Numerical Methods in Civil Engineering
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Lecture - 34
Orthogonal Polynomials-III

In lecture 34 of our series on numerical methods in civil engineering, we will continue with our discussion on orthogonal polynomials and hopefully we will wrap it up and next class, we will move on to something else, which uses orthogonal polynomials.

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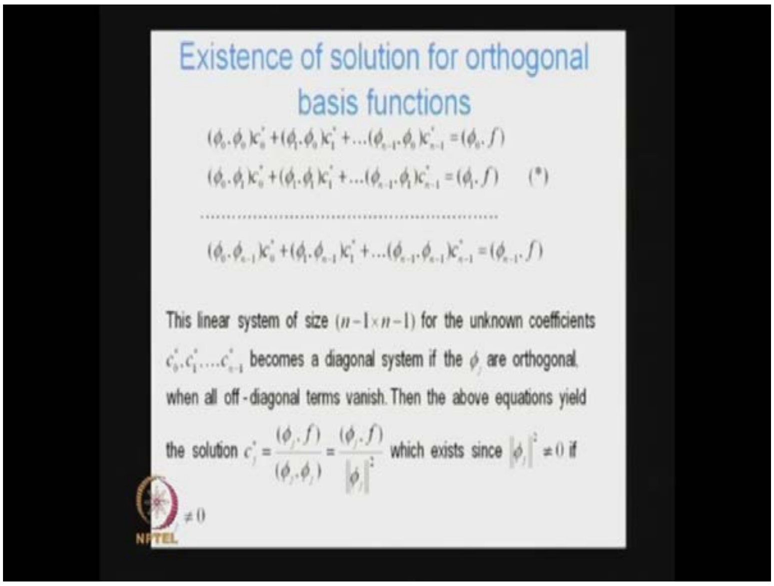

Last time, we ended our lecture talking about the solutions for orthogonal basis functions where, my polynomials, when I have a series of polynomials and each member of the series is orthogonal to the two other members of that series and we showed that for such a series for such an orthogonal series of polynomials are the solution to the best fit problem always exist. The solution to the best fit problem always exist and recall finding the solution to the best fit problem means, finding these coefficients c star, c star. I goes from 1 to n and we showed that for an orthogonal polynomial the this left hand side, it becomes a diagonal system all the off-diagonal terms cancel each other out because, this polynomials are orthogonal to each other.

So, we have a diagonal system and we can solve that diagonal system, we can invert that diagonal matrix to find these coefficient matrices like this and this is always going to have a solution, because norm of phi j square is always going to be different for greater than 0. So, in that case, we always get a solution. What about the situation, we end of the lecture talking about linearly independent systems, we did not go into detail but, what about the situation when instead, where, instead of the polynomials being orthogonal to each other, all we know is that, they form a linearly independent basis a set of functions, which are linearly independent but, not necessarily orthogonal in that case are we assure that our solution to the best fit problem exists.

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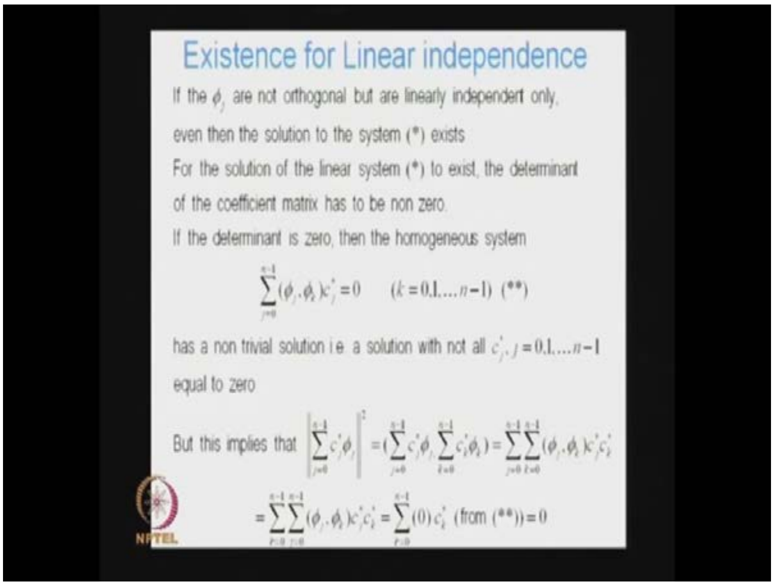

So, if the phi j is not orthogonal then the solution to the problem also exists and it can be proved like this for the solution of the linear system to exist the determinant of the coefficient matrix has to be non-zero.

That is this coefficient matrix, that determinant has got to be non-zero. So, if the determinant is 0, suppose it is 0, if the requirement has to be non-zero but, suppose that is 0, we will show that, if it is 0 then, we get a result which is not possible, which means that it cannot be 0. So, if the determinant is 0 then, the homogeneous system has a non-trivial solution. So, if the determinant is 0 this thing is equal to 0 then, this system has got a non-trivial solution. So, if this equal to 0, if we try to solve that system, if the determinant is equal to 0 then, we can find values of c, c star j, j equal to 0 to n minus 1 where, not all the c star j's are 0 and not all the c star j's are 0. So, has a non-trivial solution, that is a solution with not all c star j. j is equal to 0, 1 through n minus 1 equal to 0.

So, what does this imply? Let us see what it implies. So, sigma j equal to 0 to n minus 1 c j star phi j star if I take the L 2 square norm of that I can represent. It is an inner product sigma j equal to 0 n minus 1 c j star phi j inner product with sigma k equal to 0 n minus 1 c k star phi k, where I have instead of using j twice, I have replaced j with k in the 2 term in the inner product and then I pull out the summation signs outside, my inner product and I have inner product of j comma phi k times c star j c star k and then, I change the order of the summations. So, this I can write, I replace, I put sigma k equal to 0 to n minus 1 first. So, I just interchange the order of the summations and I get this and then, I look at this term the sigma j equal to 0 to n minus 1 phi j, phi k, c star j.

I look at these two, I look at the summation involving these two and this t has got to be equal to 0, because that was our assumption, that is the determinant is 0 then, the homogeneous system has a non-trivial solution and the homogeneous system is just this. So, if this is equal to 0 then, I have sigma k equal to 0 to n minus 1, 0 c star k which is got to be equal to 0. So, I have this equal to 0. So, what does that mean?

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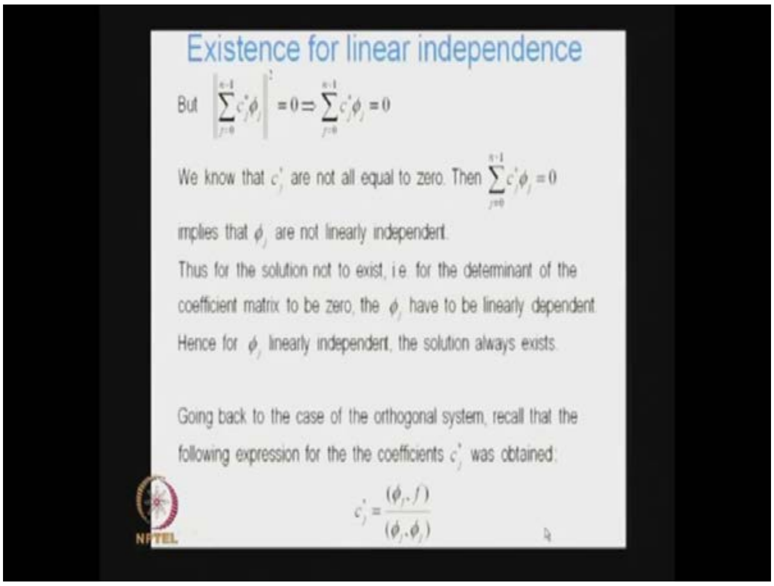

That means, that sigma j equal to 0 to n minus 1 c j star phi j equal to 0, because Inner product of this with itself is equal to 0 therefore, this itself must be equal to 0. So, then this is equal to 0 but, we know that all the c j stars are not equal to 0, why because, my homogeneous system has a non-trivial solution. So, all the c j stars are not equal to 0 and now I am showing that sigma j equal to 0 n minus 1 c j star phi j is equal to 0, what does that mean. That is impossible, if all my phi j's are linearly independent. If this is satisfied as well as the fact that not all my c j stars are equal to 0 is true, then not all the phi j's are linearly independent. The phi j's are not linearly independent. Thus, for the solution not to exist that is for the determinant of the coefficient matrix to be 0 in which, case only we get a non-trivial solution to the homogeneous system. The phi j's having to be linearly dependent, hence for phi j linearly independent, the solution always exists.

For, why does the solution always exist, because the determinant cannot be equal to 0 we saw that, if the determinant is equal to 0, we get a result which implies that, they are not linearly independent. So, if they are not linearly independent determinant cannot be 0 and a solution has to exist. So, let us go back to the case of the orthogonal system and recall that, this was the expression we obtained for the coefficient c j star which we just saw couple of slides back and let us look at it closely again.

So, c j star is equal to inner product of phi j comma f divided by inner product of phi j comma phi j. So, what does that mean? that means each of the coefficient c j are do not depend only on quantities with index j and of course, on the function f they do not depend on other polynomials, they do not depend c j star does not depend on phi k where, k not equal to j c j star, just depends on phi star j, it is just depends on the coefficient for the polynomial phi j in the orthogonal expansion, just depends on the polynomial phi j, it does not depend on phi j plus one phi j minus 1 or any other polynomial . What does that mean?

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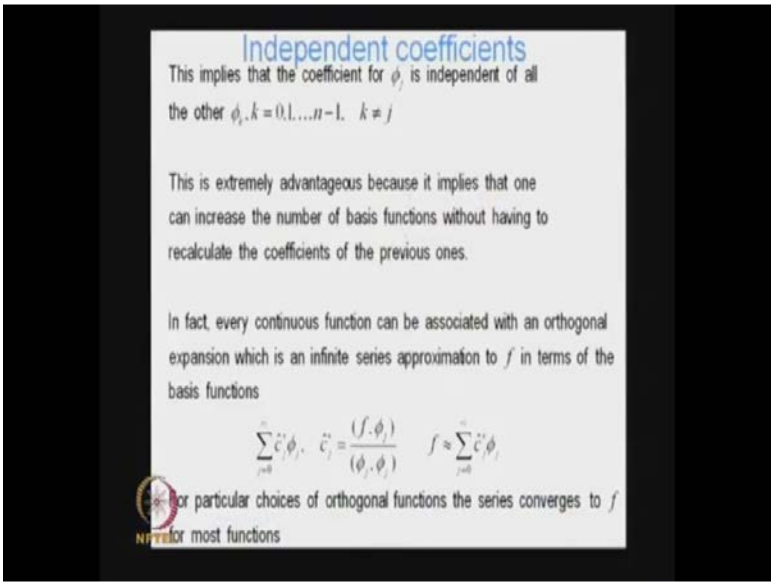

That means that, these coefficients are independent of other phi j's and this is extremely advantages, because it implies that, one can increase the number of basic functions without having to recalculate. The coefficients of the previous one suppose, I do an orthogonal expansion up to 5 members of the series and then, I find the error is still not sufficiently small, I need to reduce the error further and I know that, my series is convergent. We are going to talk about convergence slightly later but, if I know my series are convergent then, if I add more terms to that series I am certain, that I am going to get closer to the solution. So, I add more terms to that series and if I add more terms to that series. I need to evaluate those coefficients c j for those additional terms but, when I evaluate the new coefficient c j, I am assure that, I do not have to recalculate the coefficients of the polynomials, which I am already using, which makes it good, which makes it efficient.

So, in fact every continuous function can be associated with an orthogonal expansion which is at infinite series approximation to f in terms of the basic functions up till now, we have just taken a finite number of basic functions but, it is an every an any orthogonal series, we can take an infinite series is finite, because the function space is actually infinite-dimensional. So, the infinite-dimensional function space is going to have infinite number of basic functions orthogonal to each other. So, suppose we take such an expansion c j star phi j and we evaluate this c j star like this and we say f is approximately equal to sigma j equal to 0 to infinity c j star phi star j.

For most functions, it can be shown that, this orthogonal function, the series representation is convergent. It is going to if we take a sufficiently large number of terms, it is going to converge to f well, I said for most functions it should be obvious to give that, it cannot be true for all functions, what sort of functions will not even if I take an infinite number of terms in the series as still would not converge to the function, there are certain functions like that for instance functions, which have singularities, you cannot expect suppose, I have a delta function what even if I take an infinite number of terms in my series I am not going to converge to a delta function. So, given that my function is well behaved by well behaved I mean it possess sufficient degree of smoothness sufficiently high continuity requirements I can find a convergent infinite series to that comprising of orthogonal basis functions.

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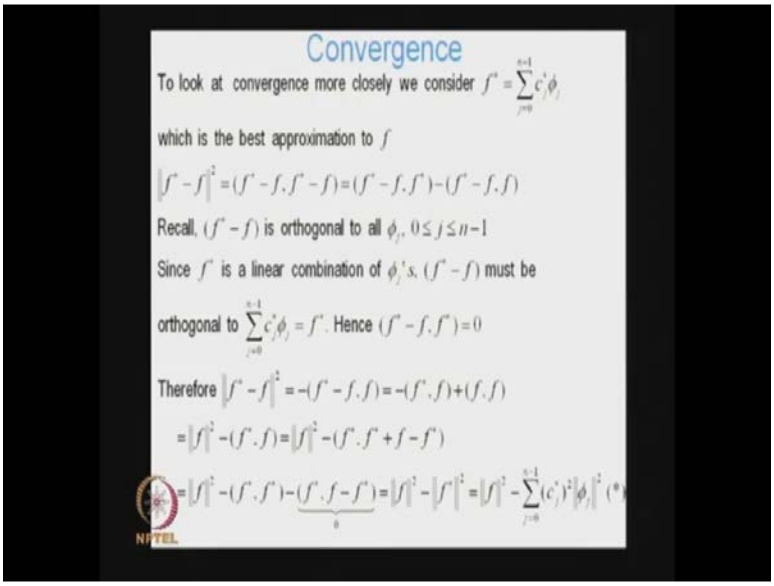

So, to look at convergence more closely, let us consider f star which is my best approximation to f which is my best approximation to f and let us look at the norm of the error the L two norm of the error f star minus f norm of square, which I can represent as the product of as an f star minus f, this I can rewrite as f star minus f inner product with f star minus, f star minus f inner product with f. so, I just split it up.

The count of the linearity of the inner product, I instead of writing f star minus f here, I wrote as I split it up to f star minus f here and let us recall that f star minus f is orthogonal to all of phi j why, because f star is my best approximation and the best, if it is a best approximation. The remainder f minus f star or f star minus f is going to be orthogonal to the space spanned by the basic functions phi j. It is going to be orthogonal to the space spanned by the basic functions phi j. So, we must get and since f star is a linear combination of phi j's f star minus f must be orthogonal to this sigma c j phi c j star phi j is equal to f star then. Hence, f star minus f comma f star is equal to 0, is that clear?

Well f star minus f is orthogonal to the space spanned by the phi j's and f star is nothing but, a linear combination of the phi j's. So, f star is equal to c 1 phi 1 plus c 2 phi 2 plus c 3 phi 3. So, if I take advantage of the linearity of the inner product. So, I can write f star minus f inner product of f star minus f comma c1 phi 1 plus inner product of f star minus f comma c 2 phi 2 plus inner product of f star minus f comma c 3 phi 3 and so on and so forth. And each of those inner products is going to be 0. Why because, f star minus f is orthogonal to the space spanned by the phi j's that means, it is orthogonal to each of the phi j's. So, it must be 0. So, f star minus f inner product with f star is equal to 0. So, then what we have, what do we have f star minus f norm of norm square is equal to this. So, this part becomes a 0 and we have minus f star minus f comma f.

Which I can write as minus f star comma, f plus, f comma f and minus f star comma f let it remain and this f comma f becomes norm of f square. So, norm of f square minus f star comma f is equal to norm of f square minus f star and then I am rewriting this f just to make it explicit I am rewriting this f as f star plus, f minus f star then, what do I have norm of f star square norm of f square minus f star comma f star inner product minus f star comma f minus f star and again, this is equal to 0 because, f minus f star is going to be orthogonal to this space spanned by all the phi j's f star is a linear combination of the phi j's. So, this term becomes 0, this term becomes 0 and I have norm of f star minus f whole square is equal to norm of f square minus norm of f star square, which is equal to norm of f square minus sigma j equal to 0 to n minus 1 c j star square norm of phi j square, because that is the representation of f star in terms of phi j's. So, I have this norm of f star minus f whole square is equal to norm of f square minus this.

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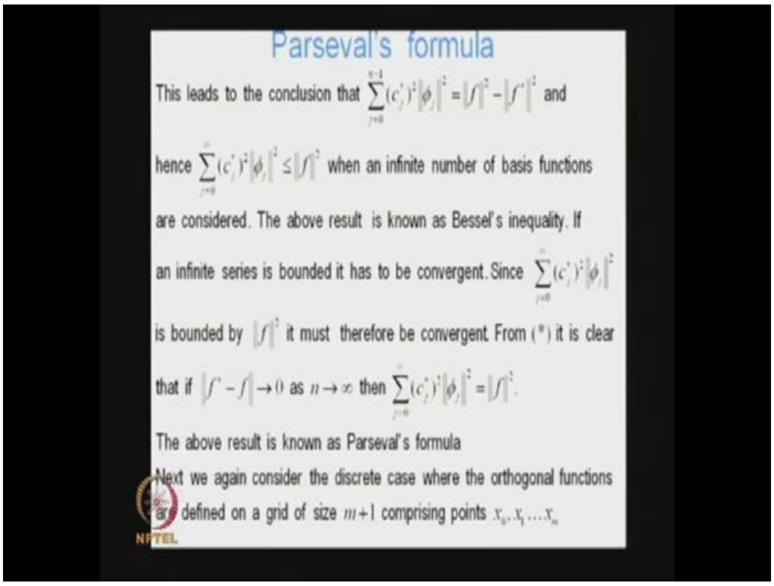

So, this is very important because, it tells us that sigma j equal to 0 to n minus 1 c j star norm of phi square is equal to norm of f square minus norm of f star square. So, this is equal to norm of f square minus norm of f star square, is that clear? So, I have that. So, what does this means that, this is always less than norm of f square. So, the best approximation is always lesser than or equal to norm of f square. So, this is true for n minus1 is also going to be true, if a when you have infinity.

So, c j star square norm of phi j square is going to be lesser than or equal to norm of f square, when an infinite number of basic functions are considered and this above result is known as Bessel's inequality this is known as Bessel's inequality. So, this is an infinite series you can see this is an infinite series and it says that, this is lesser than or equal to norm of f square. So, when infinite series is bounded if an infinite series is bounded, it is always got to be convergent. It is convergent and since sigma c j square norm of phi j square is bounded by norm of f square. It must be therefore, be convergent and from this, is that clear? In that case, what are we going to get that norm of f star minus f is going to 0 as n goes to infinity. In that case, this is going to be equal to that, if this is going to be equal to that.

That means, this thing is going to be equal to 0 therefore, norm of f star minus f is going to go to 0 as n goes to infinity. And this result is known as Parseval's formula. So, this was about the continuous case, when I had these phi j's defined, when these phi j's are basically continuous functions in the interval, in the domain, in which they are defined, they are continuous functions in the domain. However, let us now look at the case when, we have orthogonal functions, which are non continuous that is the discrete case, when these orthogonal functions are defined on a grid of size m plus 1 comprising points x 0, x 1 through x m can we say anything similarly, interesting in that case.

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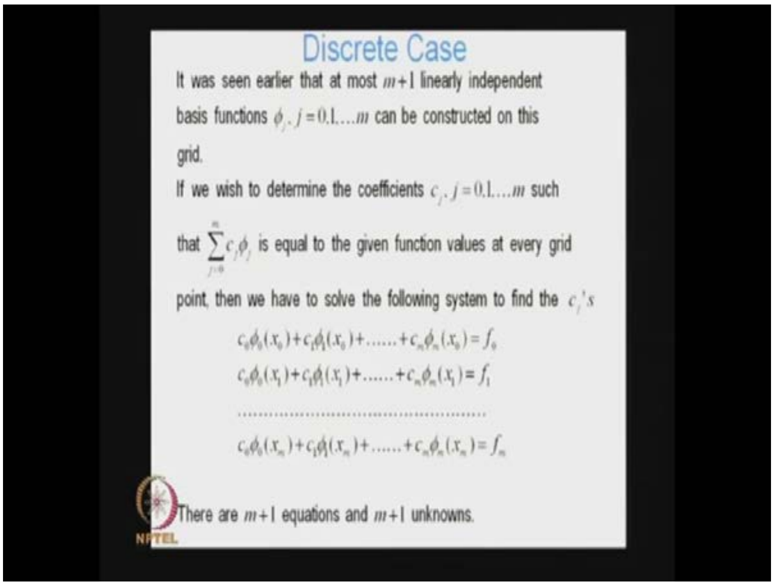

It was seen earlier, that in case we have a grid with m plus 1 points then, we have at most m plus1 linearly independent basis functions phi j. j is equal to 0, 1 through m in that case, there is no possibility of has having infinite number of basic functions because, we have a finite number of grid points and on a finite number of grid points. We have seen earlier that, we can have at best the same number of independent basis functions, same number of independent basis functions.

o, if we wish to determine the coefficient c j. j is equal to 0 through 1 to m such that sigma j is equal to 0 to m c j phi j is equal to the function values at every grid point then, we know that, we have to solve the following system of equations to find the c j's basically, I say that at each grid point my series must give me the function value. So, at x 0 this is going to be c j c 0 phi, phi 0 at x 0 plus c 1 phi 1 at x 1, x 0 and so on and fourth, plus c m phi m at x 0 . So, this has got to be equal to f 0. Similarly, at x 1 it has to satisfy that condition and at x m, it has to satisfy that condition. So, we are interested in finding the m plus 1 coefficient c. So, there are m plus 1 unknown and there are m plus 1 equation, but this coefficient matrix.

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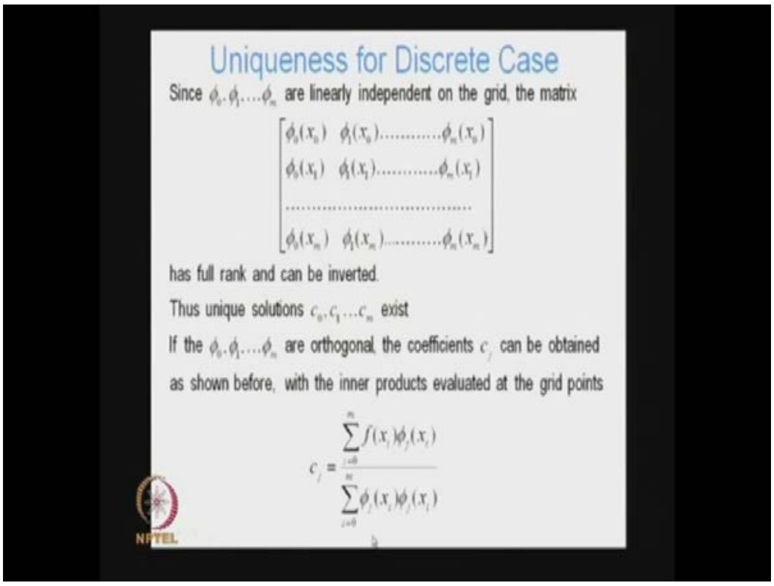

So, then my coefficient matrix is phi 0 x 0 phi 1 x 0, phi m x 0, phi 0 x1 through phi m x 1. So that is my coefficient matrix, we know that coefficient matrix if it is linearly independent. In that case we are going to get solutions. We are going to get unique solutions for my coefficients c 0 through c m and we know that coefficient matrix is going to be non-singular why because, it has got full rank why does it have full rank because, all my vectors phi 0 phi 1 through phi m are linearly independent. It has got full rank and can be inverted and therefore, you have, I have unique solutions c 0 c 1 through c m.

So that is in case, it is linearly independent in case, it is orthogonal that, we can obtain the coefficients in a very similar manner to the way we obtained the coefficients for the continuous case and instead of having an inner product. I just have a sum in the instead of having a inner product in the numerator and inner product in the denominator. I just have a sum in the numerator and a sum in the denominator and that sum is over the grid points. So, I have f at evaluated at the grid points, my function value are evaluated at the grid points divide that by this sum, I get my coefficients.

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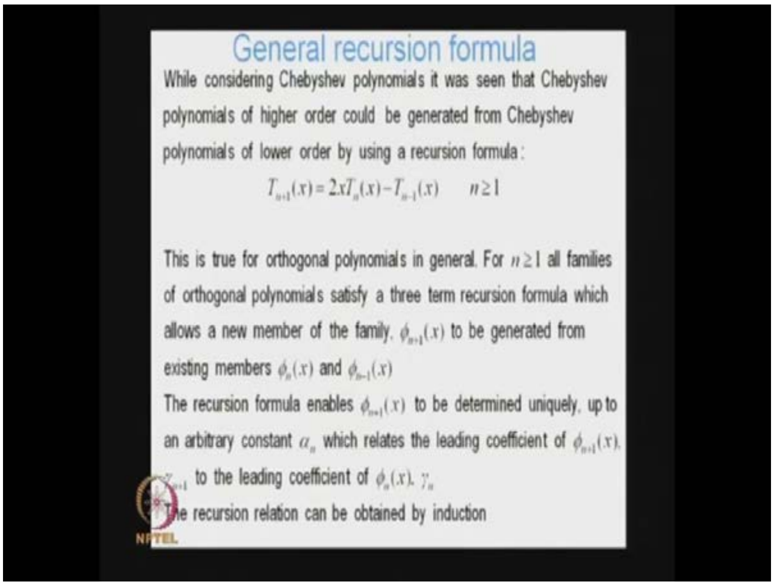

So, when we were, that is enough about finding the coefficients for an orthogonal system Now, we are going to talk about the Recursion formula, Recursion formula is very useful because, if some of these as you can see some of these polynomials they may be complicated. So, they may have complicated, they may be higher powers of x it is very coefficients which are not, which can be basically any real numbers. So, it is not possible for us to remember them but, if we have a Recursion formula as we said in case of the chebyshev polynomials, where if we know some terms in the series it is possible for us to generate other terms in the series.

We saw that, for the chebyshev polynomials if we know the first 2 terms in the chebyshev polynomials, we can generate all the terms in the chebyshev polynomials all the higher order chebyshev polynomials are using the Recursion formula. It turns out that for all orthogonal polynomials, it is possible to get write a Recursion formula and that Recursion formula has a standard form, the form is the same for all orthogonal polynomials, whether they may be Legendre polynomials gram polynomials, any polynomials, any set of orthogonal polynomials if we know the first two terms in that series, it is possible for us to generate the other terms in that series and that is because, they have a general Recursion formula for n greater than or equal to 1, all families of orthogonal polynomials satisfy a 3 term Recursion formula, which allows a new member of the family phi n plus 1 x to be generated from existing members phi n x and phi n minus 1 x .

The Recursion formula enables phi n plus 1 x to be determined uniquely, that is up to an arbitrary constant alpha n, we know it up to an arbitrary constant, which relates the leading and that arbitrary constant relates the leading coefficient of phi n plus 1 x, which is going to involve a term x to the power n plus 1, it relates that to the leading coefficient of phi n x. So, I am calling the leading coefficient of phi n plus 1 x comma n plus 1 leading coefficient of phi n x comma n and we can obtain the Recursion relationship by induction. So, let us quickly look at that. So, suppose we know the polynomials up to order j in an orthogonal series, we know the polynomials up to order j. So, I know phi 0 phi 1 phi 2 up to phi j and I am interested in finding the next higher order member of that series or suppose, I said n suppose I know up to n.

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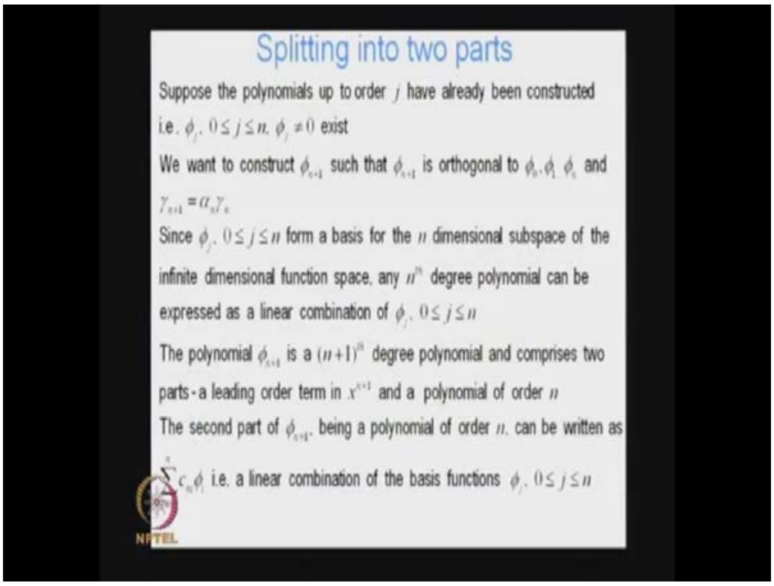

So, I am interested in finding phi n plus 1 and I have 2 conditions that I have to satisfy. My phi n plus 1 has to be orthogonal to all my previously generated phi's. So, it has to be orthogonal to phi 0, phi 1 through phi n and also, it is leading order coefficient gamma n plus 1 the coefficient of the leading order term in phi n plus 1 will is going to be a scalar multiple of the leading order term in phi n. So, these are the conditions, which under, which we can get the Recursion formula.

So, since phi j forms a basis for the n dimensional subspace of the infinite-dimensional function space, I know phi 0 up to phi n. So, I know n basis functions, I know the basic functions for my n dimensional subspace of my infinite-dimensional function space any nth degree polynomial can be expressed as a linear combination of the phi j's from j equal to 0 to n. So, any nth degree polynomial can be expressed as a linear combination of my known basis functions for the n dimensional subspace function of the infinite-dimensional function space.

The polynomial phi n plus 1 is a n plus nth degree polynomial and it comprises two parts, another n plus nth degree polynomial must have a term, which involves powers of x where, x is my independent variable x raise to the power n plus 1 and it must have other terms which, I can represent as a nth degree polynomial. So, it is an nth degree polynomial plus a term which is to the power n plus 1 and comprises 2 parts a leading order term in x n plus 1 and a polynomial of order n the 2 part of phi n plus 1 being a polynomial of order n can be written in terms of my known basis functions, which I know up to order n. So, I can write it as sigma I equal to 0 to n c n i phi I, that is a linear combination of the basic functions phi j. j greater than or equal to 0 lesser than or equal to n. So, phi n plus 1 comprises 2 parts, 1 part can be written as a linear combination of the basic functions phi 0 through phi n and it also involves another part, which is a power which is x to the power n plus 1 times some coefficient.

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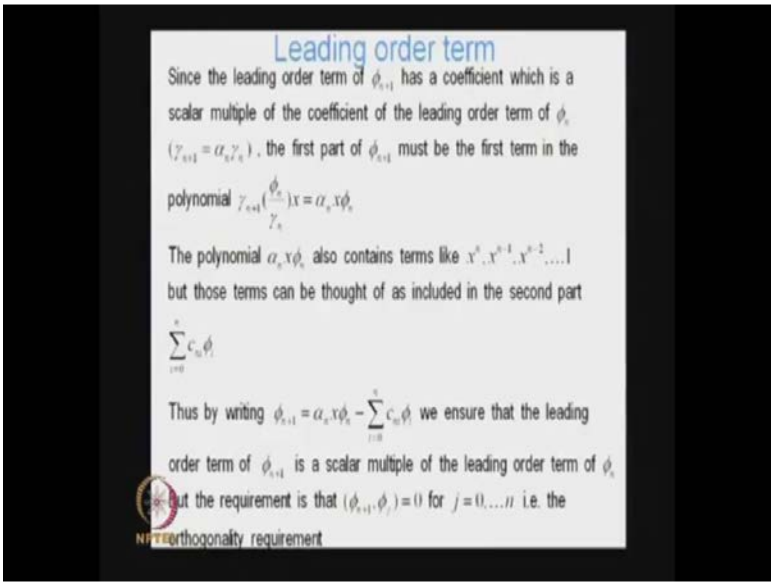

Since the leading order term of phi n plus 1 has a coefficient which is a scalar, this is our 2nd condition, which is a scalar multiple of the coefficient of the leading order term of phi n that is gamma n plus 1 is the coefficient of x n plus 1 in phi n plus 1 gamma n is the coefficient of x n in phi n and I know that gamma n plus 1 can be written as alpha n times gamma n. The first part of phi n plus 1 must be the first term in this polynomial suppose, I generate another polynomial phi n, I take my original polynomial phi n, I divide every term in that polynomial by gamma n the coefficient of its leading order term and then I multiply it by x I multiply it by x and I scale it by gamma n plus 1. So, this is going to be a n plus 1th order polynomial is phi n is n nth polynomial, I multiplying each term by x. So, this going to be a n plus another order polynomial. I am dividing each term in that polynomial by the leading order by the coefficient of the leading order term of phi n and I am multiplying each term also by gamma n plus 1. So that means, the first term of this polynomial is going to be gamma n plus 1 times x n plus 1 is that clear.

The first term of this polynomial must be the first term of phi n plus 1 and this polynomial I can write as gamma n plus 1 by gamma n which is equal to alpha n. So, alpha n x phi n, the polynomial alpha n x phi n also contains terms like x n x n minus 1 x n minus 2, 1. it also contains the term phi x n to the power n x to the power n plus 1 because, there is x phi n it also contains terms like this but, these terms can be thought of these terms are part of my these are all part, these are all, these can also all be included in a polynomial of order n.

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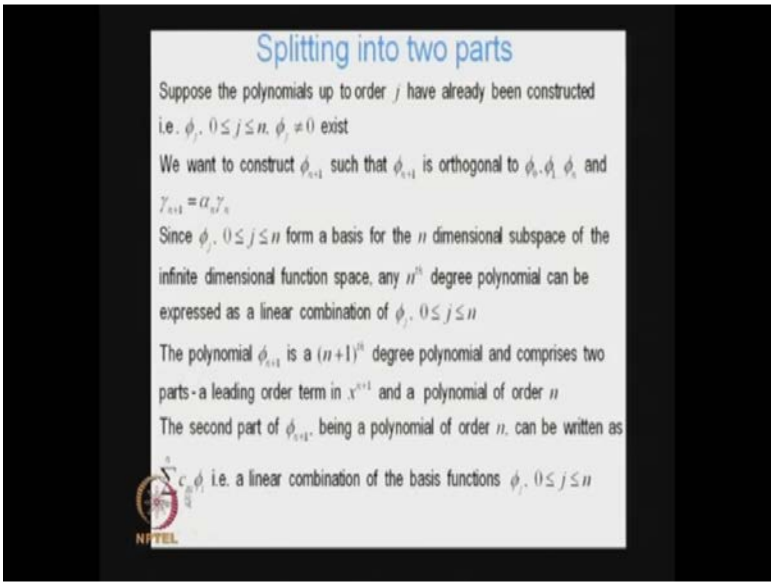

So, these can be thought of as part of my two terms. These are part of my two terms. So, I have a leading order term which is given by the first term in this series, first term in this polynomial and then I have this part. So, we can write, if we write phi n plus 1 is equal to alpha n x phi n minus sigma I equal to 0 to n c n i phi i, we are going to ensure that, the leading order term of phi n plus 1 is a scalar multiple of the leading order term of phi n, why because, the term which involves x n plus 1 is nothing but, alpha n gamma n times x n plus 1. So, the leading order term is a scalar multiple of the leading order term of phi n, we have ensured that if we write it like this. The term which involves x n plus 1 in this expression can be obtained from my old expansion my last term phi n, if I take the leading order term of phi n divide it. It is coefficient and multiplies by this thing gamma n plus 1.

So that is and that is equal. So, and that is equal to alpha n x phi n. So, the first term can be obtained by taking a scalar multiple of the leading order term of phi n and then, I have other terms which are all polynomials of order n which is a polynomial of order n. So, if this is my next term in my series, in my orthogonal series, it has to satisfy another very important condition and what is that condition? It has to be orthogonal to all my phi j's j equal to 0 to n all the lower terms in that series, it has to be orthogonal to that. So, this inner product has got to be satisfied. So, what does it mean if I have to satisfy the inner product well? Let us see.

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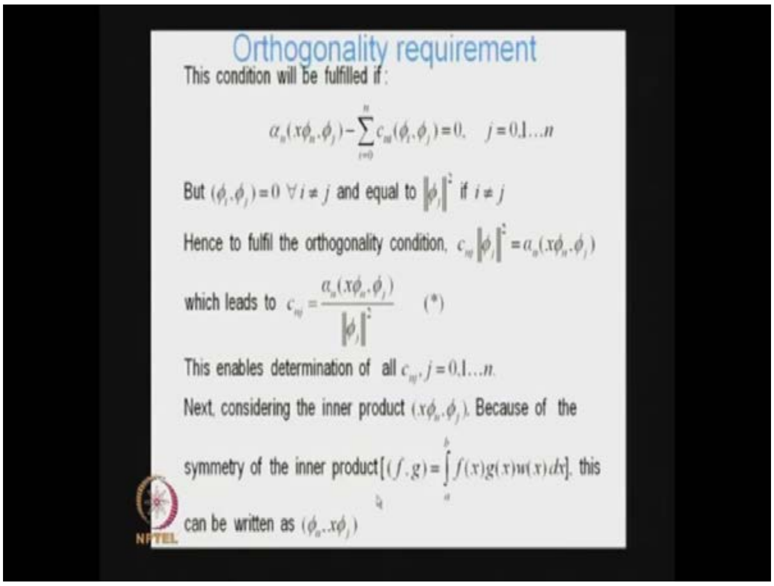

So, in this if this inner product is to be fulfilled. What it must satisfy is alpha n x phi n comma phi j minus sigma i equal to 0 to n c, n i phi i comma phi j has got to be equal to

0. So, this condition has to be satisfied. But, I know that phi i comma phi j is equal to 0 for i not equal to j and is equal to norm of phi j square if I is equal to j sorry, this is a type. So, phi is equal to phi j square is phi is equal to j hence in order to fulfill the orthogonality condition I know. So, only the term that is going to survive here, for each of this j's for j equal t 0 to n is going to be c n j norm of phi j square where, as all the other terms are going to be 0. So, c n j phi j square must be equal to alpha n x phi n comma phi j which tells me that c n j must be equal to that. So, if c n j satisfies that condition if my c n j is obtained like this then, my phi n plus 1 is a valid member of my series of my orthogonal series.

It turns out that, this many of these c n j's actually become 0, many of these c n j's become 0 why, well let us look at the inner product x phi n comma phi j, because of the symmetry of the inner product. I can write x phi n comma phi j as phi n comma x phi j, because the inner product is actually an integral and within the integral I can change the order of that, the terms in the integrant. So, I can write x phi n comma phi j as phi n comma x phi j.

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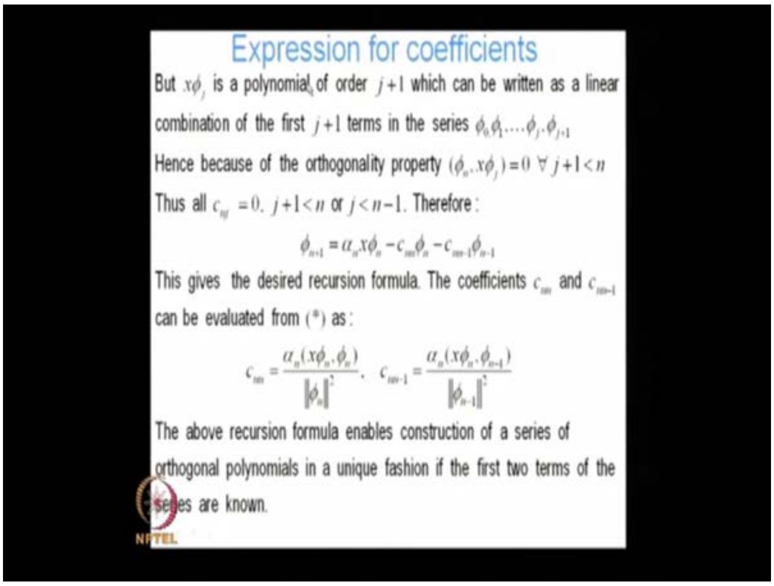

But, x phi j is a polynomial of order j plus 1; phi j is a polynomial of order j. So, x phi j is a polynomial of order j plus 1 which can be written as a linear combination of the first any polynomial of order j plus 1 can be written in terms of my j plus1 basis functions, in terms of my j plus 1 basis functions. So, this polynomial x phi j which is a polynomial of order j plus 1 can be written in terms of the j plus 1 term in the series phi 0 phi 1 phi j through, phi j plus 1.

Hence, because of the orthogonality property therefore, what do we get? We have phi n comma x phi j. So, in this part, I can represent as a sum of j plus 1 basis functions. So, if n is greater than j plus 1 then, what do I have? Then, this inner products got to be 0, because if n is greater than j plus 1 n contains phi 0 phi 1 through phi j plus 1 plus some additional basis functions and those additional basis functions are all orthogonal to the j 0 through j plus 1 .

So, phi n comma x phi j is going to be equal to 0 for n greater than j plus 1. For n greater than j plus 1 this is going to be 0, thus all c n j is equal to 0 for j plus 1 less than n or j less than n plus 1 therefore, what do we have? Let us go back to our little series here. So, we have shown that this is true, this is my representation. Because all the terms in that sum all my c n i's or my c n j's are given by this inner product and I know that, this inner product is going to be 0 for all j less than n minus 1.For all j less than 1, this inner product is going to be 0.

So, the only terms that is going to survive. The only c n j's that are going not going to be 0 are when j is equal to n and j. j is equal to n and j is equal to n minus 1, because for j less than n minus 1 all these coefficients are going to 0. So, these are only non-zero coefficients are going to arise when, j is equal to n minus 1 or j is equal to n. And therefore, my series I get rid of all the terms in that series up to 0 to n minus 2 the only terms, which survive are n minus 1 and n so I can write phi n plus 1 in terms of phi n and phi n minus 1 and thus I get my recursion formula.

So, I can write phi n plus 1 is equal to alpha n x phi n minus c n phi n minus c n, n minus 1 phi n minus 1 that tells me that, I can generate any new member in that series just from my last two members of the series using this recursion formula and this recursion formula is true is holds good for all orthogonal polynomials, whatever be the orthogonal polynomials, whatever be the type. It is true for all orthogonal polynomials; they are going to satisfy recursion formula like this.

And these c n, n and c n, n minus 1, we can evaluate from this formula from this formula.

So, we get c n, n is equal to alpha n x phi n phi n by norm of phi n square and c n, n minus 1 is equal to did I made mistake? No, I am fine, so alpha n x phi n phi n minus 1 norm of phi n minus 1 square. So, the above recursion formula enables construction of a series of orthogonal polynomials in a unique fashion, if the first 2 terms of that series are known if we know the first 2 terms if I know phi 0 phi 1 I can generate phi 2, if I know phi 1 phi 2, I can generate phi 3 and so on and so forth.

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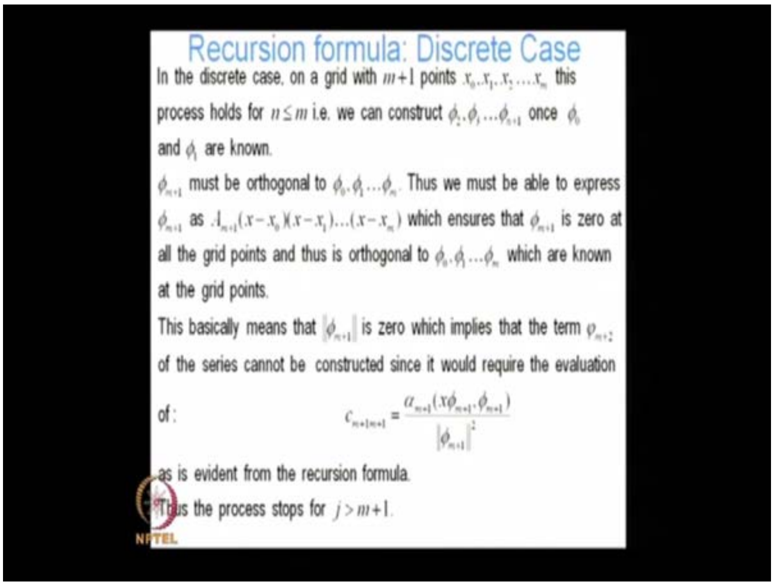

Similarly, for the discrete case also, this can be true in the discrete case on a grid with m plus 1 points x z 1 through x m this process holds for n lesser than or equal to m, that we can construct phi 2, phi 3 up to phi. This should be m plus 1 m plus but, once phi 0 and phi 1 are known once phi 0 and phi 1 are known, so phi m. So, let us look at the last term in that series in a discrete set of polynomials orthogonal polynomials phi m plus 1. So, phi m plus 1 has got to be orthogonal to phi 0, phi 1 through phi n.

Thus, we must be able to express phi m. So, because it has to be orthogonal to all those polynomials and has to be orthogonal to them at all the grid points that means, it has to be 0 at all the grid points. So, because of that phi m plus 1 has this form a m plus 1 x minus x 0 x minus 1 through x minus x m which ensures, that phi m plus 1 is 0 at all the grid points and thus is orthogonal to phi 0, phi 1 through phi m which are known at all the grid points, since phi m plus 1 is 0 at all the grid points. It basically, means that norm of phi m plus 1 is 0. So, norm of phi m plus1 is 0 so the last term the last term phi m plus 1 phi m plus 1 is got to be.

So, I had x 0, x 1 through x n. So, I have m plus 1 point. I know that, I can only have m plus 1 linearly independent basis functions. So, phi 0, phi 1 through phi n those are my m plus 1 basis functions and I find that phi m plus 1 is always going to be equal to 0. If phi m plus 1 is equal to 0, I cannot generate phi m plus 2. Because, I have to evaluate this term norm of phi m plus 1 square, norm of phi m plus 1 square for orthogonal polynomials. So, this I have to divide and this becomes 0. I cannot evaluate the next term, is that clear? So, the process stops for j greater than m plus 1. So, even for the discrete case, it is possible to evaluate the coefficients recursively, once I know my first two coefficients, is that clear?

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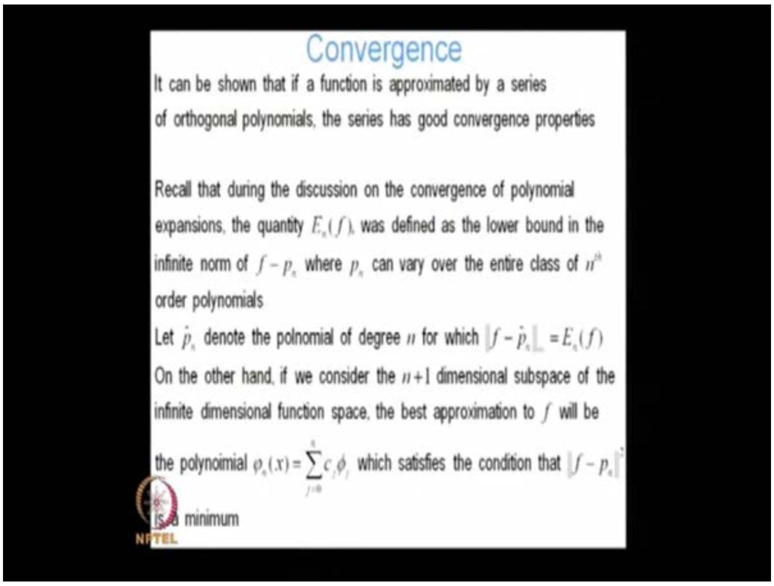

So, it can be shown that if a function is approximated by a series of orthogonal polynomials the series has good convergence properties. So, recall when we are discussing convergence of polynomials not orthogonal polynomials, we came across this quantity E n f, we came across this quantity E n f, which was defined as the lower bound in the infinite norm of f minus p n, p n is my polynomial approximation to f of order n and I said that E n f is the lower bound over all the set of another order polynomial lower bound in the maximum norm E n f E n f is the lower bound in the maximum norm of maximum or infinite norm of this difference. So, I have a function f, I am trying to approximate it by a polynomial of order n and I want to find the polynomial which gives me the lowest difference between f and itself throughout the interval, the lowest difference in the maximum norm and that difference, that quantity is known as E n f.

We talked about that earlier lower bound in the infinite norm of f minus p n, the lower bound value on this. We have seen that and let p. Now, let us say, that suppose p hat n denotes the polynomial of degree n for which norm of f minus p n hat infinity is equal to p n f. So, we are trying to find the polynomial among all the set of another order polynomials which gives me the smallest value of f minus p n throughout my interval and suppose that polynomial is p n that then, norm of f minus p n that the infinite norm is equal to E n f.

On the other hand, we know if we consider the n plus 1 dimensional subspace m plus 1 dimensional function space. We know the best approximation to that function f has the smallest norm, smallest L 2 norm in that case the error has the smallest L 2 norm, f minus p n norm L 2 norm square that is the smallest. So that is the best approximation in the L 2 norm, the best approximation in the L 2 norm may not be the same as p that n because, p that n is the gives me the best approximation in the infinite norm, it gives me that value of the function, that value of the another order polynomial, which gives me the maximum, the minimum error in the infinite norm.

But, phi n which is obtained by taking a linear combination of the n plus 1, basis functions of the n plus 1 dimensional function space has the best fit. It has the lowest error but, that lowest error is in the L 2 norm, it is in the L 2 norm. So, what is the relationship between e n f and this function phi n which is my best fit in the L 2 norm or what is the relationship between phi n and p hat n that is what we are interested in finding.

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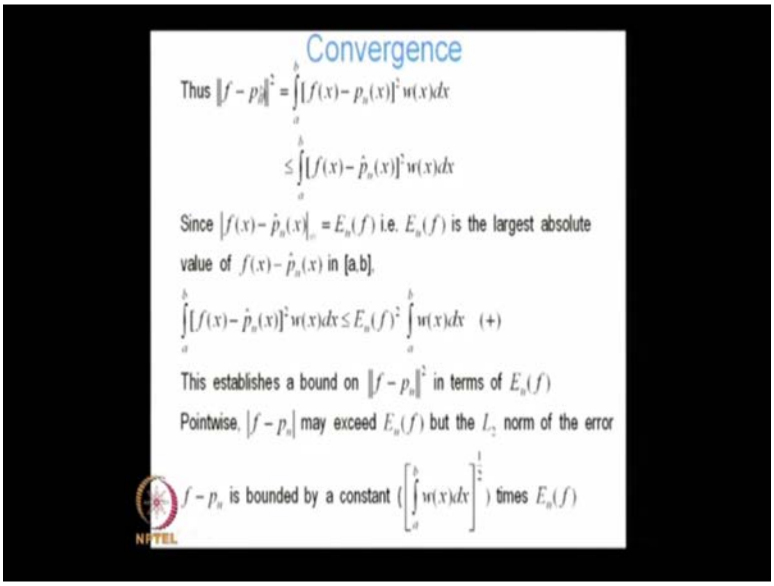

So, what to do that, we write norm of f minus p n square in the L 2 norm, which I can write like this f x minus p n x square w x d x in terms of the inner product, and this is going to be lesser than or equal to f x minus p that n x square w x d x y, because I know that f x is f x minus p hat n x is the has the smallest, absolute value in the interval. So, if instead p hat n minus f has the smallest absolute value among all another order polynomials in that interval. So, this has got to be less than that, because p that n x minus f x or f x minus p hat n x in the infinite norm is the lower bound. So, this has to be less than that but, since f x minus p hat n x the infinite norm is equal to E n f that is E n f is the largest should be the smallest I apologize is the smallest absolute value of f x minus, p n p n hat x in a b so this has to be less than that. So, smallest, it is the minimum, it is the smallest absolute value f x minus p hat n x in a b, that is why, this has got to be lesser than or equal to that.

Integral of f x minus p hat n x square w x d x is therefore, going to be lesser than or equal to E n f square, w x where I have now. So, this is the lower bound, this is equal to E n f is the lower bound. So, this has always got to be less than that is that clear. So, f x minus p x so this has got so I can. So, I can, if I bring out the bound, I can bring it outside. I am taking the maximum value. So, I can take that outside of the integral E n f square, integral a b w x d x. I can write it like that. So, this establishes a bound from norm of f minus p n square in terms of E n f is establishes a bound on this in terms of E n f.

So, what does this mean? this means that, a mod of f minus p n f minus p n look at this f minus norm of p n square is going to be lesser than or equal to E n f square double integral a to b w x that means point wise f minus p n x might exceed E n f why? because, this is p n this is not p n hat if it for p n had that would not no longer be the case but, since this is f minus p n, this may be greater than E n f but, the L 2 norm of the error is bounded by this times E n f.

So, point wise f minus p n may be greater than E n f at each point. if I look at points in that interval at any point this may greater than E n f but, the if I take the integral of that difference, the squared norm if I take the square and I integrate it over the interval after weighing it with the proper weight function then, I know that, it has got be less than the weight function square, weight integral of the weight function half times E n f. So, we were, we are going to continue with this because, it is important that, we go through this carefully.

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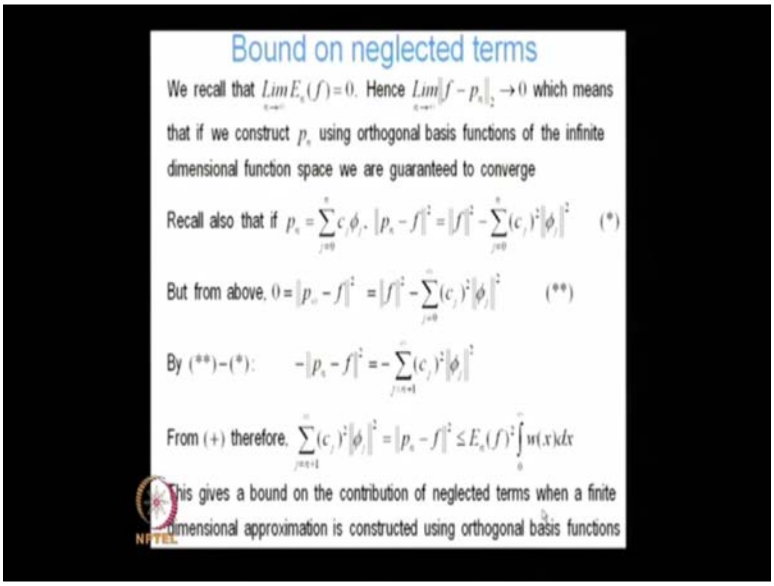

So, we are going to continue with this, next class I was hoping, I would finish the orthogonal polynomials but, we will finish this little bit and then we are going to talk about spline functions, which I have very another very important approximation, another very important set of basic functions, which can be used for approximations and then after doing that, we are going to move on to using these orthogonal polynomials when, we talk about gale kin’s method and weighted residual forms. Thank you.