Numerical Methods in Civil Engineering
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Lecture - 32
Orthogonal Polynomials

In lecture 32 of our series on numerical methods in civil engineering, we will talk about orthogonal polynomials.

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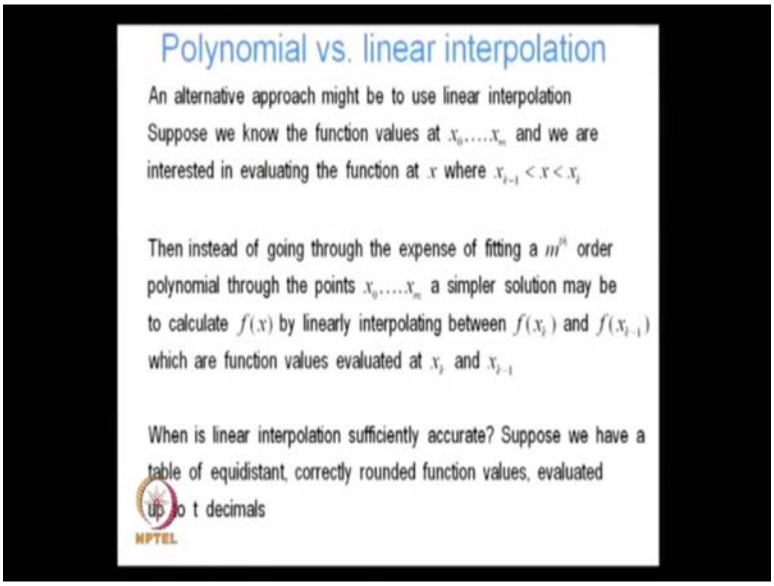

Before we do that let me recapitulate what we talked about at in our last class and particularly what we talked about at end of the last class, when we were talking about the relative merits of polynomial versus linear interpolation. We looked at polynomial interpolation and we suggested that an alternative instead, if we have a great m plus 1 point instead of trying to fit an m-th order polynomial through this points, an alternative approach might be used might be to use linear interpolation. Suppose we know the function values at x 0 through x m, and we are interested in evaluating the function at any point x, where x lies between any 2 grid points x k minus 1 and x k.

Then, instead of going through the expensive fitting and m-th order polynomial through the points x 0 through x m a simply solution may be to calculate f of x by linearly interpolating between f of x k and f of x k minus 1, which are basically the function values evaluated at x k and x k minus 1. We can always do that, but we have to find out there are two things, we have to consider; first is how accurate is our linear interpolation number one that should be the main one of the main considerations, the other main the other consideration should be how expensive, how computationally expensive or how computationally inexpensive, it is compare to using a polynomial interpolation.

So, when first we want to consider (( )) accuracy when is linear interpolation sufficiently accurate. Suppose we have a table of equidistant correctly rounded function values meaning that we have rounded the rounded them correctly. So, that we know that they are accurate up to t decimal places. So, that we know that those function values are accurate up to t decimal places.

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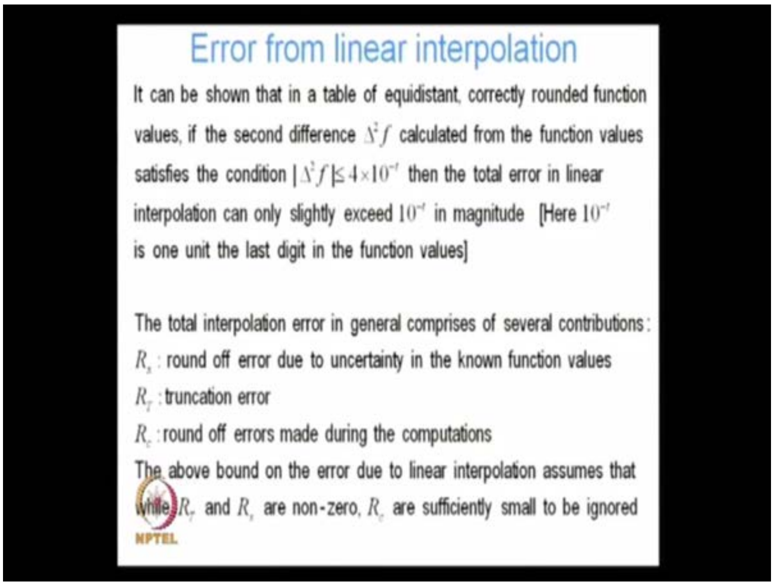

Last time I talked about a particular result which states that if given a table of equidistant correctly rounded function values, if we are assured that the second difference calculated from the function values satisfies this condition that is the second difference is very small and it is as small as 4 into 10 to the power minus t , where t is denotes the last digit in the function value then the total error in linear interpolation can only exceed slightly exceed 10 to the power minus t in magnitude, but when is this condition going to be satisfied, When is the second difference going to be so small that it is lesser than or equal to 4 into 10 to the power minus t, when my function values are varying relatively slowly, when my curve when the difference between the function values is not large only in that case when I take the second difference the second difference is going to be really small.

So, the assumption in built assumption here is that the function values are not varying very much. In that interval only then is my linear interpolation going to give me accurate results, accurate up to this level. The linear total error in linear interpolation can only slightly exceed 10 to the power minus t. In magnitude and what do we mean by the total interpolation error which I also talked about last time.

The total interpolation error has basically got 3 components it is the round off error due to uncertainty in the known function values, there is some round off there then there is a truncation error basically, we are using an interpolate of a certain order a polynomial of a certain order inbuilt in to that is the assumption, that there is an error term is a reminder term, which contributes to the error, so that is the truncation error and finally, there is a round off error made during the computations during the actual calculations you do, to find out your interpolated value. So, the above bound on the error due to linear interpolation assumes that while R t and R x are non zero. R c is sufficiently small to be ignored. So, we are just considering R x and R t when we talk about this bound.

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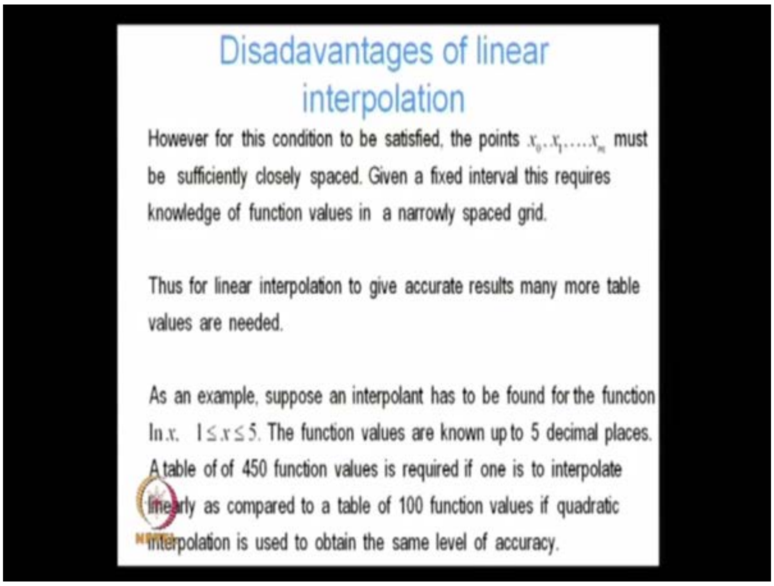

However, for this condition to be satisfied the points x 0 x 1 through x n must be sufficiently closely spaced I initially told you that the function should not, function value should not vary, that much in that interval that is equivalent to say that my interval is very small. That my points are closely spaced, so given a fixed interval, this requires the knowledge of function values in a narrowly spaced grid.

Thus, for a linear interpolation to give accurate results many more table values are needed because our grid has, we have to a very narrowly spaced function narrowly spaced grid function values at many points. That the function is varying slowly and that means if I have a large interval that means, I have many function values that need to be evaluated, So. I have many my grid has many points.

As an example suppose I interpolate has to be found for the function log, natural log of x over the interval x greater than equal to 1 less than equal to 5. So, we know this function value, we know this function and we know and we can calculate the function values at points in a grid and then we try to fit a polynomial to this function values. It turns out if we try to do it with linear interpolation. We try to fit lines between the function values and then try to interpolate the value, at an unknown point from my Interpolate. In that case if I use a linear interpolation I will need 450 function values, while if I just use a quadratic interpolation. I need 100 function values in order to get the same level of accuracy.

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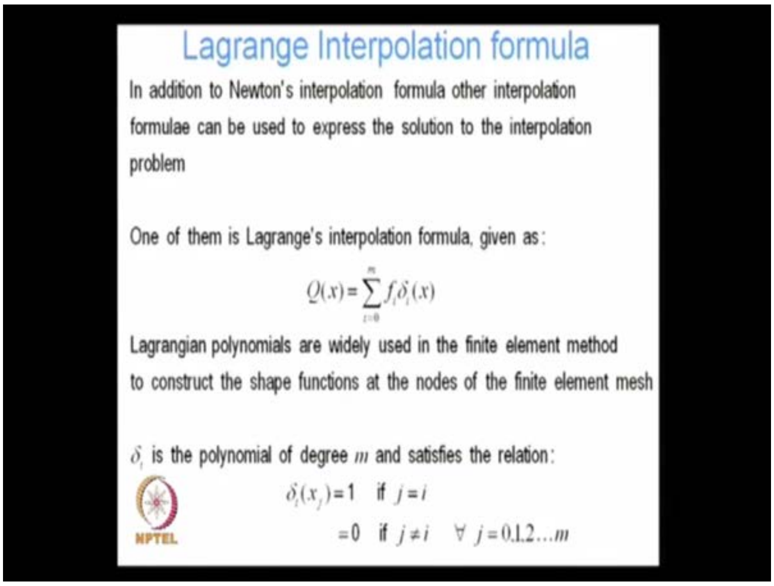

So, are we have talked about Newton's interpolation formula basically that is let us recall Newton interpolation formula is a way to figure out the polynomials to calculate the coefficients of the polynomials, of the polynomial which we are using to interpolate instead there is 1 route force way to which I told you t in the beginning.

Where you actually go ahead and invert the matrix to find the coefficients of the polynomials but instead of that we talked about the linear Newton's interpolation formula which allows you to calculate the coefficients of your polynomial, of your interpolate relatively easily. There are other interpolation formulas. For instance, the lagrange interpolate formula which is particularly popular in the context of finite element methods in addition to Newton's interpolation. We are going to talk about that and the lagrange interpolation formula goes like this, It says that my interpolation formula, Is given by this sort of a function f, f i are the known function values known function values at the grid points and delta i x is basically interpolate.

Lagrangian polynomials are widely used in the finite element method to construct shape functions at the notes of the finite element mesh delta i is the polynomial of degree m and satisfies the relation delta i x j is equal to1. So, delta i for delta each of these polynomials, are associated with the grid point and that polynomial assumes the value 1 at the grid point and is 0 everywhere else and my total interpolate is obtained by summing those polynomials, scaling each polynomial by the function value at its grid point, at the grid point at which the polynomial is1 scaling that polynomial with that function value and summing it together. So, delta i x j is equal to1, If j is equal to i and is equal to 0 everywhere else, at all every phenomena when I say everywhere else, I mean at all the other grid points.

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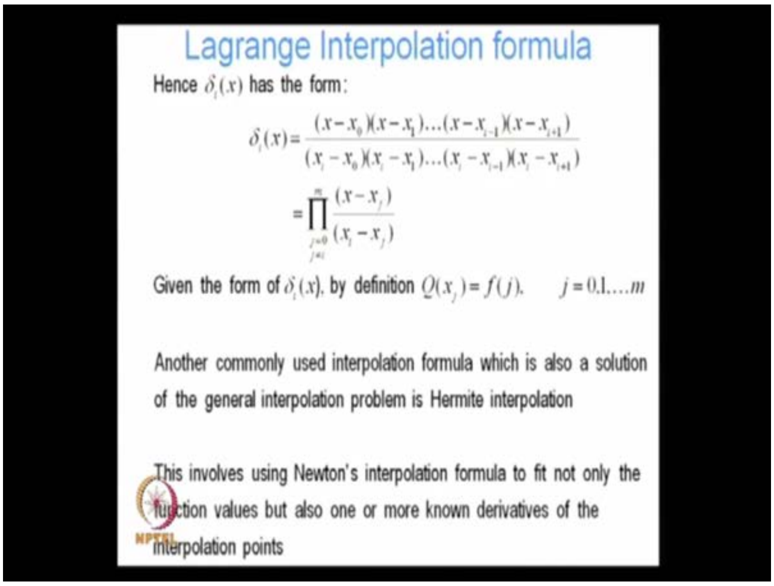

Hence, delta i x must have this following form in order to satisfy that condition delta i x must have the following form. x minus x j is pi denotes the product x minus x j divided by x i minus x j and the top for delta i x we have everything, if we look at the second term in each of this first brackets there is x 0 x 1 x i minus 1 but there is a missing x i the bottom I have x i minus x 0 x i minus x 1 through x i x i plus 1.

Actually, it should need not stop at x i plus 1 it can go up to m; it can go up to m I have missed; I have not included the higher terms. So, this gives me something like that. So, given this form of delta i x by definition if I define Q x like this if I define Q x like this. Q x j will be equal to f of j. So, that is your Lagrange interpolation formula again is the same thing I know the function values at certain grid points.

So, question of fitting a polynomial to those function values. Another commonly used interpolation formula which is also a solution of the general interpolation problem is hermit interpolation. The advantage of hermit interpolation is that we can using hermit interpolation. We can interpolate not only the function values but also the derivatives of the function values. We can you can interpolate the derivatives of the function values this involves using Newton’s interpolation formula to fit not only the function values but also 1 or more known derivatives of the interpolation points. So, if we know not only the function values at the interpolation points but we also know the derivatives, then we can use hermit interpolation and as we will see hermit interpolation is really a special case of Newton’s interpolation.

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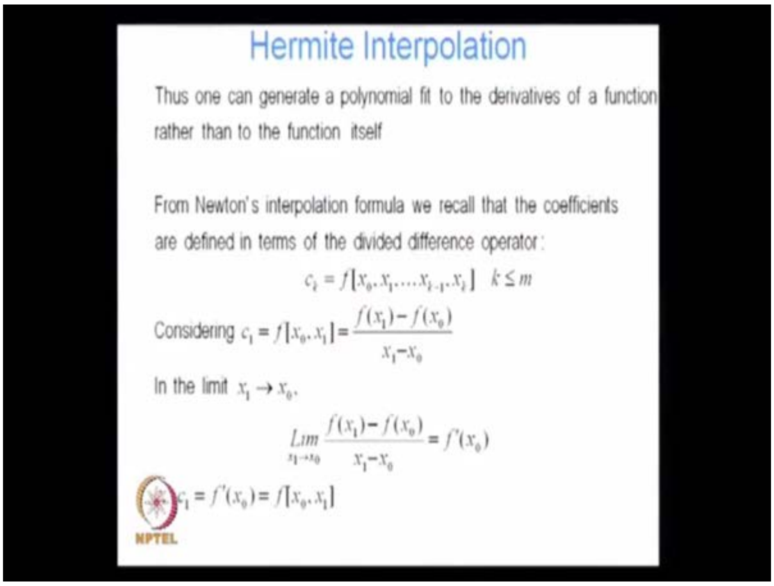

Thus, one can generate a polynomial fit to the derivatives of a function rather than to the function itself, from Newton's interpolation formula. We recall the coefficients are defined in terms of the divided difference operator, when through a lot of pain to show that for Newton's interpolation you can find the coefficients by evaluating this divided difference operator and we showed how to evaluate those divided, difference operators and in particular if you want to evaluate c 1. The coefficient of the term liner n x then that can be obtained like this f of x 0 x 1 which is basically, the first order divided difference operator which is given by f of x 1 minus f of x 0 x 1 minus x 0.

Now, take the limit of this, if we take the limit of this as x 1 tends to x 0 what do we get here, we get f of x 1 minus f of x 0 divided by x 1 minus x 0 the limit x 1 tends to x 0 that is equal to the derivative of the function at x 0. So, what does that tell me, that tells me in the limit that x 1 becomes, very close to x 0. In that case my first coefficient I can get that by equating that to the derivative, to known derivative value at x 0.

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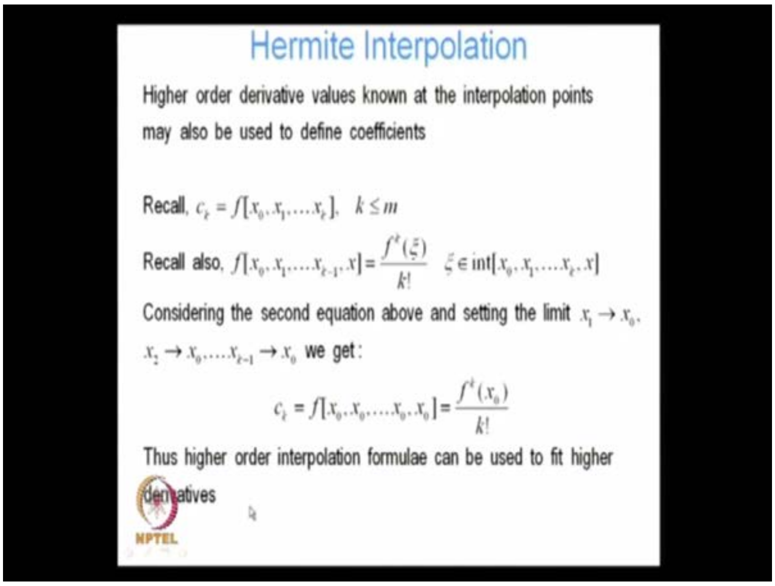

Similarly, I can do that with higher order derivatives as well. So, higher order derivative if I know higher order derivative values at interpolation points I can also use them to define the coefficients, how do we do that? Well, let us recall that c k is equal to this for k less than or equal to m we also recall that we showed this f of x 0 x 1 x k minus 1 x is equal the k-th derivative of f evaluated at xi evaluated by factorial k where xi belongs to the interval span interval, which includes basically x i belongs to the smallest interval which includes the points x 0 x 1 x k and x.

So, we consider this equation and we set the limit x 1 tends to x 0 x all the grid points the grid become smaller and smaller. So, I interest in taking the derivative. So, these things x one tends to x 0 x 2 tends to x 0 through x k minus 1 x 0. We get c k is equal to the k-th derivative of f evaluated at x 0 divided by factorial k. So, I can get the c k-th term in my polynomial expansion using hermit interpolation by this value right.

So, why does i becomes exit because everything basically collapses to x 0.So, my interval becomes, smaller and smaller and so i get f of k x 0. So, that is how we can use hermit interpolation to fit a polynomial. Not only the function values but also the derivatives this where those of you, who are the structural mechanics background this is used if our Bernoulli Euler beam’s for Bernoulli Euler beam’s.

When we want to interpolate the slopes when the particularly a finite element context when we are using, when we are trying to solve a Bernoulli Euler beam, we need to interpolate not only the displacements at the grid points and a known, we also need to interpolate the slopes at the grid points. So, in that case we need to fit that interpolate to not only the function the displacements that is the function values, at the grid points but also the derivatives of the function values at the grid points which are my slopes.

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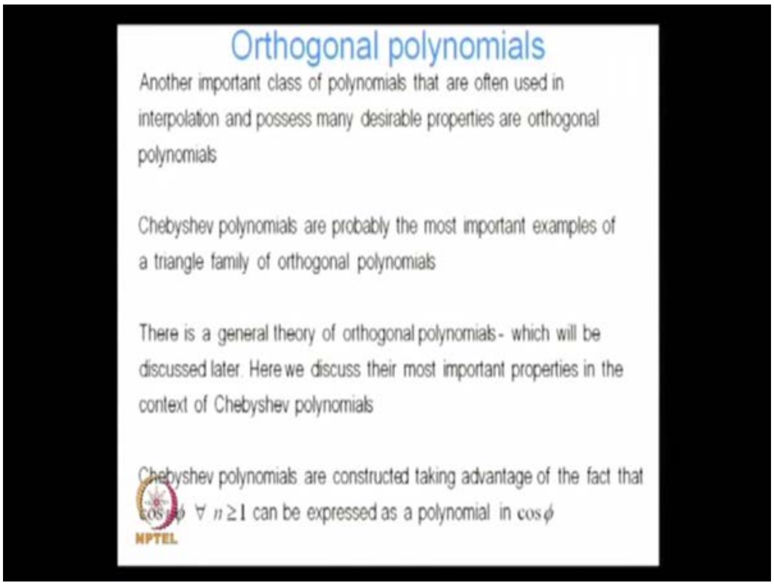

Orthogonal polynomials: Another important class of polynomials that are often used interpolation and possess many desirable properties are orthogonal polynomials and Chebyshev polynomials, The Chebyshev, spelling you can find a different spelling in different books, Some book spell with that Chebyshev, I am used the spelling Chebyshev but they are probably the most important examples of a triangle family of orthogonal polynomials. Now, these Chebyshev polynomials are just a particular instants or something which is much broader and much more general, that is the family or general theory of orthogonal polynomials.

There are particular types of orthogonal polynomials. We will talk about the general theory not in great detail but at least the basic features of the general theory. Some point later on in this lecture but first I want to motivate the lecture by talking about 1 particular type of orthogonal polynomials. These Chebyshev polynomials they are these are very important because they possess some very desirable properties as we will see later on how we construct Chebyshev polynomials. Well, Chebyshev polynomials are constructed taking using the cosine function and taking the advantage of the fact that I can write cos of n phi for any n greater than 1 has a polynomial in cos of phi I can write cos of n phi as a polynomial in cos of phi.

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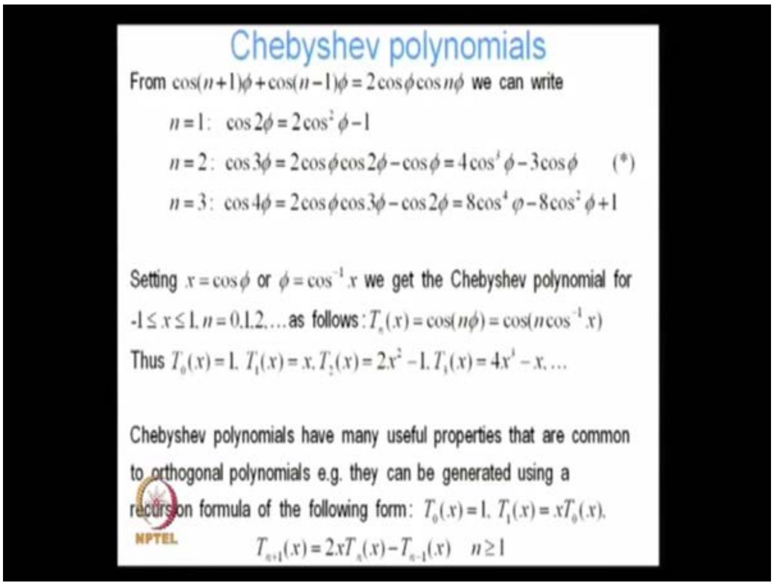

It is an example, well how can we do? Why can we do this? Well, it is just because of this particular formula. I can always write cos of n plus1 phi plus cos of n minus 1 phi has cosine of phi and times cos of n phi. So, for n is equal to 1 I can write using this formula here I can write n is equal to 1 cosine of 2 phi is equal to 2 cos square phi minus 1 this part becomes 1 1 minus 1, 0. So, to the left I got cos of 2 phi is equal to 2 cos square phi minus 1 for n is equal to 2 I have cos of 3 phi plus cos of 2 phi is equal to 2 cos of phi times cos of 2 phi So, I know cos of 2 phi in terms of cos of phi. So, I have cos of 2 phi here I know it terms of cos of phi So, I can write cos of 3 phi entirely in terms of cos of phi.

So, cos of 3 phi is equal to 2 cos of phi times cos of 2 phi minus cos of 2 phi. I know cos of 2 phi in terms of cos of phi. So, I can write cos of phi cos of 3 phi entirely in terms of cos of phi. Similarly, I can write cos of 4 phi entirely in terms of cos of phi and similarly, for a higher order terms as well any n any integer n cos of n phi I can always represent that in terms of cosine phi and then I set x is equal to cosine of phi or phi is equal to cosine inverse of x to get the Chebyshev polynomials. In the interval x it must be less than or equal to minus 1 greater than or equal to minus 1 less than or equal to 1 why does x has to have to satisfy this bound. Well, because I am setting x is equal to cosine of phi and n can be equal to 0 can be any (( )) which has talked about.

So, then we define the Chebyshev polynomial of order n with argument x as cosine of n phi which is equal to cosine of n times cos inverse of x. Thus T 0 of x is nothing but cosine of 0 which must be equal to 1 T 1 of x is equal to cos of n times cosine inverse of x which is equal to x T 2 of x is equal to T 2 of x is going to be 2 cos square phi minus 1 which is equal to 2 x square minus 1 2 x square minus 1, T 3 of x is equal to 4 cos cube phi minus 3 cos phi. So, that is equal to 4 x cube minus x. So, these are my Chebyshev polynomials, So, I am can generate the nth Chebyshev polynomial keeping in mind. That x is equal to cosine phi x is equal to cosine of phi is that clear.

So, T 3 of x is cosine of 3 times phi cosine of 3 times phi which is cos of 3 phi which is equal to 8 cos of 4 phi minus 8 cos square phi plus 1 which is equal to 8 x to the power 4 minus 8 x square plus 1. So, that is how I can generate my Chebyshev polynomials. The Chebyshev polynomials have many useful properties that are common to orthogonal polynomials, For instance, they can be generated recursively later on. I will go to talk about the recursive formula the general recursive formula for orthogonal polynomials.

What does the general recursive formula do, well it tells you that if you know the first few terms of that series of orthogonal polynomials you can generate higher order terms using that recursive formula. So, similar, so like for instance for the Chebyshev polynomials. We know that T 0 x is equal to T 1 x is equal to x times T 0 x because x T 0 x is 1. So, x times T 0 x and any higher order term in higher order Chebyshev polynomial, we can generate by multiplying the next term the next dual order term by 2 x and subtracting the previous one 2 x times T n x minus T n minus 1 x is going to give me T n plus 1 x you can try it out here.

So, if I want to evaluate T 3 x instead of doing this, instead, of doing this substitution here instead of writing out the expression for cos of cos of 3 phi I can use this recursive relationship. So, I am going to take 2 x times T 2 x minus T 1 x that is going to give me T 3 x. So, that is so that can so the higher order terms higher order Chebyshev polynomials can be generated relatively easily, without having to evaluate this the higher order the cosine terms, cosine terms for higher integer high integer values n. So, as can be seen from this.

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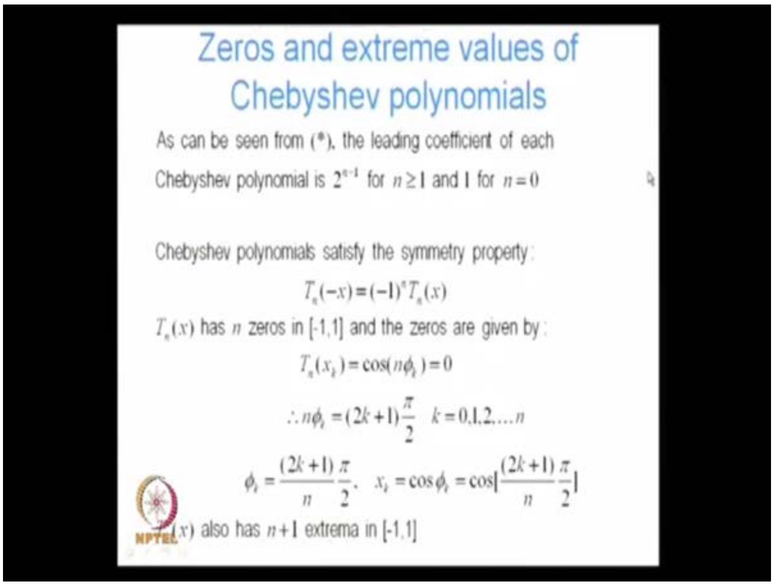

As we can see from here, the leading coefficient of each Chebyshev polynomial is 2 to the power n minus 1. So, let us take a look. So, for the 3 Chebyshev polynomial, the leading order term is x cube and what is the coefficient of x cube is 2 to the power 3 minus 1, 2 to the power 3 minus 2 to the power 2, 4 for t 2 x what is the leading order term x square. What is the coefficient of that it is 2 to the power 2 minus 1 to the 1 that is 2. So, the leading order term in each Chebyshev polynomial has coefficient 2 to the power n minus 1.

Chebyshev polynomials also satisfy the symmetry properties. So, what is the symmetry property? It says that T n of minus x is equal to minus 1 to the power n T n of x .They satisfy the symmetry property and each Chebyshev polynomial of order n has n 0. In the interval minus 1 to 1 and the 0 are given by the following we know T n of x k is equal to cosine of n phi to the power k and this has to be equal to 0. Where the Chebyshev polynomial is 0 this has got to be equal to 0.

So, when is this equal to 0 this is equal to 0 when n phi k is a multiple of pi by 2 is an odd multiple of pi by 2. So, n phi k must be equal to 2 k plus 1 times pi by 2 for cos of n phi k to be equal to 0. That is for the Chebyshev polynomial to have a 0. So, that gives me phi k the value of phi k. So, these are the values of phi k for which the Chebyshev polynomial is 0. So, I know phi k is equal to this so that means x k which is cos of phi k is given by this.

So, these are the points, these are the grid points where my Chebyshev polynomial is going to be 0. So, these are the 0 of the Chebyshev polynomial it turns out that Chebyshev polynomial also have n 0 T n. The Chebyshev polynomial of order n has n 0 in the interval minus 1 to1. It also has n plus 1 extreme in minus 1 to 1 extreme meaning minimum or maximum.

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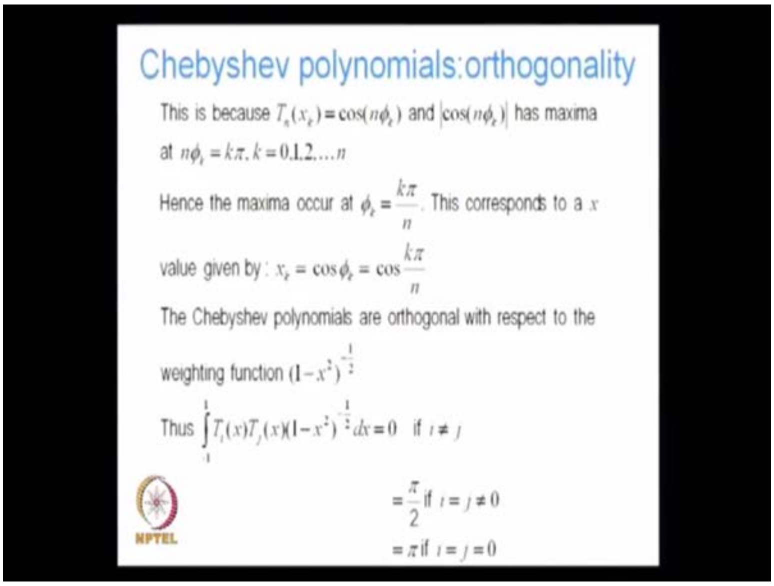

This is because the Chebyshev polynomial T n x k is equal to cosine of n phi k and I know that mod of cosine of n phi k between minus 1 and 1 and it is equal to minus 1 since it is minus 1 at pi it is equal to 1 at 2 pi it is one at two pi.

So, it has got maxima at n phi k at where n phi k is equal to k phi k equal to 0, 1,2 through n hence, the maxima occurred at phi k is equal to k pi by n. So, again we can get the location of the maxima in terms of x using the relation that x k is equal to cos of n phi k. So, the maxima, so the extrema not the maxima the extrema of the Chebyshev polynomial occur at cos of k pi by n.

So, we have talked about the, 0 of the Chebyshev polynomial. We have talked about the extrema of the Chebyshev polynomial, we shall see that these are the location of these of the 0 location of this extrema are extremely important. They are very valuable the Chebyshev polynomials.

I said are orthogonal polynomials but they are orthogonal with respect to a certain weighting function. So, if I integrate if I integrate 2 Chebyshev polynomials of different orders and multiply them with this weighting function, 1 minus x square to the power minus half and integrate them within the limit minus 1 and 1 I am going to get 0 always So, long as i is not equal to j so they are orthogonal but with respect to a certain weighting function.

If I just take integral within minus 1 to 1 of T i x T j x then I am not going to get orthogonality they are orthogonal, only with respect to this weighting function. But, if i is equal to j then they are then this integral is equal to pi if both are equal to 0 and if both are not equal to 0 but both are equal then I get pi by 2.

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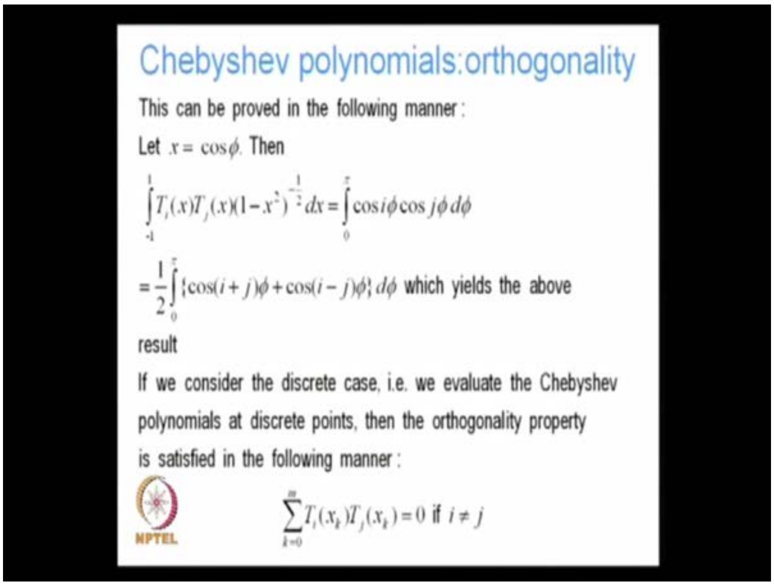

Well, you can prove that which yields relatively trivial. So, I just spend a few minutes. So, let x is equal to cos phi then I am interested in evaluating T i x T j x 1 minus x square to the power minus half then I get recalling. That I can write the Chebyshev polynomial in terms of cos phi So, I get integral between 0 to pi cos of i phi cos of j phi d phi this term actually disappears because 1 minus x square 1 minus cos square phi half gives me sin phi sin phi d x gives me d phi.

So, that term disappears. So, I have integral of this that if I evaluate that if I write it out so this is equal to cos of i plus j phi plus cos of i minus, i minus j phi. We integrate this between 0 to pi I will find, I get that previous result. So, that is for the Chebyshev polynomial in the continuous case. Where I am trying to integrate, it over the interval minus 1 to 1 so it is assumption is that the function is continuous in that interval but suppose I do not know the function values at all the points in that interval minus 1 to 1 I only know it at certain grid points then I can evaluate this, I can impose this orthogonality condition. At the grid points the orthogonality condition is satisfied at the grid points.

So, if we consider the discrete case that is we evaluate the Chebyshev polynomials at discrete points. Then the orthogonality property is satisfied in the following manner I think I have made a mistake here, because I have omitted the weighting function. So, sigma k equal to 0 to m T i x k T j x k times the weighting function evaluated that point is equal to 0 if i is not equal to j is equal to half m plus 1.

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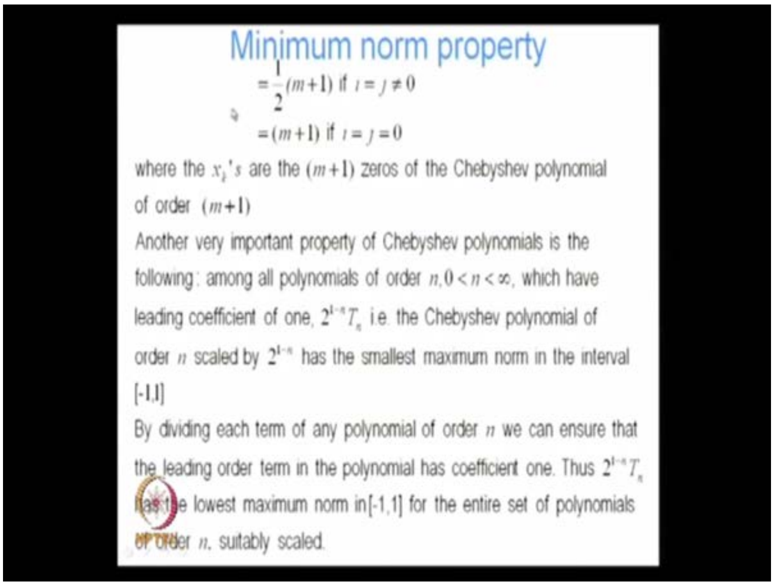

If i is equal to j is not equal to 0 is equal to m plus 1. If i is equal to j is equal to 0. Where the x k's we note where, the x k's are the m plus 1 0 of the Chebyshev polynomial of order m plus 1. Let us go back and take a look.

The sum is from k is equal to 0 to m and I am evaluating the Chebyshev. I am taking the product of the Chebyshev polynomials of order i and j but I am evaluating each of them when I am imposing the orthogonal, the orthogonality constraint is satisfied at x k where x k are the 0s of the other m are the 0 of the m-th order Chebyshev polynomial. It is they are the 0 of the m-th order Chebyshev polynomial. Is that clear? Rather than m plus 1-th order Chebyshev polynomial. Is that clear? So, the m plus 1th order Chebyshev polynomial is going to have m plus 1, 0 there are actually m plus 1 0.

If you go back and take a look from k is equal to 0, to m so it is not m it is actually there m plus one zeros so we need the zeros of the m plus 1'th order Chebyshev polynomial. The m-th order Chebyshev polynomial is only going to have m 0.

Another very important property of Chebyshev polynomials this is what this is the probably the most important property of Chebyshev polynomials and why it is so valuable is among all the polynomials of order n where n stretches. From 0 to infinity which have leading coefficient 1. So, it can be any order polynomial of any order x to the power 15 x to the power 20. It can be any order polynomial only criteria only requirement is that the leading order term in that polynomial. Has coefficient of 1 for all those polynomials 2 to the power 1 minus n T n. That is the Chebyshev polynomial of the same order scaled by 2 to the power 1 minus n is always going to give me the smallest maximum norm in the interval minus 1 to 1.

The Chebyshev polynomial t n, we saw that is leading order term. So, if was something so if I scale that scale the Chebyshev polynomial with 2 into the 1 to power minus 1 n my leading order term in the Chebyshev polynomial is also going to be 1 is also going to be 1. Because, we saw what did we see that the Chebyshev polynomial leading coefficient of each Chebyshev polynomial is 2 n to the power minus 1.

So, If I scale each Chebyshev polynomial by this factor 2, 1 to the power minus n then the leading coefficient is always going to be equal to 1.So, the Chebyshev I have, a set I have suppose I am I have n is equal to 20. So, and I have the Chebyshev polynomial of t 20 Chebyshev polynomial for a 20 t 20 and I scale the Chebyshev polynomial with all the terms in the Chebyshev polynomial by 20 1 minus 20.So, 2 to the power minus 19. So, the first order in my first the leading coefficient is going to have the term the coefficient 1 it is going to have a coefficient 1.

Now, that polynomial which I get after scaling the Chebyshev polynomial by 2 to the power 1 minus n that polynomial is going to have the smallest norm in the maximum norm in the interval minus 1 to 1 among all polynomials of order 20 provided. They have leading coefficient of the leading order term is 1. So, it is guaranteed that it is going to have this smallest norm, So, by dividing each term of any polynomial of order n we can ensure that the leading order term in the polynomial has coefficient 1 thus 2 to the power 1 minus n t n has the lowest maximum norm I when I said maximum norm I mean norm in the I mean the norm maximum. The infinity norm I mean the infinity norm for the entire set of polynomials of order n suitably scaled. So, let us look at it again.

By dividing each term of any polynomial of order n we can ensure that the leading order term in the polynomial has coefficient 1 thus 2 to the power 1 minus n t n has the lowest maximum norm in minus 1 for the entire set of polynomials of order n suitably scaled suitably scaled. That is divided by it is by the coefficient of its leading order term and we are going to show this but this is very important and is very useful.

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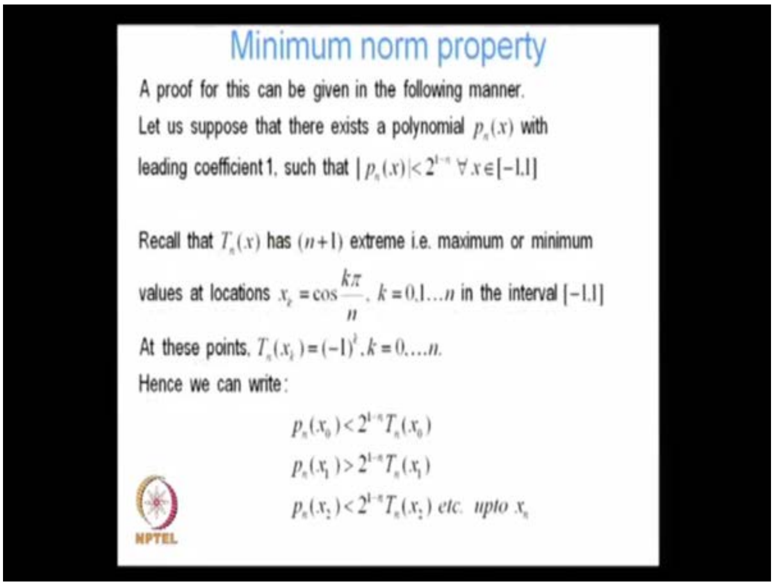

A proof for this can be given in the following manner. Let us suppose that there exist a polynomial p n of x with leading coefficient 1 such that mod of P n of x is less than 2 to the power 1 minus n for all. x belonging to that interval minus 1 So, suppose we have a polynomial P n of x who’s leading coefficient with highest order term has coefficient 1 and we have evaluated P n f P n of x at every point x in the interval minus one to one, and we have seen that the magnitude of the polynomial at each at after evaluation at each of those points is always less than 2 to the power 1 minus n.

So, we have evaluated that polynomial at all points x belonging to minus 1 and we have found that the magnitude of the polynomial is less than 2 to the power 1 minus 1 that is our supposition. Recall that T n of x has n plus one extrema that is maximum or minimum values, we know that we have just seen that and the location of those extreme of those maximum and minimum values is given by x k is equal to cos of k pi by n k is equal to 0, 1 to through n in the interval minus 1 So, at these points T n of x k is equal to minus 1 to the power k T n T n of x. So, the extrema and recall that is given with the Chebyshev polynomials are basically cosines. So, the maximum minimum has to be either minus 1 and 1 or 1 and at these points at these extrema points T n of x k is equal to minus 1 to the power k where k is equal to 0. Through n hence we can write P n of x is less than 2 to the power 1 minus n times T n of x 0. Why, we know that P n of x the magnitude of P n of x is less than 2 to the power 1 minus n and at k is equal to 0. We know T n of x 0 is equal to 1.

So, in that case P n of x 0 must be less than 2 to the power 1 minus n it is absolute value is less than 2 to the power 1 minus n. If, I remove the absolute value sin I know that this is positive so P n of x 0 must be less than 2 to the power 1 minus n times T n of x 0.

What about x 1 t x 1 T n of x 1 is going to be negative x 1 is the 0 corresponding to k is equal to 1 at this point minus 1 to the power k. So, I know that is negative, So, I am guaranteed that P n evaluated at x 1 must be greater than 2 to the power 1 minus n times this is negative and if I know that the this is bounded, this norm of this is less than 2 to the power 1 minus n. So, P n of x 1 must be this is the lower bound that is the upper bound that is the lower bound. So, P n of x 1 must be greater than this value similarly, P n of x 2 must be less than. This value because this becomes positive then So, P n of x 2 must be less than 2 to the power 1 minus n times this so what do we see, that P n of x 0 is basically alternating in sin.

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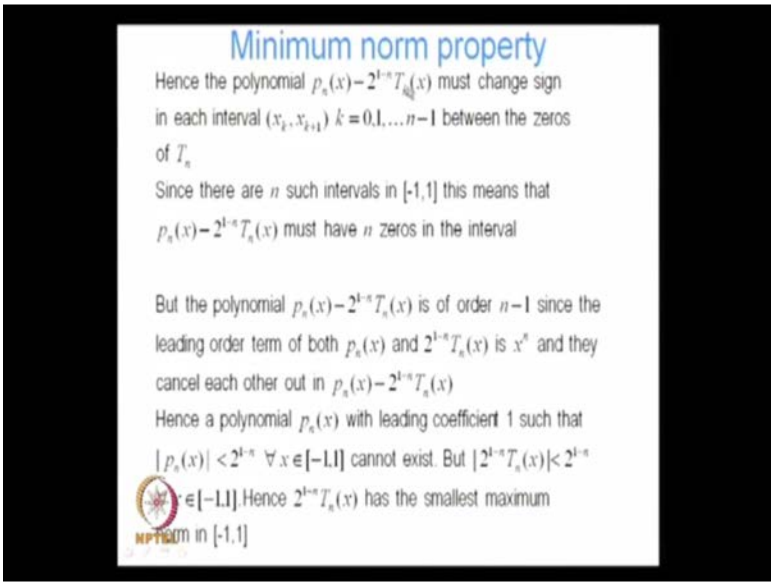

Hence, the polynomial P n of x minus 2 to the power 1 minus n T n x must change sin in each interval. Let us go back and take a look again. So, P n of x is less than this P n of x is greater than that. So, P n of x 2 is less than that. So, if I construct a second polynomial basically, P n of x minus 2 to the power 1 minus n times T n of x that polynomial is going to be alternating inside. Here, it is going to be negative here it is going to be positive; here, it is going to be negative again. So, it basically alternates in sign.

Hence, the polynomial P n x minus 2 to the power 1 minus n T n x must change sign in each interval x k x k plus 1 k is equal to 0 through 1 through n minus 1 between the 0 of T k . So, if I draw if I if on the x axis I have I mark out the 0 of my polynomial of my polynomial T n if I mark out the 0 and then if I evaluate this function. If I evaluate this function then I will find that this function is always changing sign between the 0 between it is changing sign it is become positive it is become negative.

So, in those intervals since there are n such intervals in minus one to one since the n such intervals in minus one to one, this means this function must have n zeros in the interval it is changing sign in each interval it is changing sign between each between these 0. It is changing sign, that means between these 2, 0 it is it has a 0 between the 0 of the Chebyshev polynomial this polynomial has a 0 between each any 2, 0 of the Chebyshev polynomial this polynomial has a 0 and since there are n intervals n intervals between the

0 of the Chebyshev polynomial that means there are n 0 of this polynomial has got n zeros. There are n plus 1 extremes, the n plus 1 extremes.

So, between those 2 there are it is having a 0. So, there must be n 0, n 0 I hope I made myself clear. So, there are n 0 but let us look at this polynomial again P n of x minus 2 to the power 1 minus n T n of x we know that the leading order term of 2 to the power 1 minus n T n of x is 1 because we have divide it by 2 to the power 1 minus n the leading order term of P n of x is also 1 that is what we assumed.

P n of x is a polynomial with leading coefficient 1. So, what does that mean? That means that the leading order terms, they cancel each other out. So, that means that this polynomial P n of x minus 2 to the power 1 minus T n of x is not really a polynomial of order n it is a polynomial of order n minus 1 because it is cancel out the leading order term since, it is a polynomial of order n minus 1 it can have at most n minus 1 routes.

A polynomial of n minus 1 a quadratic equation has 2 routes cubic equation has 3 routes polynomial of order n minus 1 can at most have n minus 1 routes. It cannot have n routes since so since a hence a polynomial P n. So, that means such a polynomial cannot exist hence a polynomial P n x with leading coefficient 1 such that mod of P n of x is lesser than 2 to the power 1 minus 1 in for any x belonging to this interval cannot exist because that was what we assume and if we assume that we got a conclusion, which is Peyton Craig are not be true, which is absurd, so that means such a polynomial cannot exist so that means, what does that mean, but mod of 2 to the power 1 minus 1 minus n T n of x is always less than 2 to the power 1 minus n; why because I know that mod of T n of x is T n of x has got extrema 1.

So, mod of 2 to the power 1 minus t 2 to the power 1 minus n times T m, T n of x must always be less than 2 to the power 1 minus x, but we know that there can be no polynomial with leading coefficient one such that this is less than 2 to the power 1 minus n but in case of this polynomial we know that this is always less than 2 to the power 1 minus n. That means this is the minimum, no other polynomial satisfies this condition, no other polynomial there exist with leading coefficient 1, which satisfies this condition that it is mod is less than 2 to the power 1 minus n, but I know that the Chebyshev polynomial scaled with this quantity satisfies that condition which shows that this polynomial 2 to the power 1 minus n T n of x has the smallest maximum norm in the interval minus 1 to n which is very important.

This is the smallest that is got the smallest maximum norm in the interval minus 1 to 1, it does got very important implications, for error for finding out for optimum location of the grid points to minimize error. We shall see that.

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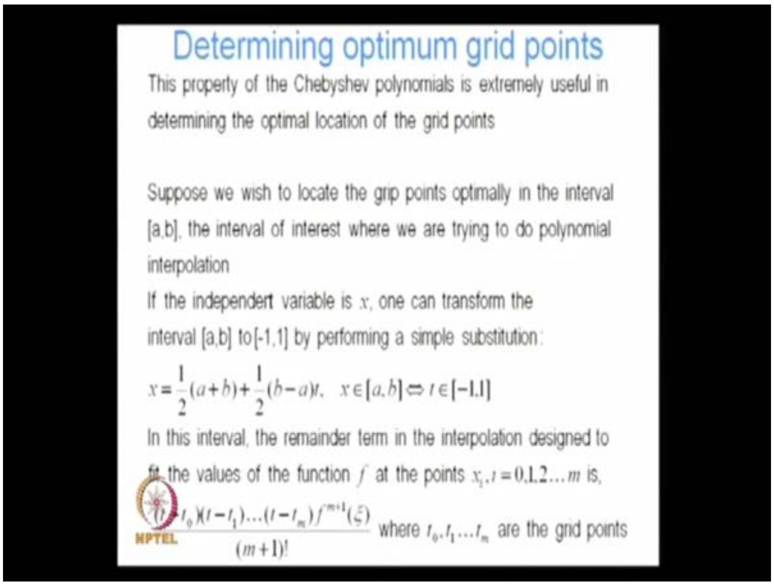

So, this helps us to determine the optimum grid points this property of the Chebyshev polynomials is extremely useful in determining the optimum location of the grid points up till now. We have considered equidistant grid points but it turns out if I looked at an example a couple of lectures back where, we were trying to interpolate, a we were trying to find a fit a polynomial to the function 1 by 1 plus x square I think that was the function and we were using a 10th order polynomial and even then we found that there were very large mismatches near the boundary, if we used a equidistant grid I do not know if you remember that but we saw that.

So, often times if we use the equidistant grid the errors specially for a higher order polynomials and specially at the ends of the interval become, very large therefore, that is why people want to find what should be the optimum location of the grid points.

In order to minimize the error how should I locate my grid points? So, that my interpolate, where error I get from my interpolate is the minimum possible error and that is where Chebyshev polynomials play a very important role. Suppose, we wish to locate the grid points optimally in the interval a b the interval of interest where we are trying to do a polynomial interpolation if the independent variable is x 1 can always transform the interval a b to minus 1 to 1 where performing a simple substitution like that if righting x in terms of t. I can transform this interval a b to minus 1 to 1. So, it does not matter if my that the interval that I am doing, am I interested is not minus 1 to 1. I can always do a transformation in variables and convert that interval to minus 1 to 1 in this term the remainder term in the interpolation designed to fit the values of the function f at the points x, i is equal to 0, 1 through m is given by this.

This we have seen earlier, the remainder term is given by this t minus t 0 t minus t 1 f m plus 1 xi and factorial m plus 1 where t 0 t 1 t m are the grid points this we have seen earlier. We have proved earlier. So, this is known and we recall that xi belongs to xi this function this value xi here, this value xi belongs to the interval which is the smallest interval which contains all the routes as well as x which includes t 0, t 1 through t m as well as t in this case t not x.

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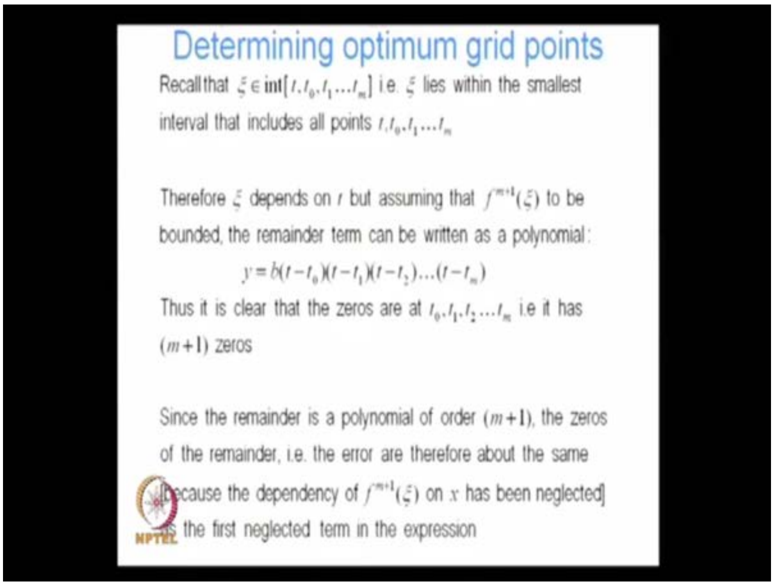

Therefore, xi depends on t. Here, depends on t because xi belongs to the interval. xi belongs to this interval. So, there is a dependence on xi on t but assuming f m plus xi to be bounded that we can give a maximum value to f m plus 1 xi in the interval of interest. We can write the remainder term as a polynomial. Where, b by b I am representing an upper bound to the maximum value of this quantity. So, we can write the remainder term as a polynomial like this, y is equal to b times t minus t 0 t minus t 1 t minus t 2 through t minus t n thus it is clear that this remainder term has 0 at t 0 t 1 t 2 t n that is it has got m plus 1, 0 remainder has got m plus 1, 0.

Since, the remainder is a polynomial of order m plus1 the 0 of the remainder that is the error are therefore, about the same as the first neglected term of the expression. So, if my if I did not curtail it at m, if I did not curtail it at m the next higher order term it would have had m plus 1. It would have had m plus 1, 0 it could have been a higher order polynomial, It would have had m plus 1, 0s and those 0s are relatively close to the 0s of the remainder, Is that clear. But, so you bare with me. Assume that that is true. But, what I want to I will come back to this in the next class.

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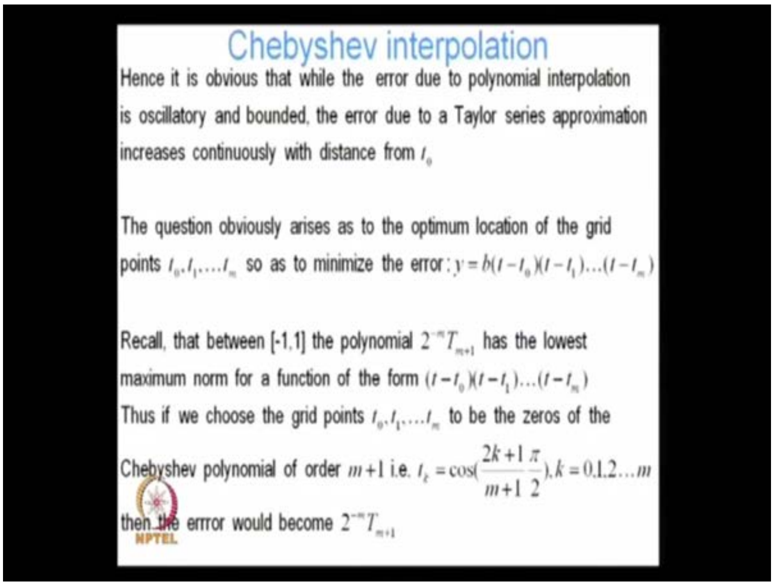

But, before I move to before I end this lecture. I want to talk about why the Chebyshev polynomial is important. So, we know that this is my remainder term and I know that the Chebyshev polynomial the Chebyshev polynomial the 0s of the Chebyshev polynomial give me the smallest have the smallest maximum norm the Chebyshev polynomial has the smallest maximum norm in the interval minus 1 to 1. So, if I locate my grid points t 0 t 1 t 2 t n took of inside with the 0s of the m, Is that clear, f the m plus 1th order Chebyshev polynomial then in that case this remainder term this remainder term I am guaranteed that the remainder term has the smallest possible error in the interval minus 1 to 1 that is why Chebyshev polynomials are very important. That is why the 0s are very important, because I know that the remainder term is given by this and these are coinciding with my grid points t 0 t 1 through t m are coinciding with the grid points.

Now, if I make sure that my grid points coincide with the 0s of the Chebyshev polynomial, and then I am guaranteed that this polynomial which is basically, the remainder has the smallest possible maximum norm. In the interval minus1 to1, So, basically I am minimizing the error, in the maximum norm if I make sure that my grid points are located at the 0s of the next higher order Chebyshev polynomial is that clear that is why Chebyshev polynomials are so important and that is why the 0s give the optimum location of the grid points they give me the where I should locate the grid points in order to minimize my error term. In order to, minimize my error term in order to have the smallest possible error in the maximum norm is that clear.

So, we will stop here, and we will continue with a further some further discussion Chebyshev polynomials which we rewind up, and then we will talk about orthogonal polynomials in general.

Thank you.