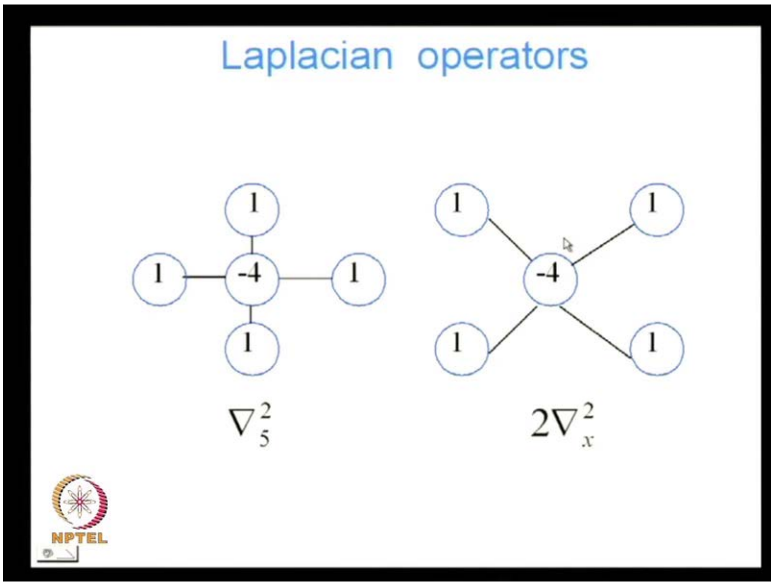
Numerical Methods in Civil Engineering

Prof. Arghya Deb
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 30
Interpolation

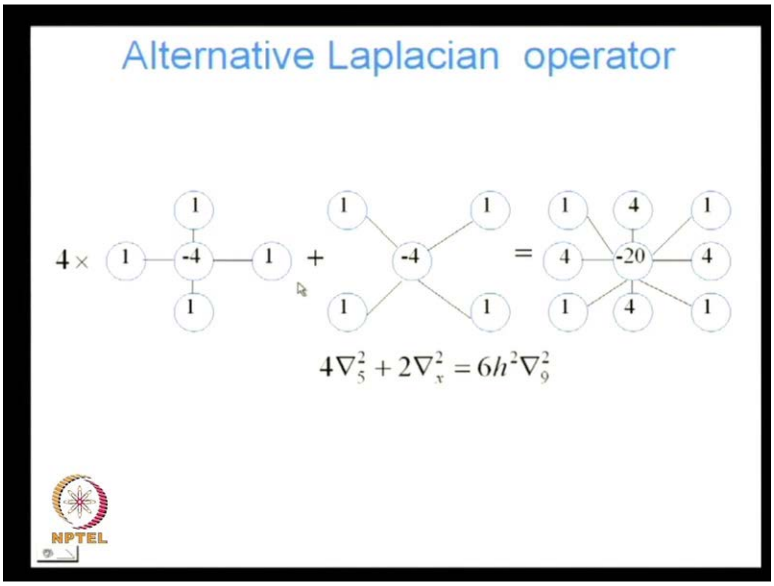
(Refer Slide Time: 00:26)



In lecture, thirty of our series on numerical methods in civil engineering; we are going to talk about interpolation. In the previous lecture, I talked about various difference operators and I said how you are you can approximate various differential expressions, which appear in the partial differential equations, which commonly occur in civil engineering.

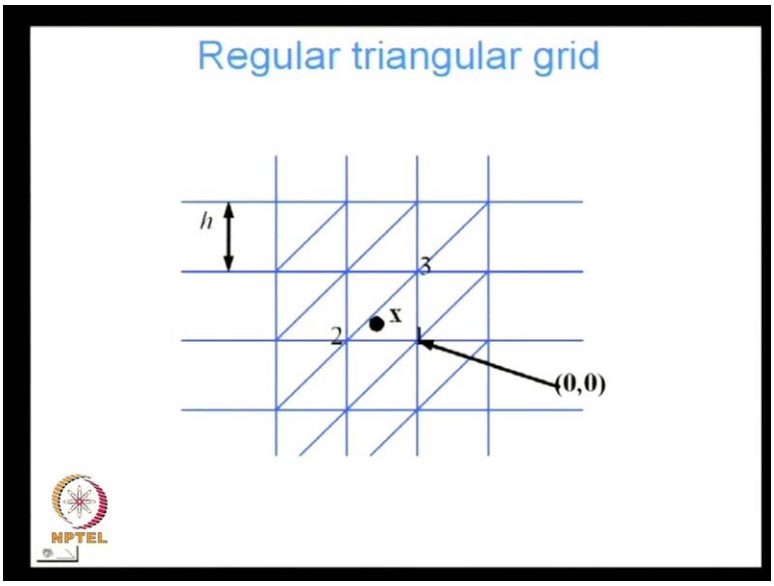
And, we saw how we can use difference expressions to approximate those differential operators and one of the particular differential operators that we talked about was a Laplacian operator, which as we saw earlier occurs in almost all the major equations of interest partial differential equations of interest in civil engineering, the wave equation; the diffusion equation as well as the heat equation. So, the Laplacian operator that we talked about was the first operator, which is on the left hand side which is centred and with and the other operator that we talked about was the diagonal Laplacian operator.

(Refer Slide Time: 01:30)



And, I also mention that you can construct a composite operator by using, weighted by; weighing each of these different approximations to the Laplacian operator to come up with a composite operator. For instance, by multiplying the first operator that we solved with 4 and adding to that the second operator; we get this operate and you can see this is 6 times a 6 h square times Laplacian. So, as I said as I told you that it is always better to add them together and then divided by 6 to get the Laplacian operator rather than dividing it because that is going to accumulate round off.

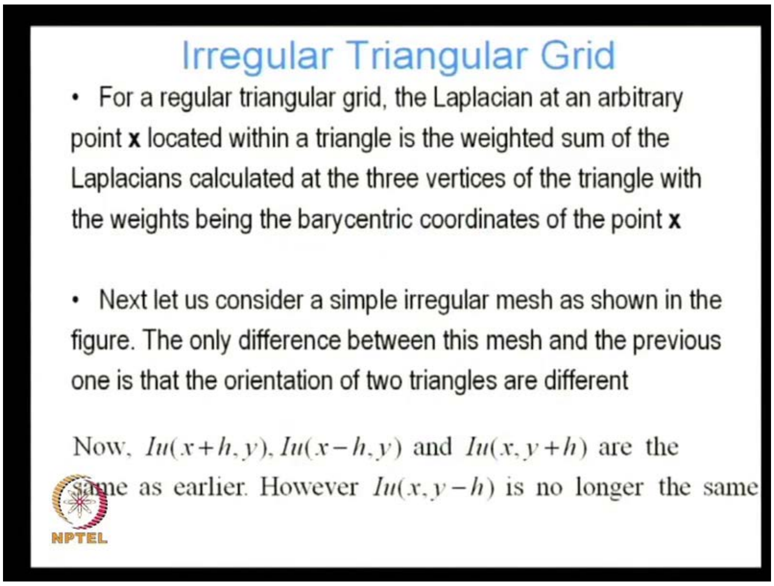
(Refer Slide Time: 02:12)



And, we also looked at; how the how we can. So, we looked at the Laplacian operator on a rectangular grid, how we can approximate the Laplacian operator in a rectangular grid? And, we also looked at; how we can approximate the Laplacian operator on a regular triangular grid. for instance, I have a triangular grid like this with each triangle of side h right side h and if I want to find out what is the Laplacian at this point x within a triangle. How can I use my difference expressions to find an expression for the Laplacian at this point we saw that.

But when it comes to irregular triangular grids that become a problem as I. And, as I mentioned it is in real world problems, it is almost always the case that the grid is irregular. First of all, because the first reason being the geometry is irregular; number one and number two the reason why we want to use triangular grids is because we want to do free meshing right in a particular region if we have sharp gradients we would like to have a finer mesh in that region. So, in that case it is inevitable that we are going to end up with irregular triangular grids.

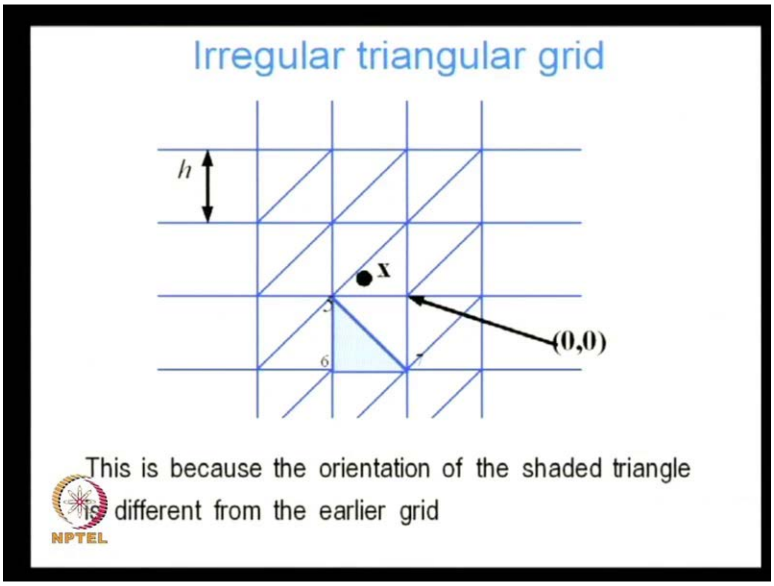
(Refer Slide Time: 03:29)



So, what does the Laplacian operator look like when we consider an irregular triangular grid. For a regular grid, the Laplacian at an arbitrary point x located within the triangle is we saw is the weighted sum of the Laplacians calculated at the three vertices of the triangles right. So, in order to find the Laplacian at this point, what we did was that we form the Laplacian here we form the Laplacian here and we form the Laplacian here and then we use the barycentric coordinate of this point; the triangular coordinate of this point to weight the Laplacians evaluated at each of those points to find out the value of the Laplacian at this point.

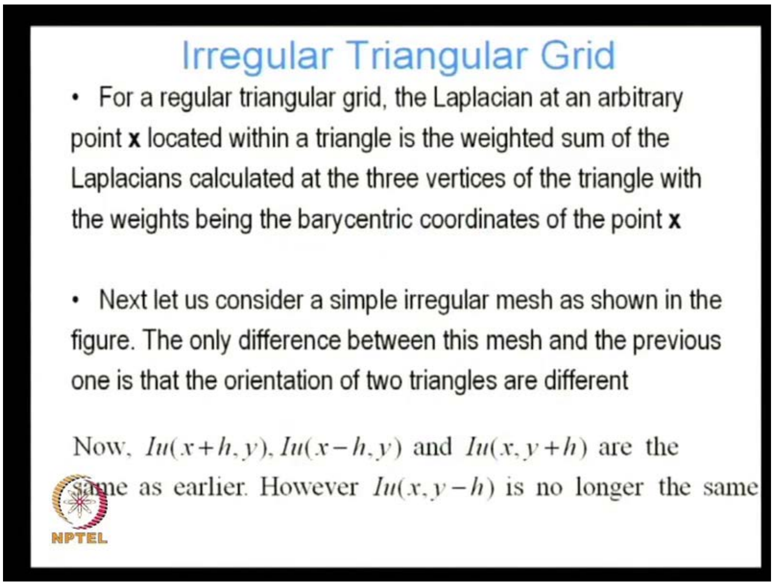
Instead of considering a regular triangular mesh we want to consider an irregular mesh right and the irregular mesh that we are going to look at is a very simple irregular mesh in that it differs from a regular mesh by only a small perturbation. But, if conceptually we understand how we have we evaluate the laplacian operator at such an irregular mesh, we can extend the idea to arbitrary irregular meshes.

(Refer Slide Time: 04:43)



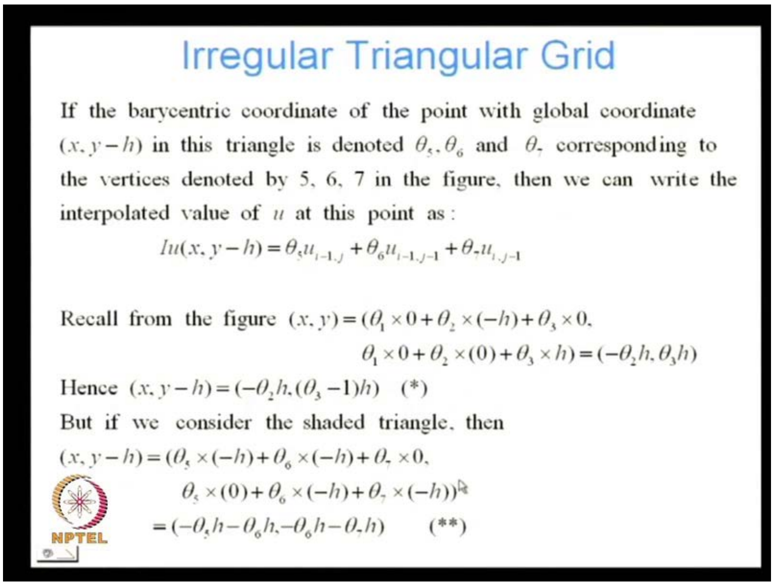
So, the irregular mesh we will consider is something like this, where it is very similar to the previous mesh except that you can see the orientation of these two triangles is different from the rest of the triangles. So, we have changed the orientation of these two triangles but kept the orientation of the reset of the triangles the same.

(Refer Slide Time: 05:00)



So, two triangles are different; so, if we recall this Iu x h plus y we encounter that in our previous lecture .it is the interpolant right. It is the interpolated value of u evaluated at x plus h y what is x plus h y well x plus h if this is x right x plus h would be somewhere here. So, Iu x plus h, y Iu x minus h y, Iu x y plus h are the same as earlier. Why lets go back and take a look; so, x plus h the triangle the shape of the triangle is the same. So, that remains the same x minus h somewhere here, it refers to this triangle. So, that triangle is the same, x if we look up there right x plus x I mean y direction plus h that triangle is also the same. So, the triangle that is different is going to be x y minus h because that refers to this triangle, which as you can see has got a different orientation than the previous triangles right.

(Refer Slide Time: 06:22)



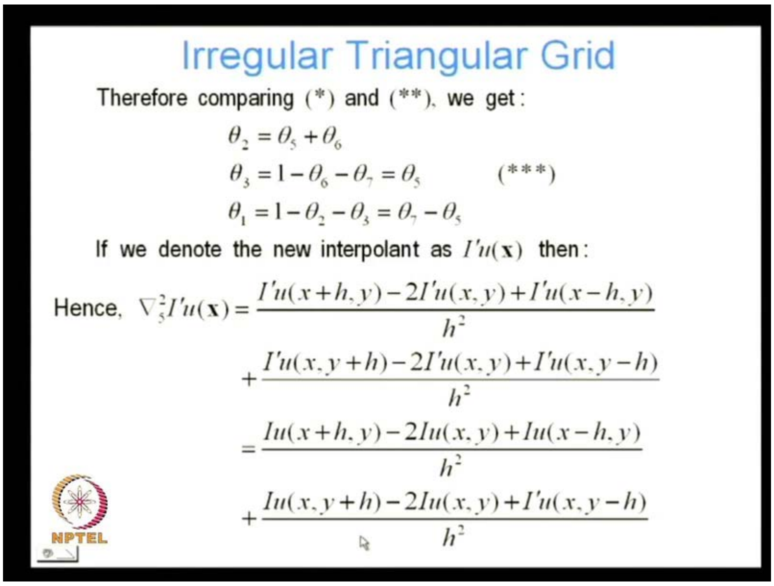
So, these are the same as earlier; however, this is no longer the same. if the barycentric coordinate of the point with global coordinate x y minus h in this triangle is denoted. So, what we are saying is that we this is the triangle of interest right. So, if we denote these nodes as 5 6 and 7 then we look at x y minus h, which lies somewhere here right x y minus h is denoted by theta 5, theta 6, theta 7 then we can write the interpolated value of u at this point as theta 5 u minus 1 j plus theta 6 u minus 1 j minus 1 plus theta 7 u i j minus one.

Lets go back again and take a look. So, it is theta 5 times u I minus 1 j right plus theta 6 u I minus 1 j minus 1. And, this is theta 7 u I j minus 1 right. So, it is the interpolated values of all these three values right. So, I u x y minus h is equal to theta 5 u I minus 1 j plus theta 6 u I minus 1 j minus 1 plus theta 7 u I j minus 1. So, let us go back to that figure and look at x and y look at this point right with coordinates x and y. So, we can write x and y as x and y is equal to theta 1; x is actually theta 1 time 0 plus theta 2 into minus h plus theta 3 into 0, why lets go back and take a look. So, if I want to find out x it is while this is my first point theta 1 it has got 0; x coordinate 0.

This point has got x coordinate minus h that point has got x coordinate 0. So, it is theta 1 times 0 plus theta 2 times minus h plus theta 3 into 0. Similarly y, the y coordinate of this point is equal to theta 1 times 0 plus theta 2 times 0 plus theta 3 times h. So, that is what I have just written here right and if I do this simplifies this I get minus theta 2 h theta 3 h that I know is the barycentric coordinate using the barycentric coordinate the x y coordinates can be written like this. So, from here x y minus h I can write as minus theta 2 h that part remains the same because the x coordinate is remaining the same, the y coordinate is decreased by h. So, that becomes theta 3 h minus 1; so, theta 3 minus 1 h.

But, if we consider the shaded triangle; if we consider this triangle again right and we look at x y minus h then in that case that is equal to theta 5 times minus h plus theta 6 times minus h plus theta 7 into 0. It is theta 5 times minus h; theta 6 minus h theta 7 0.And the y-th coordinate is equal to theta 5 into 0, theta 6 into minus h, theta 7 into minus h. So, that is what we get and if I simplify that that becomes minus theta 5 h minus theta 6 h minus theta 6 h minus theta 7 h but these two have got to be equal right you cannot have x y minus h different depending on which triangle you use it to calculate it right. So, these two have to be the same, So, this gives me a relation between theta 2, theta 5 and theta 6 and theta 3 minus 1 and theta 6 and theta 7.

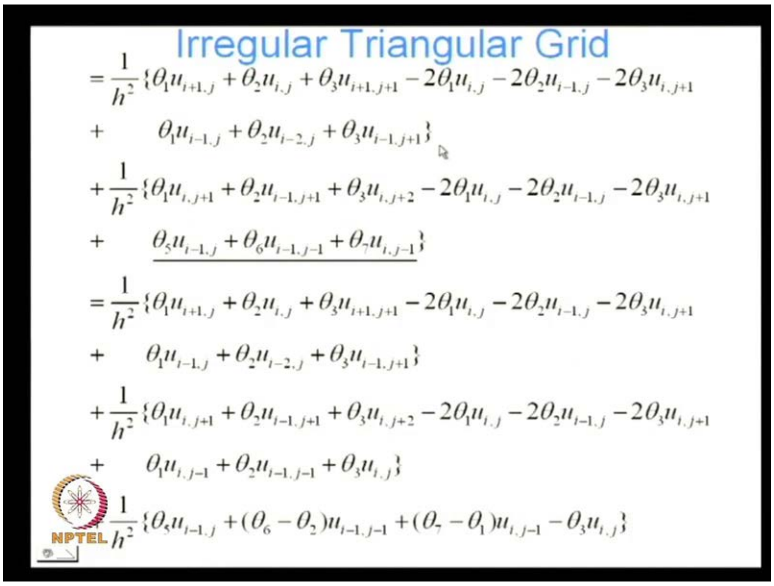
(Refer Slide Time: 10:32)



So, what do I get I get theta 2 is equal to theta 5 plus theta 6, theta 3 is equal to 1 minus theta 6 minus theta 7, which is equal to theta 5. Since, all the barycentric coordinates must sum up to 1 right and theta 1 is equal to 1 minus theta 2 minus theta 3 which pulling these two together I get theta 7 minus theta 5. So, now if I denote the new interpolant as I prime u of x. Then Laplacian of I prime u of x using R operator, which we have seen earlier can be given like this. This we have seen before right this we have seen before I have. So, this is I prime u x plus h y minus 2 I prime u x y plus I prime u x minus h y that is in the x direction and this is in the y direction. x y plus h x y, x y minus h right.

So, that remains the same I prime u x h plus y is equal to I u x h plus y that I already know; u x y I prime u x y is equal to I u x y I prime u x minus h y is remains the same. This part remains the same as the previous interpolant that only thing that is going to change is this one that we have seen earlier right. So, this I can write as this, where this I calculated using my regular grid right I calculated using the regular grid. So, the only part that is changing is this part.

(Refer Slide Time: 12:11)

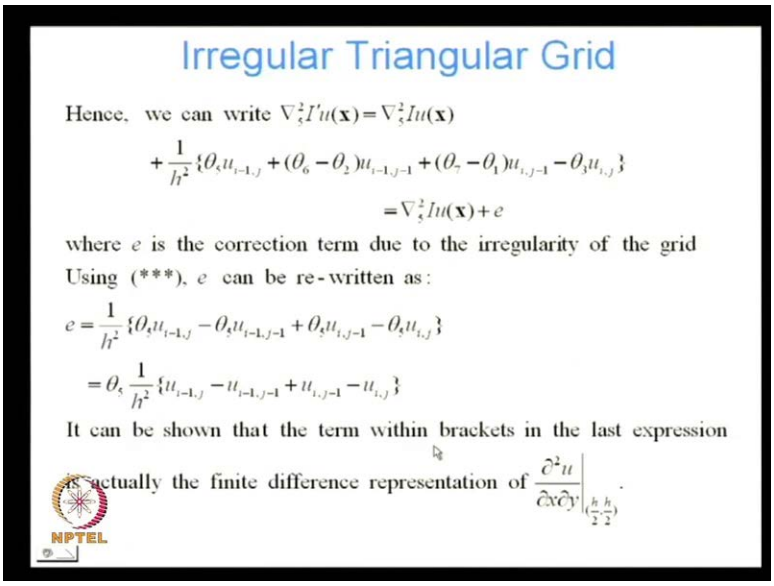


So, again this is just the expression for that rights this we have seen earlier writing it in terms of these quantities. And, now the only thing that has changed is this term right this term which I can write as from my previous expression and now this one this expression right I can use that expression to write this as like this right I can write it like this. So, this is; this part remains the same this is practically identical where then what I have done is that instead of writing this I have used the previous expression right.

I have used the previous expression and then I have, but this plus this must be equal to that right this plus this must be equal to that. So, this is the previous expression, which I got using my regular grid right this I got using my regular grid and then I have to correct it correct it to get that expression. So, I have to add to it this term right I have to add to it that term.

So, this is this is what I got using the regular grid right, but now I know that this term is different. So, I have you can see, that if I subtract theta 1 theta 2 and theta 3 are going to cancel out if I add these two terms together and I am going to get theta 5 u minus 1 j, plus theta 6 u minus 1 j minus 1 plus theta 7 u i j minus 1 these two these three terms are going to cancel out right. So, the advantage, why I want to do this because I want to write it as my previous expression my previous expression.

(Refer Slide Time: 14:02)



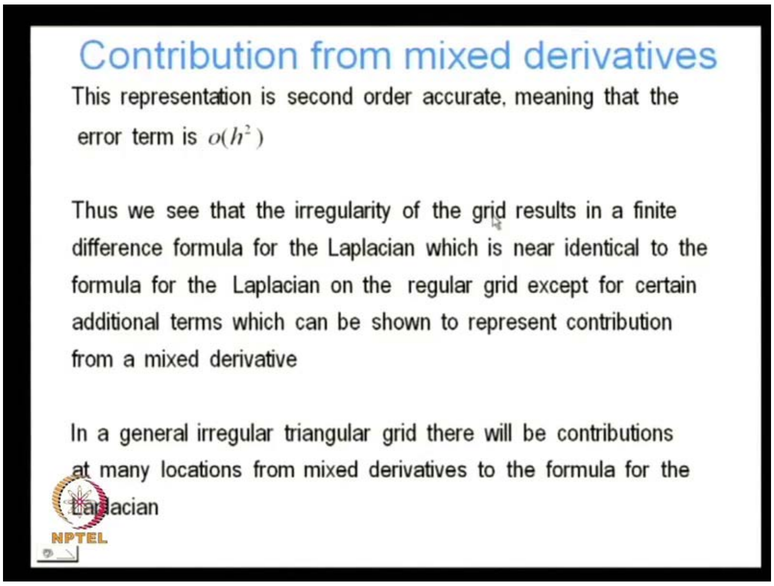
So, we can write del 2 5 I prime u of x del 2 5 using the new interpolant is equal to del 2 5 using the old interpolant plus this correction term right. Let us go back again. So, this is my del 2 5 using the old interpolant up to here right and this is my correction term right. So, that is my correction term. So, I can write it as this is evaluated using my regular grid plus a correction term, which is due to the irregularity of the grid right. this is due to the irregularity of the grid; So, using these expressions right.

There which I obtained between theta 2 theta 3, 3 theta 1 and theta 5 theta 6 and theta 7. I can write this e write this expression in like this right basically I. I can write everything in terms of theta 5 why well theta 5 u minus 1 j is theta 5 that remains the same; theta 6 minus theta 2 I can write using that expression theta 6 minus theta 2 is minus theta 5 right. So, that gives me minus theta 5 here instead of theta 6 minus theta 2 then I have theta 7 minus theta 1, which I can write using this expression theta 7 minus theta 1 equal to theta 5 right. So, that I can again write as theta 5 u i j minus 1 right and then I have theta 3, which again I can write straight away as theta 5. I can write straight away as theta 5.

So, I can get everything in terms of theta 5 and u minus 1 j you this that and I pull out theta 5 and I am left with that term right. So, my new Laplacian on the irregular grid is the old using the regular grid plus an error term and the error term can be represented by theta 5 plus this. Now, this term it can be shown that is actually a finite difference expression for the mixed derivative Right, up till we are considering the Laplacian, but now you can see that when we move to an irregular grid we also the it also involves the finite difference approximations for the mixed derivative.

So, we are evaluating the Laplacian, but since the grid is irregular we have this additional term, which is actually a finite difference approximation of the mixed derivative right. So, this representation of the mixed derivative; this representation of the mixed derivative is second order accurate. So, the error is also second order due to the irregular grid

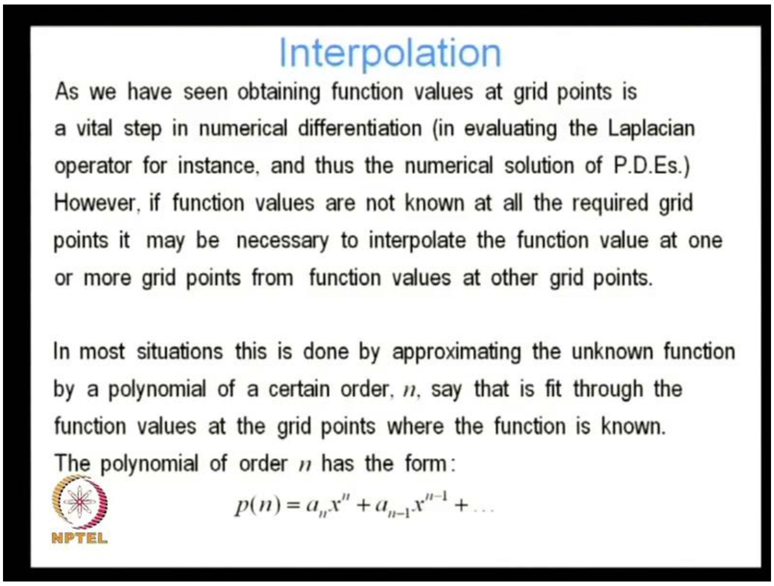
(Refer Slide Time: 17:09)



Thus we can see that the irregularity of the grid results in a finite difference formula for the Laplacian, which is near identical to the formula for the Laplacian and the regular grid except, for certain additional terms which can be shown to represent contributions from a mixed derivative right. So, this is for a really simple irregular grid but as you can understand in a real irregular grid; in a general irregular grid there will be contributions that many locations from mixed derivatives to the formula for the Laplacian right. There will also contributions to the mixed derivates that. So, if you evaluate the Laplacian using a regular grid, you have to add additional corrections for the fact that the grid is irregular and those correction terms will involve finite difference approximations for the mixed derivatives.

So, although we are evaluating the Laplacian right, which does not involve any mixed derivatives right. Our finite difference approximation is going to involve mixed derivatives, if we have an irregular triangular grid. Right, this is just to give you some idea of these operators.

(Refer Slide Time: 18:26)



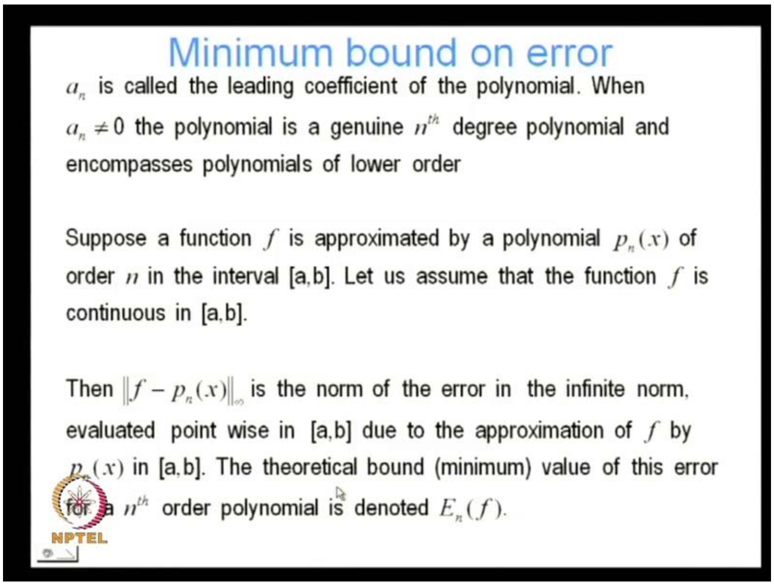
So, there are many, many complex operators and for irregular grids, but the fundamental principles are the same. So, we have looked at finite difference operators for rectangular grid right. We have looked at finite difference operators on a regular triangular grid and I have given you some idea of what the complications might be if the grid is irregular. So, as we have seen in order to get these finite difference expressions finite difference approximations for the difference operators what we need is the values of the function at many many grid points; at several grid points right.

So, now if we do not know it is possible that we might need to for instance if we adaptively refine a mesh right if we if adaptively refine a grid. That is we add grid points at locations, where of interest right; where we have sharp gradients of the solution varies very fast and things like that right when since if you have; if you are trying to find out stress concentrations right. we might want to add grid points at those locations right. So, if you do not we might not know the function values of those grid points. suppose ,we know the function values only at a certain number of grid points but then as we adaptively refine the grid, we need in order to we have we have to add know the values at those grid points in order to find my find in order to formulate my finite difference operators right.

So, how do you? So, in that case we have to find the function values at those new grid points which we just created from my old grid and how do I find the values function values at those new grid points. Well, I interpolate right I use the values at my old grid points right and try to fit a polynomial and then try to interpolate the function values at the new grid points. So, that is what we do. So, if function values are not known at all the required grid points, it may be necessary to interpolate the function value at one or more grid points from function values at other grid points. In most situations, this is done by approximating the unknown function by a polynomial of a certain order; n say, that is fit through the function values at the grid points where the function is known.

The order of the polynomial the maximum order of the polynomial depends on the number of grid points at which we know the function values. The polynomial of order n has the form, this is the general form of a polynomial of order n a n x to the power n a n minus 1 x n minus 1 and So on..

(Refer Slide Time: 21:12)



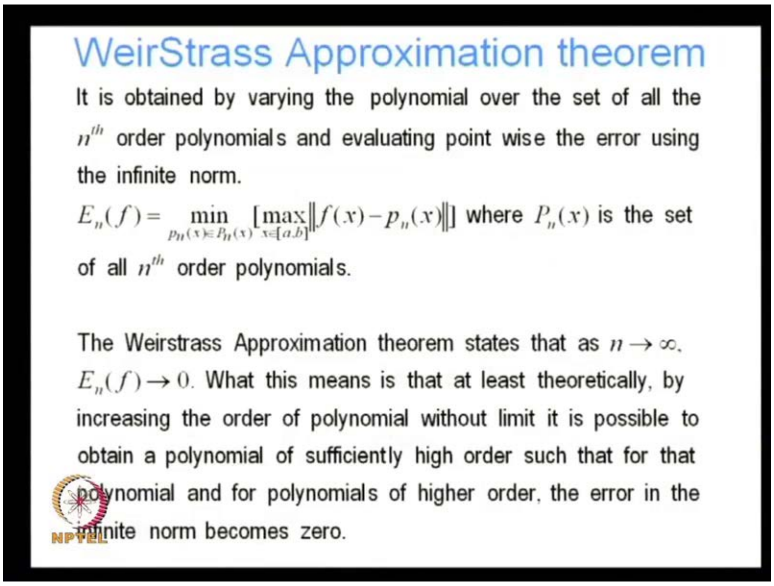
And So forth where this term is known as the leading order term of the polynomial n is called the leading coefficient of the polynomial, when a n is not equal to 0 the polynomial is a genuine nth degree polynomial and encompasses polynomials of lower order. Suppose a function, f is approximated by a polynomial p n x of order n in the interval a b right.

So, we have an interval a b in that interval I am going to I want to approximate a function f of x using that polynomial that nth order polynomial. And let us assume that the function f is continuous in that interval a b define a norm f minus p n of x in the infinite norm is the norm of the error in the infinite norm. So, it tells me how far of my polynomial approximation is from my given function at each point in that interval right. And, I am going to calculate that the infinite norms that is the largest difference over that interval right.

So, evaluated point wise in a b due to the approximation of f by p n x in a b right. the theoretical bound of this error for nth order polynomial is going to be denoted as E n f right. So, E n f is that is a bound on the error basically I want to approximate that function f x over that interval a b using a polynomial and I am of order n and E n f gives me the smallest value of the maximum error in the as I use different polynomials right I will try to find out what is the maximum error due to that polynomial in that intervals. So, I fit that polynomial in that interval and I look at each point within that interval find out what is the error at each point and then find out the point. where the error is the maximum right. And, then I loop over all the set of polynomials of nth order right.

So, if I have. So, if I am considering cubic I look at all the possible cubics and find out which cubic gives me the smallest error right; the smallest error right.

(Refer Slide Time: 23:31)

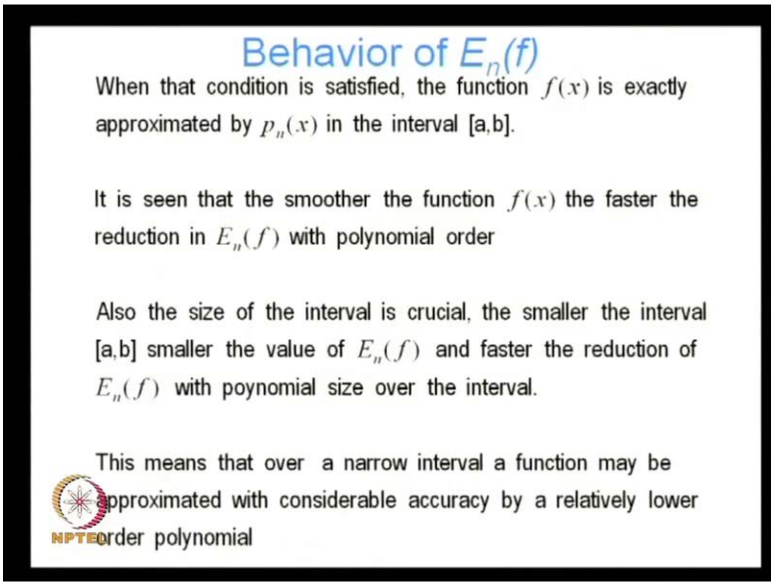


So, it is obtained by varying the polynomial over the set of all nth order polynomials and evaluating point wise the error using the infinite norms. So, what do I do I find out f x minus p n x the maximum value of this difference over all the points in that interval and then find out do this over all the polynomials all possible polynomials of order n and when that error is minimum that is my E n of f right is that clear??

So, the then there is a theorem which is called the Weirstrass approximation theorem, which states that as n goes to infinity E n f goes to 0. So, what it says that if my if the order of my polynomial becomes infinitely large, then this error is going to become 0. The error defined like this right the error defined like this is going to go to 0, what this means is that at least theoretically by increasing the order of polynomials without limit, it is possible to obtain a polynomial of sufficiently high order such that for that polynomial and for polynomials of higher order the error in this norm becomes 0.

Basically, the function is the polynomial is exactly equal to the function because it is the maximum difference between the function and the polynomial value at any point in that interval is 0 right. So, the function becomes exactly at the polynomial becomes exactly equal to that function. So, that is what the weierstrass approximation theorem states.

(Refer Slide Time: 25:21)

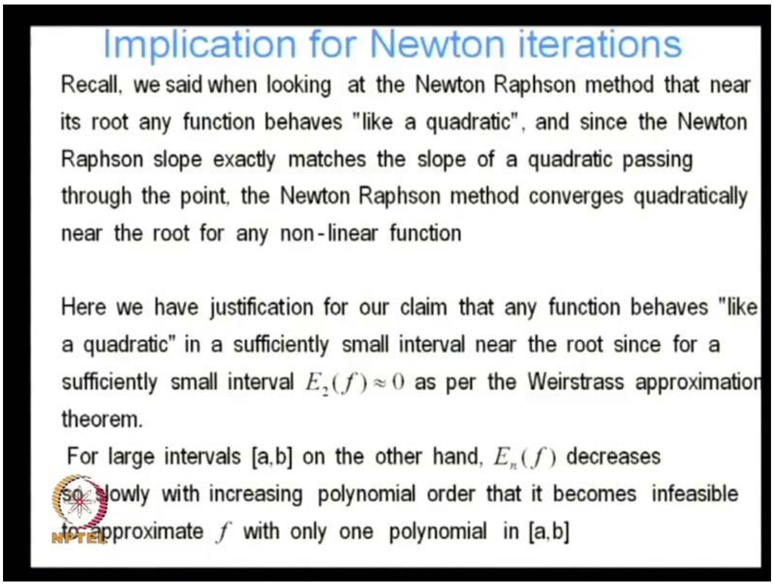


So, when that condition is satisfied the function f of x is exactly approximated by p n of x in the interval a to b. And, it has been seen that the smoother the function of f the function f; the faster the reduction E n f with polynomial order

So, if my original function is smooth right if it has got more derivative derivatives, which are continuous up to a sufficiently high order then it can be approximated the number of orders we need to go right number of higher order polynomials; we need to take to exactly approximate that function is relatively low right. So, that the number the n the order of the polynomial we need to use to get E n f sufficiently small, where in your 0 is dependent on the smoothness of my original function. It also depends on the size of the interval over, which I am trying to approximate f of x; if I am trying to approximate f of x over a very large interval then I have to take even very high order polynomials the n has to be much much higher.

But, on the other hand if I am trying to approximate f of x over a relatively small interval I can do that with a relatively lower order polynomial even as relatively lower order polynomial will give me E n f sufficiently small right. So, if the size of the interval is crucial the smaller the interval a b smaller the value of E n of f and faster the reduction of E n of f with polynomial size over the interval right. This means that over a narrow interval a function may be approximated with considerable accuracy by a relatively lower order polynomial right. And, this as I want to draw this I want to draw a connection between this result and what we have seen earlier for the Newton Raphson method; what did we see for the Newton Raphson method well.

(Refer Slide Time: 27:28)

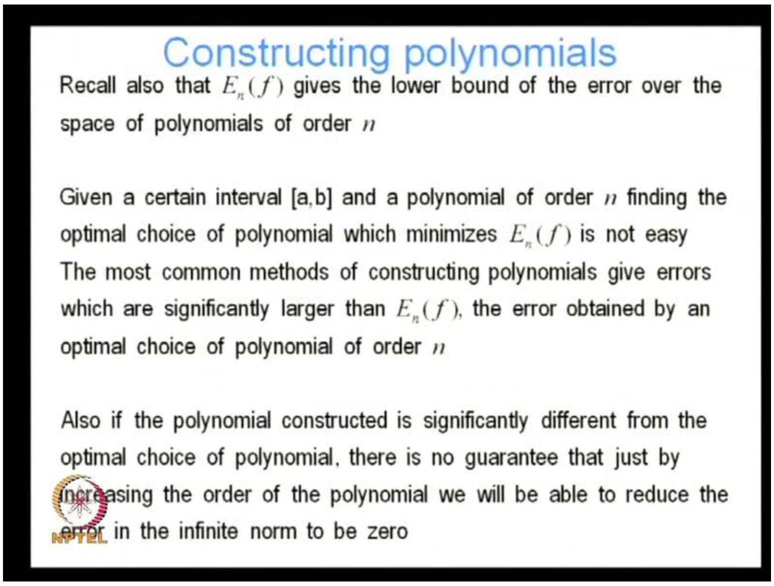


When we looked at the Newton Raphson method, we said that near it is root why does the Newton Raphson method work. Well, I said it that works because near it is root any function any non-linear function behaves like a quadratic right, I said it behaves like a quadratic and since the Newton Raphson slope exactly matches the slope of a quadratic passing through that point. The Newton Raphson method converges quadratically near the root of any non-linear function. So, near the root of a non-linear function I said the function behaves like a very similar to a quadratic and that is why the Newton Raphson method converges quadratically.

Now, here we have a justification for our claim that any function behaves like a quadratic near the root because in a sufficiently small interval centred around the root right. That function, my weirstrass approximation theorem states that function can smaller the interval the lower the order of polynomial by which that function can be approximated exactly. So, that is why in a sufficiently small interval near the root E 2 of f is approximately equal to 0 because as the interval becomes smaller and smaller I can use lower and lower order of polynomials to approximate the function.

So, near the root E 2 of f approximately equal to 0 as per the weirstrass approximation theorem and that is why a Newton Raphson method gives quadratic convergence. Any function behaves like a quadratic at sufficiently close to the root. For a large intervals, a b on the other hand E n of f decreases. So, slowly as you increase the order of the polynomial that it becomes infeasible, you cannot approximate a function with only one polynomial in an interval right. So, if I have a large interval then even if you use a very high order polynomial you it is it the error E n f is not it does not decrease enough right.

(Refer Slide Time: 29:45)



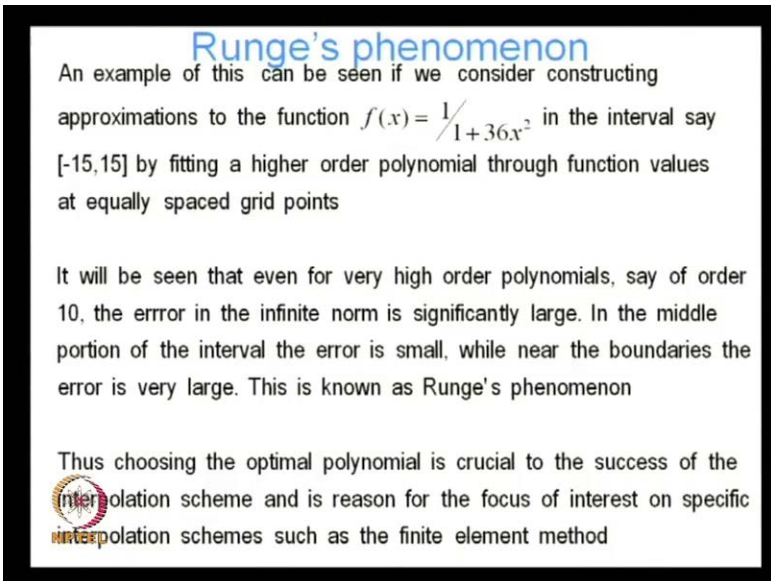
So, we are going to see an example of that slightly later on I am not sure where. So, recall also that E n f gives the lower bound of the error over a space of polynomials of order n. Now, what this means that for in order to find E n f we have to try all the polynomials of a certain order right and we have to find the optimum polynomial the polynomial which gives me the lowest error in maximum norm right. It gives me the lowest error in the maximum that is practically impossible right, given a certain interval a v a b and a polynomial of order n. finding the optimal choice of the polynomial, which minimizes E n f is not easy. The most common methods of constructing polynomials give errors, which are significantly larger that E n f the error obtained by an optimal choice of polynomial of order n.

So, if we construct a polynomial; any polynomial using you will talk about schemes for constructing polynomials, we construct a polynomial like that there is very little chance then that polynomial is the optimal polynomial right the error it is going to give is not going to be anywhere near may not be anywhere near the lowest error for, which we have to find out we have to try all the polynomials of that order. And, then find the one which gives the lower that that is a that is a complex and that extremely expensive process right.

So, if the polynomial that is constructed is significantly different from the optimal choice of polynomial there is no guarantee that just by increasing the order of the polynomial. we will be able to reduce the error in the infinite norm to be 0 right. So, as just because I increase the order of the polynomial, it does not mean that my E n f is going to be 0. If, for that higher order polynomial that polynomial that I actually use is very different from the optimal polynomial of that order right from the; so, this is very important right just because I increase the order of the polynomial does not mean that my error that I am going to get anywhere close to E n f right. And, my error is going to go down substantially right because that polynomial may be the best polynomial the optimal polynomial of that order right that is very important.

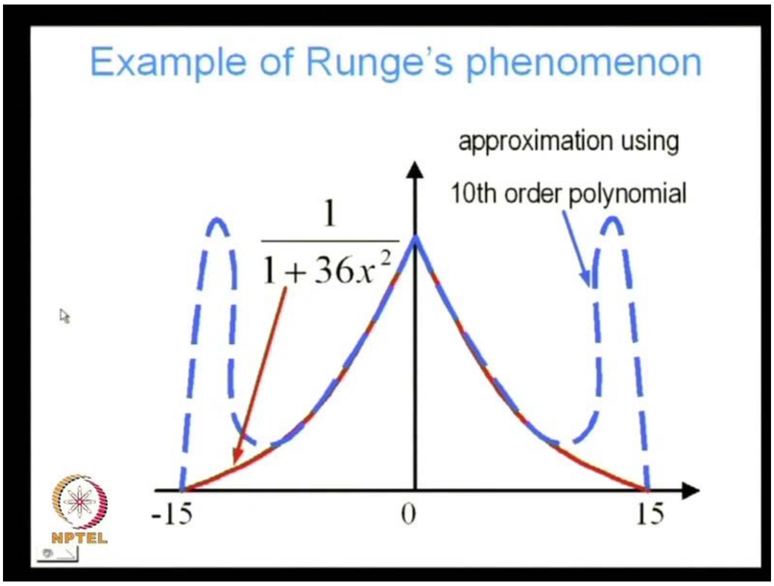
Then that is why the finite element method works. If the finite element method has something which is known as the best approximation property right it is the it is in some sense it is an optimal choice of polynomial. So, the choice of it is an optimal choice of approximation. The polynomials that we use in the finite element method they are the optimal polynomial in a certain sense right. So, that is why choosing any higher order polynomial is not going to give me a substantial reduction in error right.

(Refer Slide Time: 32:59)



Where example of this can be seen, if we consider constructing approximation to a relatively simple function f of x is equal 1 by 1 plus thirty 6 x square in a large interval in minus 15 to 15 and by fitting a higher order polynomial through the function values at equally spaced grid points right. And, what we are going to do is we are going to show you I am going to show you an example, where this function has been approximated by a polynomial of order ten right. A polynomial of order ten in order to fit that polynomial of order ten how many grid points do I need eleven function values at 11 grid points right and I fit that I fit that function this red function right.

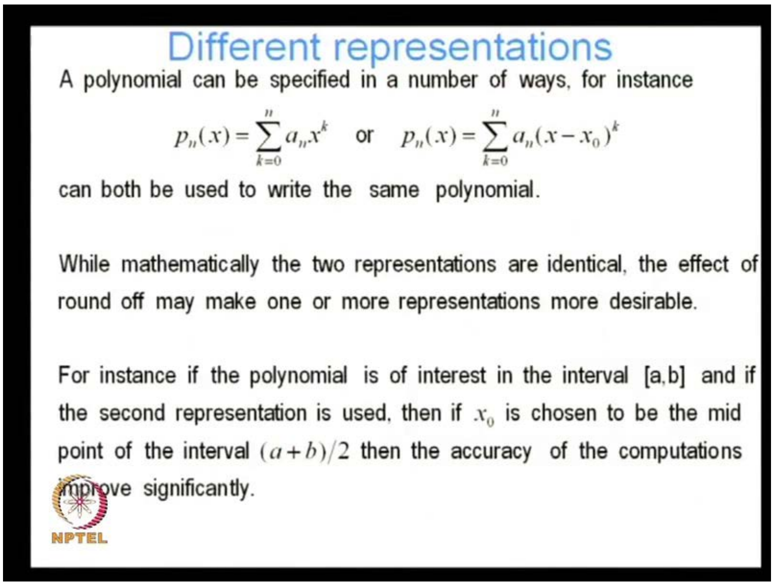
(Refer Slide Time: 33:49)



This is about 1 plus 1 1 by 1 plus 36 x square over this interval right. and, then I fit it using tenth order polynomial a tenth order polynomial right and what I find and I use equally spaced grid points, I used equally spaced grid points and what I find that near the centre of the interval my polynomial approximation is very good but near the boundaries there are very very large errors right; there are very very large errors. So, you can see even though this is a very high order polynomial and this function is relatively simple right. It is relatively simple; even then my very high order polynomial with equally spaced grid points does a very poor job near the boundaries.

For very high order polynomial say of order ten the error in the infinite norm is significantly large because infinite norm is the error over the entire interval. So, if I look at the infinite norm, it will pick out those large errors at the boundaries right. So, in the middle portion of the interval the error is small while near the boundaries the error is very large right. This is known as runge's phenomenon. Right, this is known as runge's phenomenon. Thus choosing the optimal polynomial is crucial to the success of the interpolation scheme and is reason for the focus of interest on specific interpolation for finite element method is just a special way of picking the polynomials right. So, it is a special way of picking the polynomials and that is why we are interested in these special schemes such as the finite element method because they have these optimality properties which not every polynomial possesses right it is not every polynomial possesses.

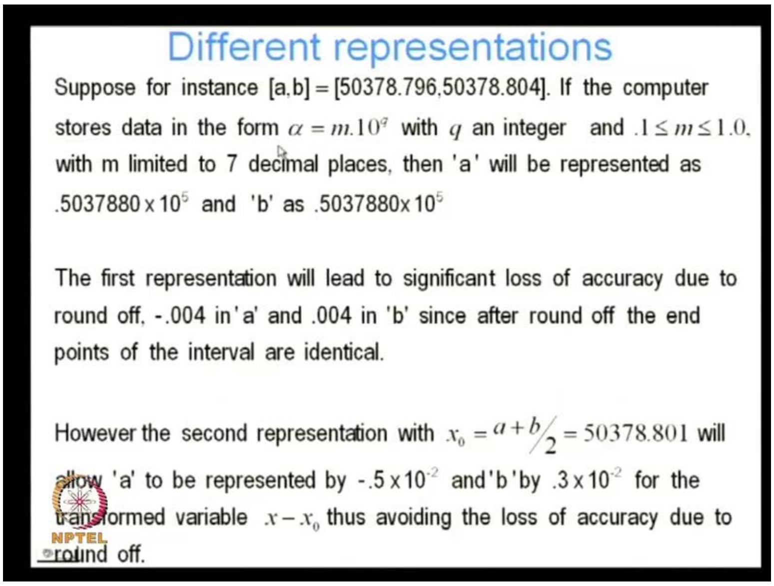
(Refer Slide Time: 35:56)



So, now let us talk about some ways of representing polynomials, we will start with the simplest ways of representing polynomials polynomial can be represented like this p n x is equal to sigma k equal to 0 to n a n x to the power k or p n x is equal to sigma k equal to 0 to n a n x minus x 0 to the power k. we can use both these representations to represent the same polynomial. While mathematically these two representations are identical they may have very important implications, if we consider numerical implementation. why is that well for instance, if we if I if the polynomial of interest is in the interval a b and if the second representation is used and if we choose x 0 to be the mid-point of the interval a plus b by which is a plus b by 2.

Then, the accuracy of the computations improve significantly right it can be seen instead of using this approximation, If I use this approximation to represent a function over an interval a b right. Then and if I choose x 0 to be the mid-point of the interval the errors due to this approximation are significantly less then due to this approximation a round off error.

(Refer Slide Time: 37:21)

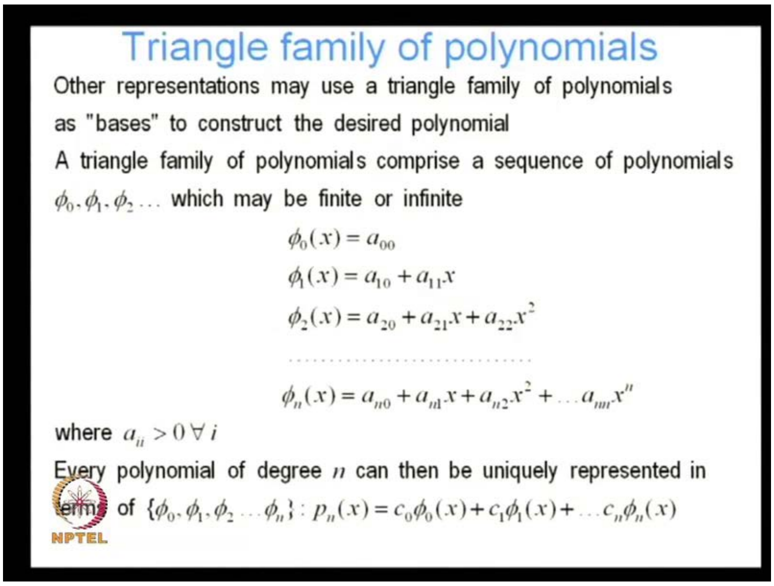


Now, let us look at an example for instance, if I have a b given like this right 50378.796 and this if the computer stores data, which we have we looked at right at the beginning of the course; we looked at how the computer stores data. And, if the computers stores data in this form alpha is equal to m times ten to the power q with q is an integer because there is a maximum for maximum value of q depending on the computer and m there is there is m is limited to 7 decimal places. Say, then a will be represented like this right a will be m can only take up to can be represented at with 7 decimals right.

So, this can be this will be represented like that and you can see that the internal computer the internal representation of these two numbers on the computer is practically identical is identical right. But, the first representation will lead to significant loss of accuracy due to round off. So, minus 0.004. So, this is point 796 then this is 88. I know what did I do it should be 800 not 880 and 800. So, that is the type of right. So, in that case that is going to lead to a loss of accuracy of minus 0.004 in a and 0.004 in b right. Since, after round off the end points of the interval are identical.

However, if I use the second representation with x 0 is equal to a plus b by two right. So, that is my x 0 then in that case this will allow a to be represented by my this because now a is going to be this minus x 0 right this minus that. So, now, a is going to become this and b is going to become this. So, now, I have an exact I have not lost any accuracy right for the transformed variable x minus x 0 these are my end points in the interval right thus avoiding the loss of accuracy due to round off. So, the way we choose to represent the polynomial is also very important right in terms of accuracy.

(Refer Slide Time: 39:46)

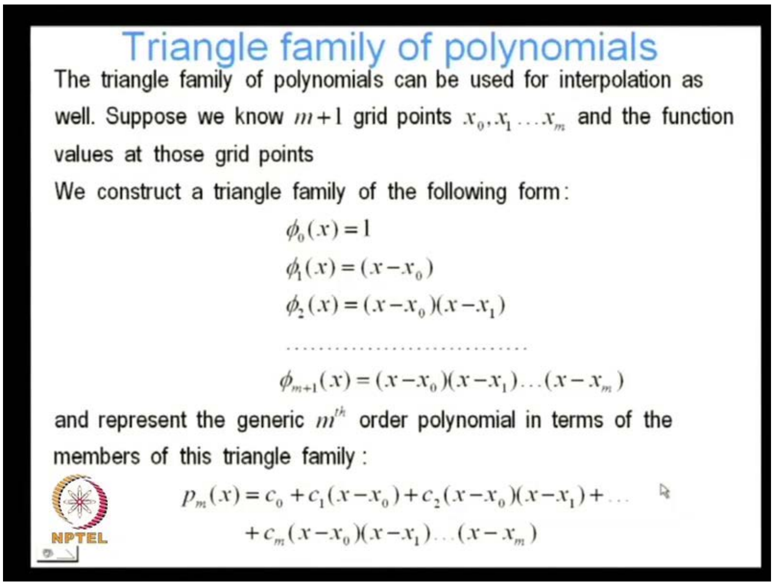


So, other representations may use a triangle family of polynomials right. We have higher and higher order; we have polynomials which have basis right and these basis form a triangular triangle family they are of higher and higher order. So, it comprises a sequence of polynomials like phi 0 phi 1 phi 2. You can see, phi 0 is a zeroth order polynomial; phi one is a first order polynomial; phi two is a second order polynomial and so on and So forth. phi n is a nth order polynomial. So, these are basis right and I can construct any polynomial by taking linear combinations of these basic basis polynomials these polynomial basis right.

So, this a i i greater than 0 because I just want to make sure that this polynomial is truly of that order right. If a n were not equal to 0 this would not be an nth order polynomial it will become a n minus one th order polynomial. So, every polynomial of degree n can then be uniquely represented in terms of these basis polynomials. So, those of you who have had some exposure to finite elements these are very similar to your basis. So, these are your basic functions right your polynomial.

So, when we choose first order elements infinite element, we make sure that our basis only involves this and this right higher order elements involve more terms right. p refinement right this we have increased the order of the basis polynomials right. So, it is exactly the same idea right. So, every polynomial of degree n can therefore, be represented in terms of these basis polynomials p n x is equal to c 0 phi 0 x plus c 1 phi one x plus c n phi n x.

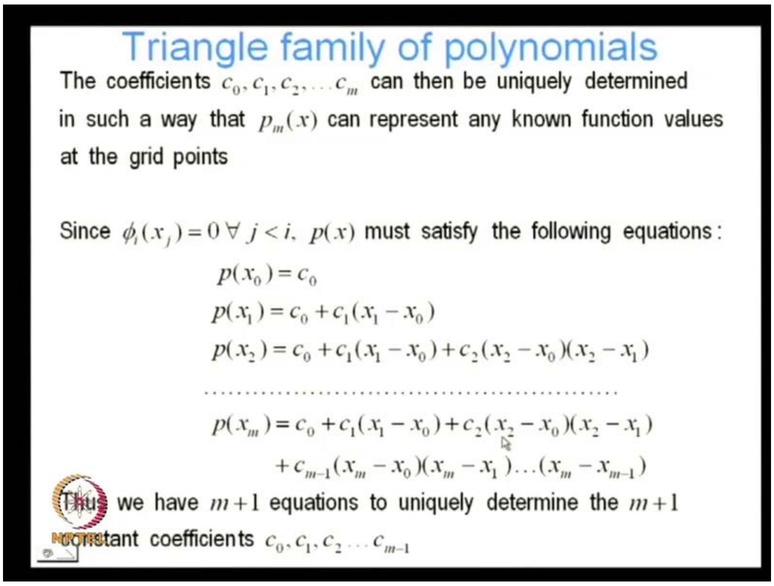
(Refer Slide Time: 41:43)



The triangle family of polynomials can be used for interpolation as well suppose we know m plus 1 grid points x 0 x 1 through x m and the function values at those grid points. So, we can construct a triangle family of polynomials using those grid points right. So, I construct my first polynomial to be equal to 1; my second polynomial to be x minus x 0, x 0 being the first grid point, phi 2 x minus x 0 times x minus x 1 and So on until I get phi m plus 1. This is x minus x 0 x minus x minus 1 up to x minus x m.

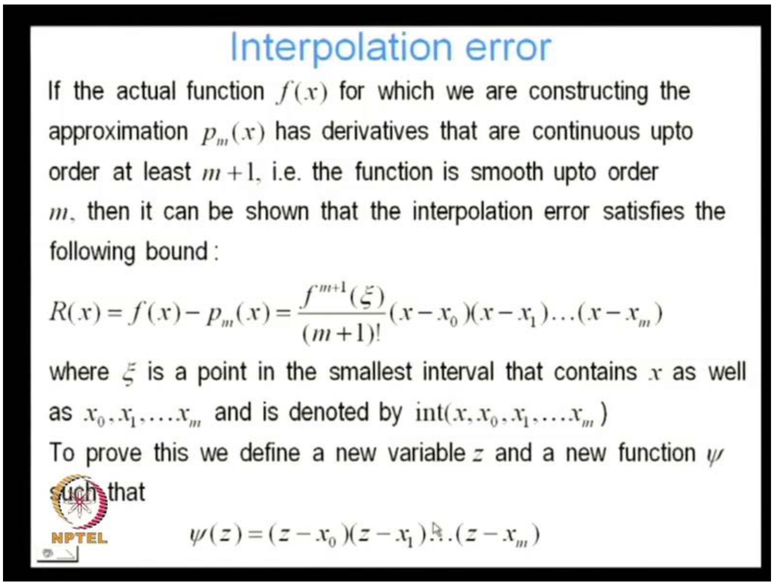
So, I using the m plus 1 grid points I construct phi m plus 1 right and then I represent a generic nth order polynomial, I know that once I construct my basis functions right. My basis functions then I can represent any polynomial of order, m th order polynomial using a linear combination of these basis functions, which I just constructed right. So, p m p m of x any polynomial of m th order is equal to c 0 plus c 1 x minus x 0 plus c 2 x minus x 0 so on, plus c m x minus x 0 up to x minus x m right.

(Refer Slide Time: 43:11)



So, the coefficients then I c 0 c 1 c through c m can then be uniquely determined from the function values at those grid points; from the know function values at the grid points. Since, phi I j x so this is just I have shown, how you determine those function values right. So, you evaluate p of x 0 which gives you c 0, p of x 1; p of x 1 which is going to give you c 1 x 1 minus x 0 plus c 0 at x 2, you got c 2 x 2 minus x 0 plus x 2 minus x 1 plus those terms and so on and so forth. So, using this right, we have m plus 1 equations to determine my m plus 1 unknown coefficients c 0 through c m and I can use this to determine my coefficients and fit the polynomial.

(Refer Slide Time: 44:17)



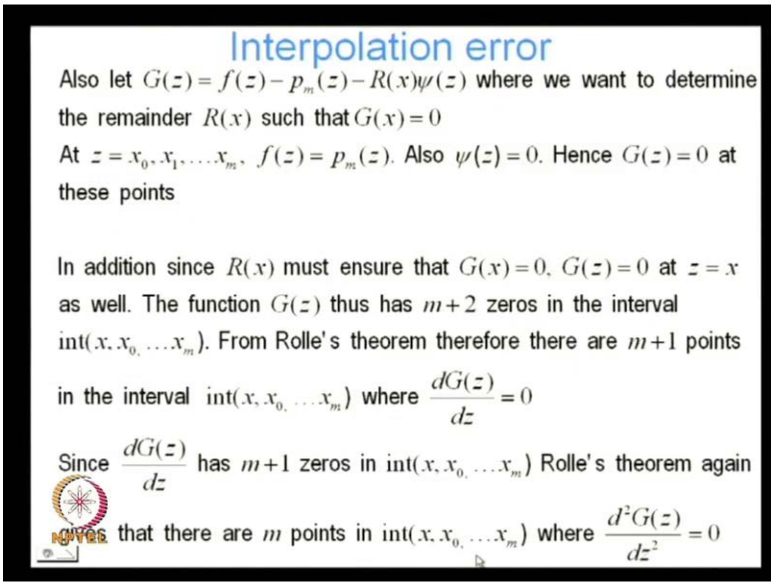
If the actual function f of x for which we are constructing the approximation p m of x has derivatives that are continuous up to order at least m plus 1 right. That is one order higher than the polynomial I am constructing to approximate that function. Then, I can write an expression for the error my function f of x is may be much much higher order, it might be it is need not necessarily be a m th order polynomial, it can be of higher order right. But, only restriction I am imposing is that, the it is continuous up to order m plus 1 if this function is continuous up to order m plus 1.

Then I can get an expression for the error right by approximate by approximating the function f of x by this polynomial of a m th order I am going to get an error right. Because, the actual function is not an m th order polynomial right. And then, I can get an approximation for the error using I can get an approximation for the error and that approximation is given by this value f m plus 1 xi by divided by factorial m plus 1 times x minus x 0, x minus x one through x minus x m.

So, this is the error this is the error due to my approximating f of x with the polynomial p m of x right. and this is evaluated at a point this is equal to f m plus 1, this is m plus oneth derivative of f evaluated at a point xi and the point xi lies within the interval which is span which is basically contains all my points x 0 x 1 x m as well as x right. Because, xi is a point in the smallest interval that contains x. x well as x 0 x one x m and is denoted like this. So, what I am saying is that I am approximating f of x with this polynomial of order m and I am guaranteed that the error the error due to my approximation is can is going to be given by this term right. It is going to be given by that term.

So, by getting some idea of this expression the value of this term, I can get an idea of the of the error right I can get an. So, if I know at the point x at which I am approximating the my function with this polynomial. I also get an idea of the error involved in that approximation right. Well, I am going to give a short proof for this. To prove this, we define a new variable z and a new function psi such that psi of z is equal to z minus x 0 times z minus x 1 through z minus x m. So, I define a new variable z and I define another function psi of f z given like this.

(Refer Slide Time: 47:40)



And, I also define another function G of z which is equal to f of z minus p m of z, f of z being my original function, p m being my polynomial approximation minus R of x times psi of z. So, we want to find out R of x such that G of x is equal to 0, when G of x is equal to 0 what does that mean; that means, f of x minus p m of x is given by R x times psi of x right. So, I want to find out R x and psi of x i have already seen is of this function psi of x is basically this part right I am saying psi of x is like this part and I want to find out what form must R take in order for me to write this right in order for me to write that.

So, that is how we define G of z; G of z is equal to f of z minus p m of z minus R x times psi of z, where we want to determine the remainder R of x such that G of x is equal to 0.So, at z is equal to x 0, x 1 through x m I know that f of z is equal to p m of z, why because my polynomial satisfies the function values exactly at the points x 0 x 1 through x m.

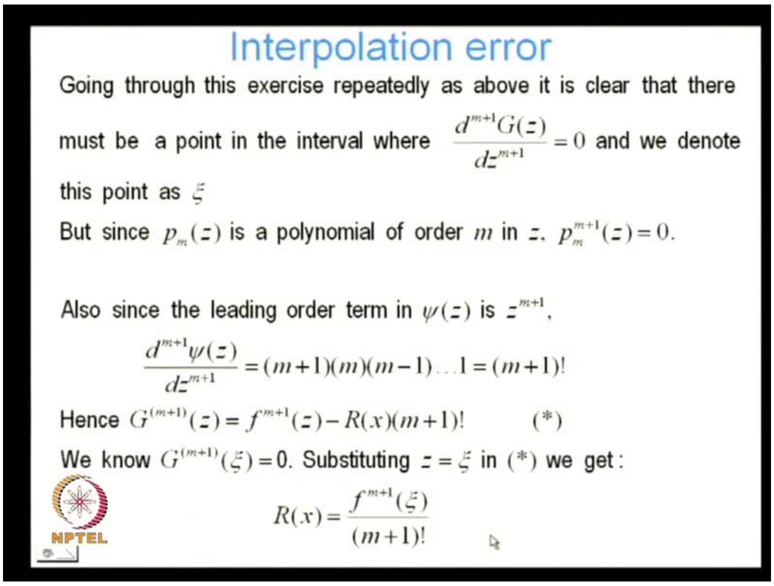
So, at zee is equal to x 0 x 1 through x m f of z is equal to p m of z also psi of z is equal to 0 at all those points at x 0 x 1 x m you can see psi of z is going to be if z is equal to x 0 this thing is going to be 0 z is equal to x 1 this thing is going to be 0 z is equal to x m that thing is going to be 0. So, psi of zee is also going to be 0 at x 0 x 1 and x m. So, since f of zee 0 at x 0 x 1 x m p m of f of zee is equal to p m of f of zee is equal to p m of zee at x 0 x 1 and x m and psi of zee is equal to 0 at x 0 x 1 and x m. So, G of z must be 0 at x 0 x 1 and x m right. In addition, since R of x must ensure that G we were we are going to choose R of x such that such that G of x is equal to 0 so; that means, that G of z is equal to 0 at z is equal to x as well right.

So, from here I know that G of z; z equal to 0 at those m plus 1 points what are those m plus 1 points x 0, x 1, x 2 up to x m right. So, G of z is equal to 0 at those m plus 1 points G of zee is also 0 at zee is equal to x right. That is because that is why that is that is how I am going to choose R of x right. So, basically G of z is equal to 0 at m plus 2 points and what are those m plus 2 points x 0 x one through x m plus the point x right. So, the function G of z has m plus 2 zeros in the interval x, x 0 through x m. Go back to your first here calculus or wherever you studied calculus right. A function, if it has two zeros if it has in an interval a b, if it is 0 at two points we know by the derivative of that function what do we know the derivative has to be 0, at least at one point right. The derivative has to be 0 at least at one point one intermediate point right. So, that is Rolle’s Theorem right from Rolle’s theorem, if a function.

So, now, let us look at look at this function G of zee. So, since there are m plus 2 zeros there are m plus 1 points in the interval where d G of zee is equal to 0 if they are two zeros there is one point in the interval where this slope has got to be 0 if there are m plus 2 zeros there are m plus 1 points in the interval where the derivative has to become has to be 0 right. So, I know that d G zee d zee has m plus 1 zeros in that interval right. Now, d G plus 1 G d zee d zee has m plus 1 zeros; that means, d 2 G d z must have m zeros right;

if you think of this function, this function has m plus 1 zeros. So, it is derivative must have m zeros right; it is derivative must have m zeros.

(Refer Slide Time: 52:34)

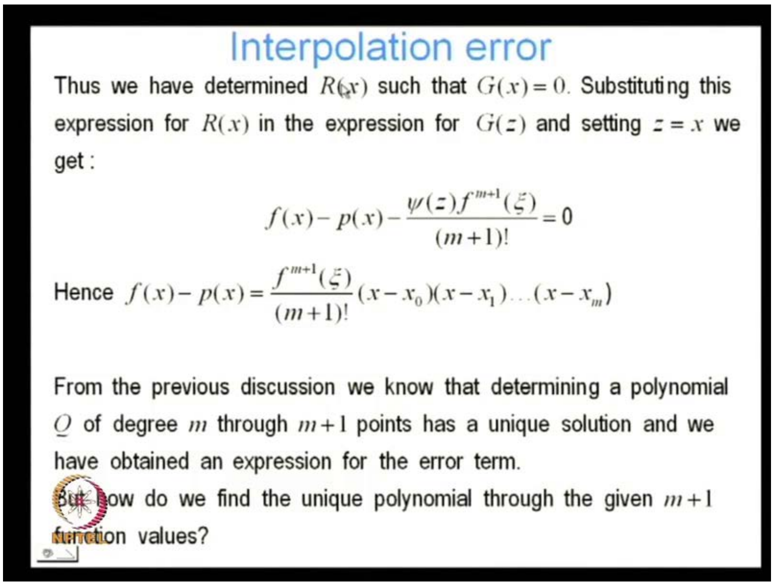


Similarly, if we go through this over and over again it is clear that there must be a point in the interval where d G zee d zee m th m plus one th derivative of that right d m plus 1 d z m plus 1 G z must be equal to 0. There must be one point in that interval, where this is got to be equal to 0. And let us denote that point as xi right; let us denote that point as xi, but we know that the polynomial p m z is a polynomial of order m right. So, if I take m plus one th derivative of that polynomial that is always got to be equal 0 right.

Also, since the leading order term in psi zee is z to the power m plus 1 where is psi z psi z. So, if I take m plus one th derivative of psi z. I am going to get factorial m plus 1 right because the other terms are going to give me 0 only the leading order term is going to remain right and that is going to be equal to factorial m plus 1. So, that term is going to be that hence if I go back to my G z and take m plus 1 derivatives then, what do I get I get f m plus 1 z right I get f m plus 1 z then p m plus 1 z I know is equal to zero. So, that term vanishes I have got R x and then I have got m plus derivative of psi z, which I just saw is equal to factorial m plus 1.

So, G m plus 1 z is equal to f m plus 1 z minus R x m plus 1 factorial. And, we now we know that G m plus 1 psi at one point xi at some point xi it is equal to 0. So, substituting z is equal to xi in this interval we get R x must be of that form if R x is of that firm then G of z is equal to f z minus p m z minus that times psi z right.

(Refer Slide Time: 54:54)



And, then I replace z by x and I get exactly my expression f x minus p x is equal to that right, which tells me that my error is given by this expression. So, if I have a poly it should be p m sorry f of x minus p m of x minus p m of x is given by that expression right. So, we have looked at polynomials the errors due to polynomials right. we have looked at the weirstrass approximation theorem and we have see on what factors the error depends right. You have also seen how we go about constructing those polynomials right and I have also mentioned that constructing the optimal polynomial is really hard right.

But, how to construct polynomials in general I have seen but it turns out that instead of actually if we if we have m plus 1 points. And, we know the function values at those m plus 1 points you need not really solve that system of equations here actually solve that huge system right for m plus 1 equations and m plus 1 unknowns to find my coefficients there are simple rules to find out the coefficients. So, if I know m point and if I know the function values at those m points there are simple rules by which I can construct the coefficients for the polynomials and that is given by something known as Newton's interpolation formula we need not solve this system to find those coefficients and next lecture we are going to talk about that.Thank you.