus 1 j minus 1 then i plus 1 j minus 1, which probably involves evaluating the function value i plus 1 j minus 1 which involves evaluating the function value here and i minus 1 j plus 1.

This involves evaluating the function value here. So, it involves the diagonal terms on the grid diagonal terms on the grid plus the function value at the location where we want to evaluate the laplacian that by h square. So this is another approximation to the laplacian. And sometimes, we put the 2 together, we use the 5 point operation as well as the diagonal approximation and combine them with some coefficients scale them with some coefficients and then, add them together making sure that the sum of the coefficients adds up to 1. So, we had we had combining these 2 operators out of weighing them, weighing the 5 point operator by certain coefficient, weighing the diagonal operator by certain coefficient making sure that the sum of the weights is 1 and saying that is a representation of my actual laplacian, that is an representation.

So, in this case for instance, we can scale it by two-third and one-third if it scale the 5 point difference operator by two-third, the diagonal difference operator by 1 third and similarly, for more complicated differential expressions, it is possible to construct difference operators in similar fashion by combining more than one difference operator.

(Refer Slide Time: 31:01)



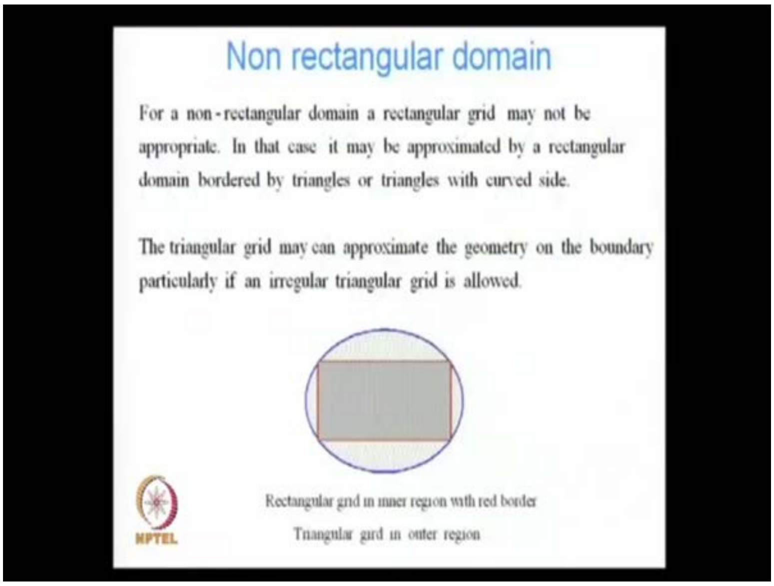
So, this is an example of that where, we have combined the 5 point with the cross differential difference operator to get what is called a 9 point laplacian, this is probably the most commonly used representation of the laplacian neither this nor this is commonly used, this is what is most commonly used the 9 point representation of the laplacian. So, here, I have said 6 times h square this is equaled to sorry, 6 times, that is equal to 4 times this plus 2 times that. So, 4 times this plus 2 times that actually, this is actually representation. I should probably have mentioned that, there is a 2 here. So, this is actually representation of 2 del square x u i j this is 2 del square u x i j is given by that by h square.

So that involves terms like this and then 4 times that, if we had, and that we get something like this. So, this is 6 h square del square by 9. So, if I divide this, if I scale all the coefficients by these appropriate values, the term here is scale by 4, the term here is scale by 1 this term. I scale by minus 20 and add it altogether and I divide it by 6. I am going to get my representation of my 9 point laplacian operator, all these 6 and 4; this is just to get these coefficients, which are instead of fractions. We have whole numbers. So, that the coefficients are whole numbers. So, that to make sure that my coefficients are whole numbers rather than fractions, because fractions as soon as we have fractions, we have to take division as soon as you take division, you have introduce errors, and you introduce errors round off errors

That is why; we prepared to deal with whole numbers as our coefficients instead of having some fractions scaling the function value. We want to scale it with a whole number. So, at least there, we do not introduce any round off errors and then, we have to evaluate the laplacian, we have to divide it by 6 atleast. We are doing the division only once after doing all these multiplications with these whole numbers. We are doing the division only once while if we used fractions for each coefficient. We would introduce round off errors every time, we add to that difference formula. So, every term in my difference formula would have a round off error, while if I scale with 41 and like minus 20, I would have round off error only when, I divide the whole thing by 6 that is the idea.

This formula is useful in many situations, because it is remainder term, again this remainder term is a second order term, that is always going to be true but, in this case the remainder term is actually del 4, which is basically del square. So that is a nice form involves this square of. So, it is like this. So, your laplacian, this is your true laplacian this is your numerical representation of the laplacian and the error is of the order of the square of the laplacian.

(Refer Slide Time: 34:50)



That is where, this is a sort of nice, and because it gives the error is of the square of the order of the square of the laplacian. So, that was the rectangular domain everything is fine, we have a regular grid point. It is you can ignore to evaluate that you know as you know your difference operators, it is very nice.

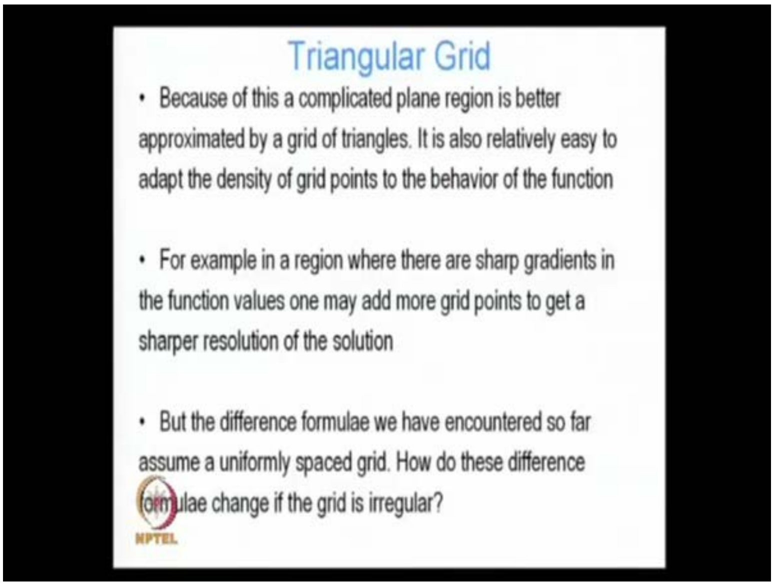
But, in most real world problems, you do not have rectangular domains, why do not you have rectangular domains well, because of two reasons. One reason is mostly, because your geometry is non-rectangular, you really have a rectangular geometry you may have an arbitrary geometry and it is very hard to divide it into rectangles. It is very easy to divide it into triangles but, it is very hard to divide it into rectangles. So, you are almost never going to have. It is a made up problem or it is a very simple problem, you neither going to have a rectangular grid but, any real domain it is almost always possible to divide it into a rectangular grid and then the part which I can where, I cannot have a rectangular grid. I can divide it into a triangle I can have a triangular grid there. So, it is very important that we be able to approximate our differential operators on a triangular grid.

And also the triangular grid can approximate the geometry on the boundary particularly if an irregular triangular grid is allowed. So, regular triangular grid again, it is restrictive but, if I have a free grid I can put points wherever I like. I do not have to put points in a particular fashion plus h minus h, I do not have to put points likes that if I put points arbitrarily then, it is much more easy for me to approximate the geometry. It is much more each for me to approximate the geometry and another reason why this, why the triangular while a free triangular grid is particularly useful, because consider a real world problem where, I have regions of sharp gradients where are my solution variable my u is not the same everywhere is not, I do not mean to say not the same everywhere but, it does not have it varies differently in different locations.

Where in some regions, you have a sharp gradient in u and in other regions u is more or less relatively flat, it does not, it varies very slowly, in other regions it varies very fast. So, in the region where u varies very fast where my function value of interest varies very fast, I would like to have more grid points there, because I know that function value is higher, that function is varying in a highly non-linear fashion. So, unless I have sufficient number of grid points in that location I would not be able to capture that highly non-linear variation. So, it is. So, with a regular grid point, it is very hard to capture all this things that non-linear regions of sharp gradient for instance in solid mechanics. One thing I have stress concentrations, we have a whole and the stress concentration around that whole. So, you know that, if you have to capture that you must have lots of grid points near the edge of the whole.

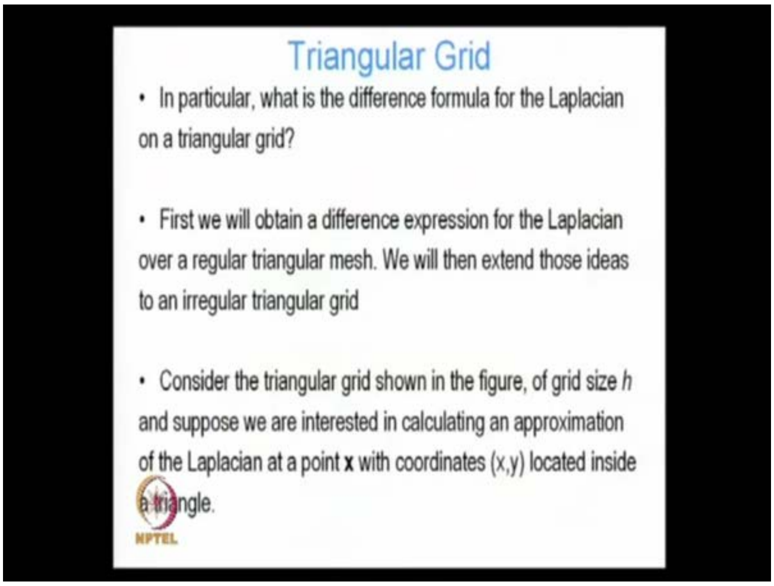
While far away from the whole where, this whether variable of interest for instance the displacement is varying relatively smoothly, you need not have that mini grid points. So, and if we have to be able to write these difference operators on, first of all in a triangular mesh number 2 on a irregular mesh and irregular grid, because of this a complicated plane region is better approximated by a grid of triangles.

(Refer Slide Time: 38:38)



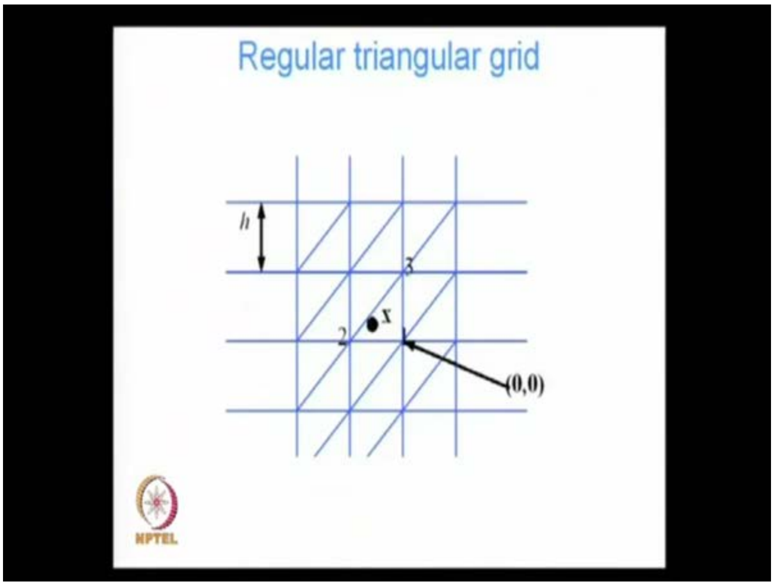
It is also relatively easy to adapt the density of grid points to the behavior of the function. What do I mean by that well, I mean exactly what I just said that, when you have a sharp gradient, it is you can introduce small grid points are there. So, for example, I just mentioned that to get a sharper resolution of the solution but, the difference formulae we have encountered. So, far assume a uniformly spaced grid how do these difference formulae change? If this grid is irregular, what will be the difference formula? if the grid is irregular.

(Refer Slide Time: 39:20)



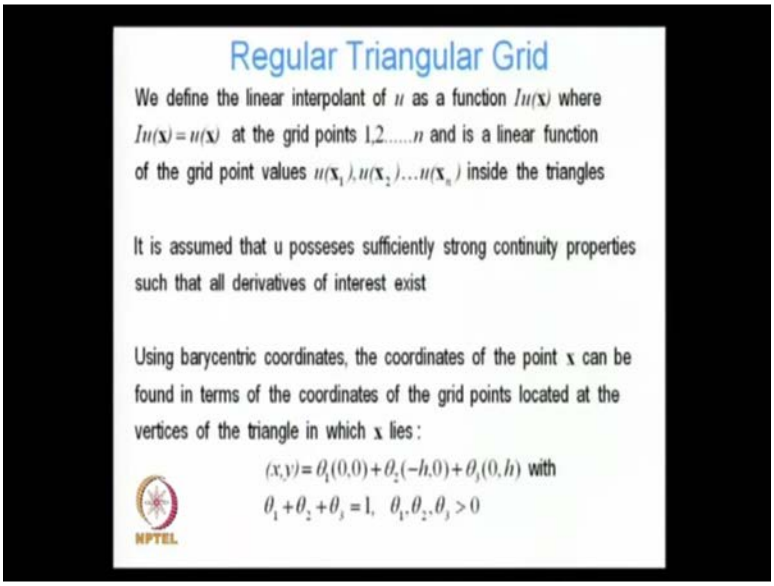
In particular since, the laplacian is the most interesting as is the most important difference formula for second order partial differential equations. What is the difference formula for a laplacian on a triangular grid? What is it on a regular triangular grid and what is it on an irregular triangular grid? So, we will first obtain a difference expression for the laplacian over a regular triangular mesh. We will then extend those ideas to an irregular triangular grid. So, I have use the word mesh and grid interchangeably though, they might have certain but, in this case we mean the same thing.

(Refer Slide Time: 40:04)



So, let us consider the triangular grid shown in this picture where, I have a grid of triangles and suppose, I want to evaluate my function value at x, I want to evaluate my function value at x calculated. I am interesting not only in the values; I am actually interest in evaluating the laplacian at x. I am x is located inside a particular triangle and the grid points. I am denoting as 1, 2 and 3. Now, in this case I have a regular triangular grid. So, my my size is h all my triangles have size h and I say that I assume that this is my origin this is the origin.

(Refer Slide Time: 40:50)

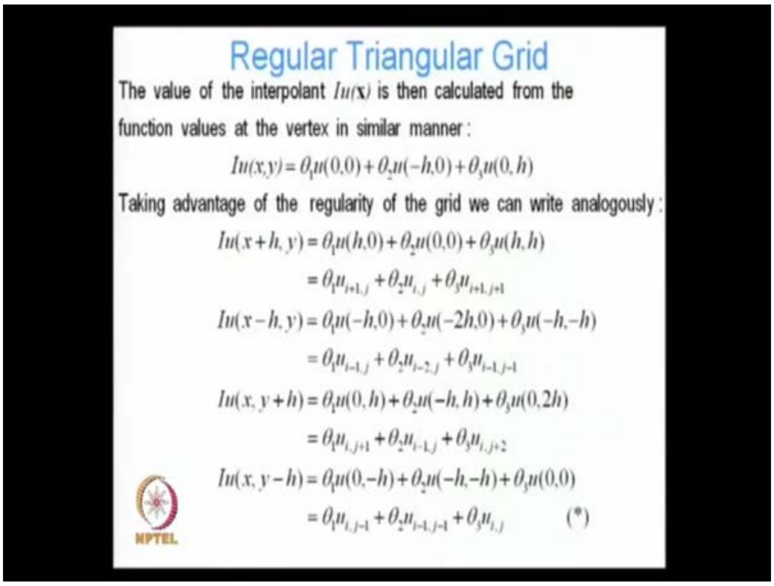


So, we define the linear interpolant of u as a function i u of x where i u of x is equal to u x at the grid points 1, 2 and 3 to n and is a linear function of the grid point values u x 1 u x 2 u x n inside the triangles. So, in order to get an expression for the laplacian first I have to define an interpolant, which tells me, what is the value of u at x at x, if I know the values of u at these grid point locations at 1, 2 and 3 if I know the values of u at 1, 2 and 3. I can use my interpolant to find the value of u at x.

And it is of course, we assume that u possess sufficiently strong continuity properties which will allow us to evaluate all which. So, that all these derivatives that we for instance for us the second order partial derivatives exist. So, using the barycentric coordinates, basically triangular coordinates like, you might be familiar with them with that name the coordinates of the point x can be found in terms of the coordinates of the grid points located at the vertices of the triangle in which x lies.

So, I can find out the coordinates of this point x in terms of the coordinates of these points 1, 2 and 3 by interpolating the coordinates. And I am saying that, those interpolant Theta 1 times, the coordinate at this location theta 1 time, the coordinate at this location plus theta 2 times, the coordinate at this location plus theta 3 times. the coordinate at this location is going to give me the coordinate of x and the condition is that theta 1 plus theta 2 plus theta 3 is equal to 1 and theta 1, theta 2, theta 3 are all positive. So, these are my barycentric coordinates theta 1 theta 2 and theta 3 are the barycentric coordinates of the point x.

(Refer Slide Time: 43:12)



So, the value of the interpolant i u x is then is calculated from the function values at the vertex in similar manner. So, I basically say i u x y I have. Basically, I am saying that if I want to evaluate u at x y x has got 2 parts, 2 components x and y. So, I want to evaluate u at x y.

I am going to do that by taking theta 1 times u 1, u at 1 which is u at 00 plus theta 2 at u at 2 which is equal to if this is 00, and that must be minus h 0 and this at 3, this must be 0 h. So, I am evaluating u at x y by theta 1 u 00 plus theta 2 u at minus h 0 plus theta 3 u at h at zero h. So, this is how I am going to evaluate my interpolants and taking advantage of the regularity of the grid. We can write analogously i u x plus h y. So, u at x y is given by that.

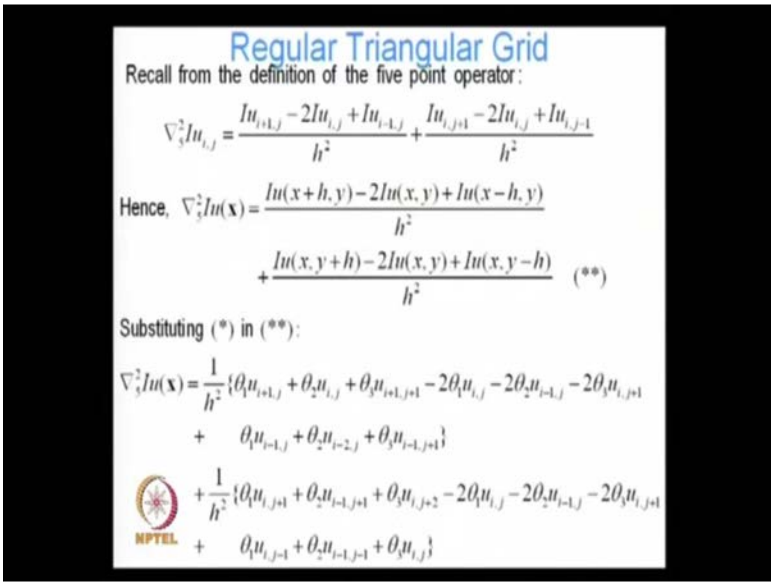
Now, I am interested in evaluating u at x plus h y where, would x plus h y be x plus h y would be exactly in this triangle. It would be x plus h in the x direction. So, x here it is x y the coordinates the cartesian coordinates of that point has small x and small y. So, if I am looking at a, if I am looking for a point with cartesian coordinates small x plus h y, that would be somewhere in that triangle just h from this point along the x direction but, what do you notice about that point, that point lies in this triangle. So, the barycentric coordinates of that point have to be theta 1, theta 2 and theta 3 because of the regularity of the grid. If I am looking at this point, that point has barycentric coordinates theta 1, theta 2, theta 3 if I am looking at the mirror image of that point in this triangle which is x plus h y then, the barycentric coordinates of that point here are again going to be theta 1, theta 2 and theta 3 but, now it will involve. Now, let us go back and take a look. So since I. So, because of the regularity of the grid, we can write i u x plus h y is equal to theta 1 u h zero earlier on. It was u at 0, 0. Now, we are evaluating u at h 0, because everything has shifted by h in the x direction, so u at h 0 plus theta 2 u at 0, 0.

So that involves this point theta 2 u at 0 and theta 3 at this point which is actually, h h which is at h h. So, theta 3 u at h h but, you can see, because of the regularity of the grid. We can still use theta 1 theta 2 theta 3, because the barycentric coordinates have not changed of that new point are the same as for the whole point. Similarly, i u x minus h y I can, this means I have to shift 1 this h to the left. So, I am going to move to this triangle. I am going to move to this triangle and x x would be somewhere here. So, it x would be somewhere here.

And that would involve theta 1 u minus h 0, theta 2 u minus 2 h 0, theta 3 u minus h minus h. So, if I write it in terms of indexes indices, I have u i minus 1 j this is u i minus 2 j, because I have moved negative 2 in the x direction but, I have not moved in the y direction. So, that remains j and here, it is I have moved i 1 in the x direction and 1 in the y direction. So, I have u i minus 1 j minus 1.

Similarly, I can write i u x y plus h like this using the same same sort of argument now in this case I am looking at a point here. I am looking at a point here, and then i y minus h when, I am looking at a point in this triangle. So, in that case I get something like that. So, I can write these i u x plus h y i u x minus h y i u x y plus h and i u x y minus h in terms of this grid point values scaled by my barycentric coordinates but, you note that, we are using the barycentric coordinates of my original point x. I am using the barycentric coordinates of the point x at which, I want to evaluate the laplacian. I do not have to use recalculate the barycentric coordinates, because of the regularity of the grid.

(Refer Slide Time: 49:00)

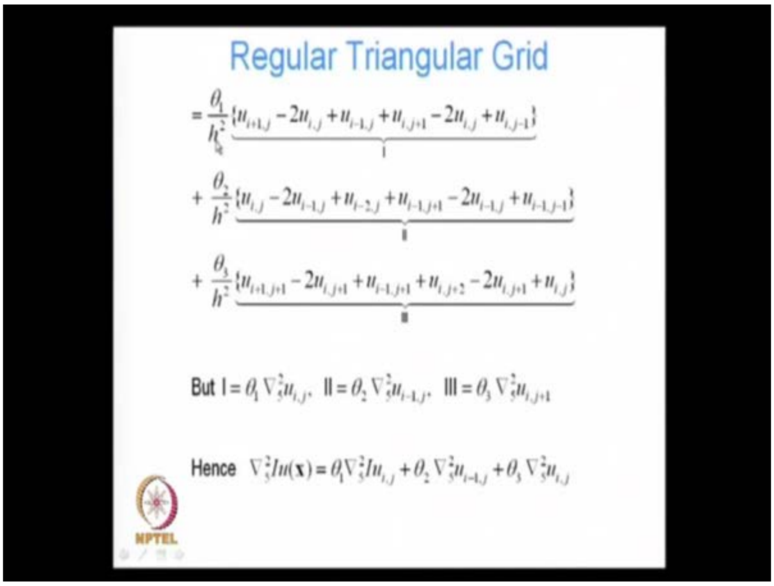


So, now what do I get. So, again let us go back to our definition of the 5 point difference operator, which was del 2, 5 u i j is equal to u i plus 1 j u i j u i minus 1 j and so on and so forth. Now, I am saying that del 2, 5 i u i j is equal to i u i plus 1 j minus 2 i u i j plus u i u i minus 1 j plus this. So, we have introduced this i, because I am going to now evaluate the laplacian on the interpolant.

So, hence I can say that del 2 5 i u i u x is equal to i u i plus 1 j I can write as i u x plus h y x index changes by 1 y index remains the same minus 2 i u x y plus i u x minus h y there and here I have i u i j plus 1. So, i u x y plus h minus 2 i u x y plus i u x y minus h by x square h square And then I substitute my previously evaluated expressions where, i u x plus h y i u x minus h y i u x y plus h i u x y minus h in this expression.

And therefore, I get as an expression like this involving u y plus 1 j u i j and so on. I get something like this and then what I do? Well i decide to pull out all terms with theta 1, I pull out all terms with theta 2 together and I pull out all terms with theta 3 together then, what do I get.

(Refer Slide Time: 50:56)

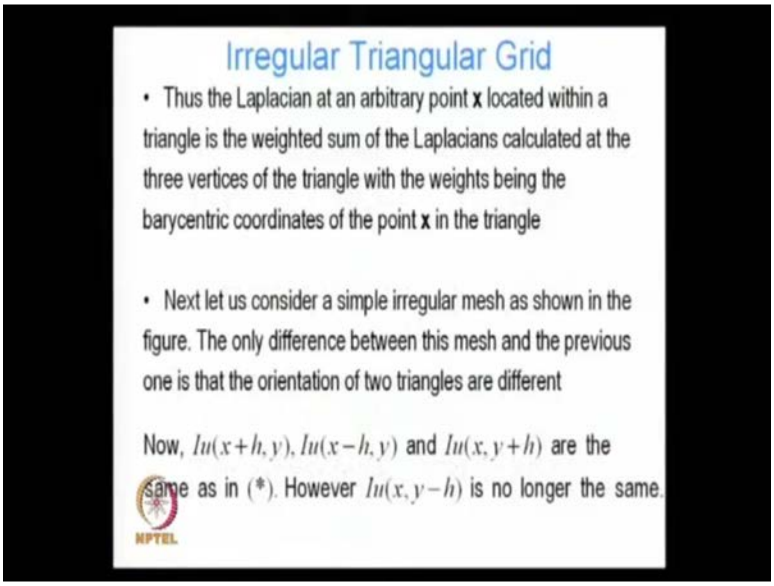


Then, I get something like this. Theta 1 plus h square times this theta 2 by h square times, that theta 3 by h square times this and what are these? This term within curly brackets here is nothing but, theta 1 times del 2 5 u i j del 2 5 u i j you can see, u i plus 1 j minus 2 u i j plus u i minus 1 j plus u i j plus 1 minus 2 u i j plus u i j minus one. So, we have seen, that is equal to del 2 5 j what about that term? Well that is nothing but, del 2 5 u i minus 1 j, because this is i. So, you replace here i by i minus 1 and you will get that. So, if i replace i by i minus 1 here, I am going to get this. So, i by i minus 1 i by i minus 1 i get i minus 2 j j j does not change i. So, only I get that i replaced by i minus 1 i by i minus 1.

So that is nothing but, my laplacian evaluated at u minus 1 j and if you look at this is nothing but, i u i j plus 1 where I have replace this j by j plus 1. So, what do I get. So, I get del 2 5 i u x the laplacian evaluated at x is equal to theta 1 times. The laplacian evaluated at i j plus theta 2 times the laplacian evaluated at i minus 1 j plus theta 3 times the laplacian evaluated at i. That is a mistake i j plus 1, that is the laplacian evaluated at i j plus 1 but, what is i j i minus 1 j and i j these are represent the points on the triangle they just represent these points.

They just represent these points, this point and that point. So, finally, what do I get? Finally, I get that my laplacian evaluated at an integral at an internal point is equal to the laplacian evaluated at my grid points scaled by my barycentric operator of the point, at which I want to evaluate the laplacian scaled by the barycentric point of the barycentric coordinates of the point, at which I want to evaluate the laplacian.

(Refer Slide Time: 53:41)



Thus, the laplacian at an arbitrary point x located within a triangle is the weighted sum of the laplacians calculated at the 3 vertices of the triangle the weights being the barycentric coordinates of the point x in the triangle. So, next I want to consider an irregular grid, I want to evaluate the laplacian on an irregular grid and I want to show you how to do that. I will do at for a simple case because for a complicated case, it is hard to explain but, once you understand it for a simple case, it is possible to extend it for complicated cases as well but, since we are out of time, we have to do that next lecture. Thank you.