Numerical Methods in Civil Engineering
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Lecture - 38
Integral Equations

In lecture 38 of our series on numerical methods on civil engineering, we are going to talk about integral equations. And basically I will give a brief introduction to integral equations for those who are not familiar with them, and then we will talk about numerical techniques for solving integral equations.

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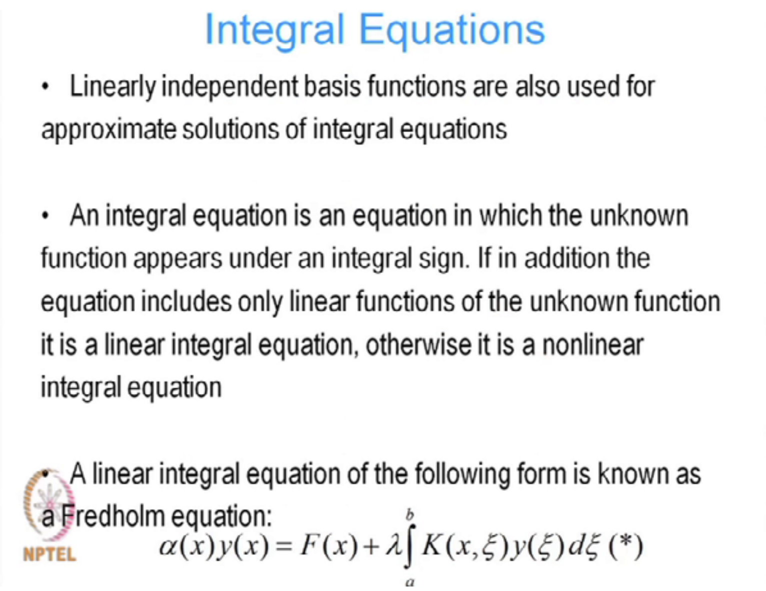

In the last two lectures, we saw that linearly independent basis functions can be used for solving partial differential equations. In fact, as we will see in this lecture, linearly independent basis functions have a wide variety of uses, they can be even used for solving integral equations. Now, what basically are integral equations? Well an integral equation is an is as the name suggests is an equation where the unknown variable, the variable we are trying to solve for is under in an equation with an under an integral sign.

So, basically an integral equation is an equation in which, the unknown function appears under an integral sign, if in addition. So, that is basically the most fundamental definition of an integral equation, in equation like the equation you see at the bottom of this slide. Where the unknown function the unknown function here being y it is inside this integral it is inside this integral as you can see there is this y of psi. So, the unknown function appears within an integral, remember what is a differential equation? In a differential equation the unknown function typically appears within a derivative right. It involves the derivative of the unknown function an integral equation on the other hand, it involves the integral of the unknown function. So, that is the most fundamental definition of an integral equation.

And remember what is a linear differential equation? In a linear differential equation the coefficients the coefficients of that differential equation, they are independent of the unknown variable. They do not depend upon the unknown variable, for instance c d x d y d y d x plus v d 2 y d x square c and b do not show any dependence on y, they are independent of y. That is why it is a linear differential equation, for instance here if in addition the equation only involves linear functions of the unknown function. It is a linear integral equation for instance if this f does not show any dependence on y, or this alpha. For instance the best way to look at it is this coefficient alpha. This coefficient alpha shows no dependence on y in the that is why it is a linear differential equation.

So, there are two types two main types of integral equations, they are known as the first type is known as the Fredholm equation. And then there is something known as the Volterra equation. The Fredholm equation always has this form right it involves some the unknown function, some coefficient times the unknown function plus you can think of this as a.

Since, we are civil engineers one can think of this term is an excitation or a low term and in addition there is this unknown function sitting inside the integral, the unknown function sitting inside the integral right. So, this is a Fredholm equation and you can see the bounds the limits of this integral are a and b. And a and b are have no dependence on either x or y right a and b are just constants, right? So, the integral the limits of the integral are constants that is a Fredholm equation and that is a fundamental difference between a Fredholm equation and a Volterra equation as we will see later on.

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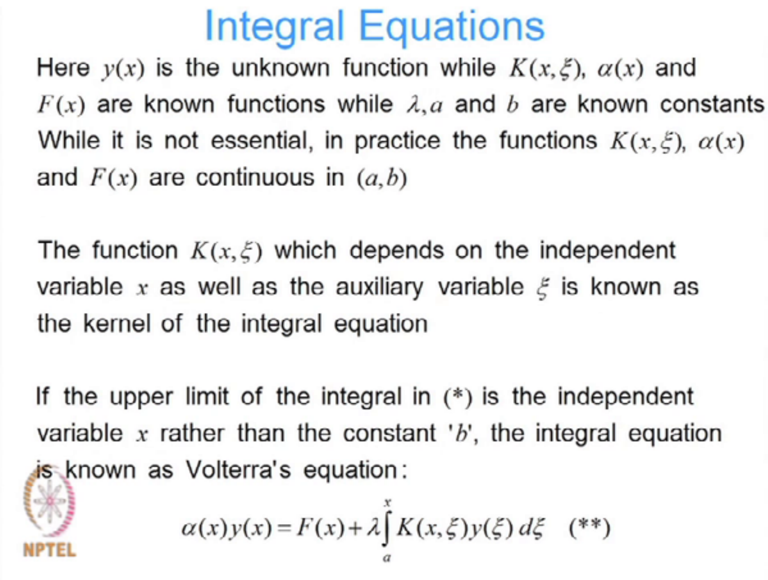

So, here y x is the unknown function while K x psi this K is a very important thing and we will see that it is the properties of this K, the Eigen functions of this K, these are I mean the properties of the k determine, what are the Eigen functions I should not go too for ahead of myself. But what I am trying to say is that the properties of this K determine many things about the integral equation, how we can solve the integral equation what are the possible solutions and things like that. So, this K is known as the kernel of the integral equation, it is the kernel of the integral equation.

So, in addition there are alpha x and F(x) which are known functions. So, all K alpha and F are known functions while y is the unknown function and we are trying to solve for y right we are trying to solve for y. So, in this equation the unknown function is y alpha f and k are known functions alpha is a function of x f is a function of x, k is a function of x and the intermediate variable psi. And in addition there are constants the constants are lambda a and b these are also known constants. And while it is not essential in practice the functions in practice when I say in practice, I mean that in physical problems right because we are trying to solve, we are trying to use numerical methods to solve physical problems. So, we are trying to set up models for physical problems.

In case of physical problems it turns out that these functions k x psi alpha x and f x are continuous in the interval a and b. So, during the course of this lecture, we will confine ourselves to situations where k x psi alpha x and f x are continuous and continuous in the interval a to b, which is the interval of interest. I mean it is just not essential, but it is, but it not practical that is why we will confine us as to that.

The function k x psi which depends you can see depends on the independent variable x as well as an auxiliary variable psi. And as I mentioned earlier it is known as the kernel of the integral equation, and you can see that, if we know if we know the kernel right and if we know how to evaluate this integral, how to how to solve the problem right for a certain kernel. Suppose, I have a certain kernel and I know how to solve the problem how to solve the integral equation for a certain problem, then whatever be the value of f of x it is very easy to solve.

Similarly as known in case of in case of differential equations if we the differential equation, the solution of a linear differential equation has got two parts, it has got the homogenous solution and the particular solution. The homogeneous solution does not change right with the with the loading with whatever be our excitation, whatever be our load the homogeneous part does not change it is only the particular solution that changes.

So, similarly we will see in case of integral equations that if we can solve the homogeneous integral equation, what is the homogenous integral equation? It basically involves an integral equation where this F(x) is equal to 0 right, where this term is equal to 0. If we can solve the homogeneous integral equation then solving the particular solving the effect including the effect of the forcing function, including the effect of the term F of x is like incorporating the effect of the particular solution incorporating the effect of the load, but the homogenous part, the part which is independent of f of x remains the same.

So, as I said if the upper limit of the integral in case of a Fredholm equation we saw that it was a constant, but in case it is not a constant if there is a for instance here you can see the upper limit is x small x right, it is our independent variable right. Since the upper limit is x then it is rather than the constant b, the integral equation is known as Volterra's equation right, it is known as Volterra's equation. So, that is the most important difference between a Fredholm equation and a Volterra equation, the form are the same right except that in the Fredholm equation the integration limits are constants. While in case of the Volterra equation the upper bound is the variable x right, the upper bound is the variable x.

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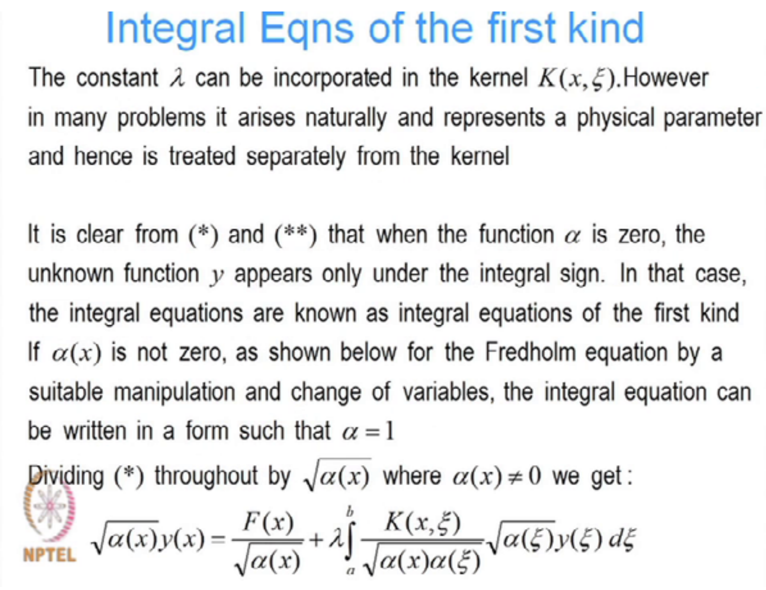

Now, one can ask that why do we have to incorporate the effect of this lambda, separately we could as well have put in inside the kernel, defined is suppose k prime or k star and included the effect of lambda within the kernel, but why do not we do that? Because there are there are reasons why including that parameter lambda separately is useful and we will see what are those reasons, as we go as we go for there in the lecture.

The constant lambda can be incorporated in the kernel k x psi; however, in many problems it arises naturally it turns out that when we model a physical problem using integral equations, there is a physical quantity like lambda right which has a physical meaning right. So, it make sense to separate it out because it might be it might the value of lambda might be measurable from experimental data right. So, it is important to measure to separate it out.

And addition mathematically it has got we will see later on it has got a certain implication as well. So, the main reason why we do we do not included in k x psi is because in many problems, it has a physical meaning. And number two it has a in purely mathematical terms we will see that it has special mathematical meaning as well. So, it is clear from the if we look at the Volterra equation, or if we look at the Fredholm equation that, if the function alpha x the coefficient of the function y of x on the left hand side is 0 then what is going to happen?

Then we have an unknown function we have an integral equation where the unknown function appears only within the integral sign right, it appears only within the integral sign it does not appear outside the integral sign. So, in that case we call these sort of integral equations integral equations of the first kind right, these Fredholm or Volterra equations, where alpha is equal to 0 are known as integral equations of the first kind.

If alpha x is not equal to 0, it is always possible to convert as we will see. So, we have this alpha x right by dividing by making by doing suitable manipulations, it is possible to make this coefficient of y of x 1 right. So, we will see how that is done. So, if we divide throughout by root of alpha x what do we have. So, we had alpha x and dividing throughout by root of alpha x. So, I have root of alpha x y x is equal to F(x) by root of alpha x plus lambda and I am considering a Fredholm equation now.

So, lambda integral within a to b k of x psi divided by root of alpha x. So, that that was the root of alpha x I am dividing and then I multiply the top and bottom with root of alpha psi right. And now you can see it is quite possible to get rid of my alphas from the left hand side altogether, because now if I define another function y prime of x which is equal to root of alpha x y x. Then I will have y prime of x is equal to f prime of x plus lambda integral a to b k prime of x psi y prime of psi.

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Integral Eqns of the second kind

Denoting $\sqrt{\alpha(x)}y(x) = y'(x)$, $\dfrac{K(x,\xi)}{\sqrt{\alpha(x)\alpha(\xi)}} = K'(x,\xi)$

we get: $y'(x) = F'(x) + \lambda\int_a^b K'(x,\xi)y'(\xi)\,d\xi$ which is a Fredholm equation with $\alpha = 1$

A similar operation can be performed for the Volterra equations. Integral equations with $\alpha = 1$ are known as integral equations of the second kind.

In case the original kernel $K(x,\xi)$ is symmetric, $K'(x,\xi) = \dfrac{K(x,\xi)}{\sqrt{\alpha(x)\alpha(\xi)}}$ is symmetric as well

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So, again we get back we again recover a Fredholm equation, but now the Fredholm equation on the left hand side, y the unknown function has a coefficient of 1 right it has a coefficient of one. So, by suitable manipulation and change of variables the integral equation can be written in the form such that alpha is equal to 1.

So, if we denote root of alpha x y x is equal to y prime x this if we denote as k prime of x psi we get y prime of x is equal to F prime of x plus this, which is exactly a Fredholm equation with alpha is equal to 1 right. So, this is a Fredholm equation of the second kind, and if it is always possible to convert a Fredholm equation of the second kind, where the alpha is not 1 to a Fredholm equation of the second kind where alpha is equal to 1 by doing this sort of change of variables right by doing this sort of…

A similar operation can be performed for volterra equations and integral equations with alpha is equal to 1 are known as integral equations of the second kind. And we saw and one thing nice about this transformation is that suppose my kernel K was symmetric was symmetric, what do I mean when I said that the kernel is symmetric it is symmetric in its arguments right. So, it is symmetric in x ze. So, it will involve terms like x ze x square ze square x cube ze cube right x 4 ze 4 it is symmetric in x ze.

So, if it is symmetric if my kernel was symmetric in x ze by performing this operation, I am going to preserve the symmetry of the kernel right because you can see I am dividing by root of alpha x root of alpha ze. So, I am preserving symmetry it is symmetric it is still symmetric in x ze. So, in case the original kernel k x ze is symmetric k prime x ze which I define like this is going to be symmetric as well.

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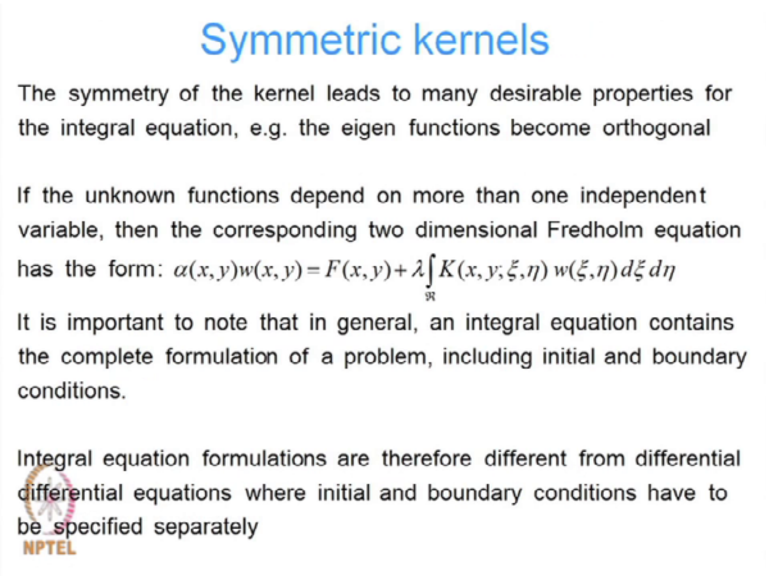

Now, the symmetry recall when we when we were looking at many, many classes back when we were looking at linear systems and we were considering the properties of linear system, we looked at the properties of the coefficient matrix. And we said that when the coefficient matrix is symmetric very nice properties follow, why? What are the what are the nice things about it well if it symmetric it turns out that all its Eigen Eigenvectors are orthogonal. For a symmetric matrix the Eigenvectors are orthogonal to each other right. So, similarly for a symmetric kernel for an integral equation the Eigen functions of that integral equation become orthogonal, if an integral equation has a symmetric kernel, the Eigen functions of that integral equation are going to be orthogonal right, so that that is the most important benefit of having a symmetric kernel.

Now, we have looked at integral equations where there is one independent variable x right, there was only one independent variable x. Look at those equations there was only one independent variable x, the variable psi is just an intermediate variable because it gets integrated out you can see that after you evaluate this integral psi, psi vanishes right. So, there is just one independent variable x and the dependent unknown function y right, in case there is more than one independent variable, it is also possible to have integral equations with more than one independent variable.

For instance if we have a two dimensional Fredholm equation then it is of this form, you can see in addition to x, there is another independent variable y right in addition to x there is another independent variable y. So, here which is the unknown function; obviously, w right w which is the function of x and y is what we have to try to find right, we have our physical problem give rise to this equation. We model our model for the physical problem, gives rise to this integral equation and by solving this integral equation we will find that unknown function w right, we will find our unknown function w.

A fundamental difference between an integral equation and a differential equation is how the how an integral equation incorporates boundary conditions. Well you known that if you have a differential equation if you are solving any differential equation, for instance differential equation in a in a commonly encountered in civil engineering, for instance the vibration of a rod or the vibration of a string right. So, in all those differential equations you know that you have to specify the boundary conditions right. So, in addition to having the differential equation you need the boundary conditions. So, unless you specify the boundary conditions in addition to the differential equations, the problem is not well posed you cannot find a solution to that problem.

However, it is quite different in case of integral equations because integral equations in this are actually self sufficient, in the sense that they do not need to specify boundary conditions separately, for integral equations. They include the boundary conditions are included as part of the integral equation itself, it is part of the formulation of the integral equation itself, you do not need separate boundary conditions. So, integral equations are different from boundary from differential equations because initial and boundary conditions do not have to be specified separately. So, the integral equation contains a complete formulation of the problem including initial and boundary conditions. And we will see how that happens by looking at some examples.

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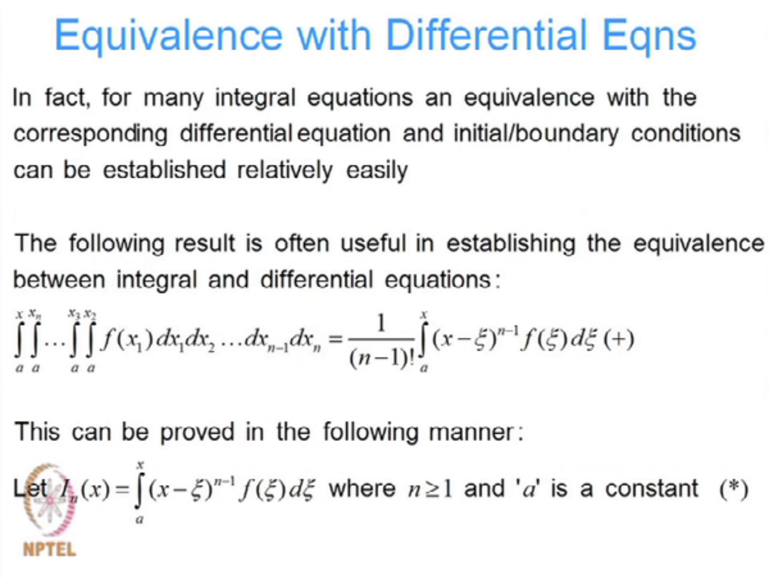

And before we do that I want to establish some sort of equivalence. So, typically in most in most cases if you have a differential equation, which defines a which describes a physical phenomena. It is possible to have a corresponding integral equation describing that physical phenomena as well, only thing is that the integral equation not only incorporates the information that is carried by the differential equation it incorporates the boundary and initial conditions as well right. So, for many integral equations and equivalence with the corresponding differential equation, and initial and slash boundary conditions can be established relatively easily.

But in order to do that we have to look at little bit of formalism right, in particular we have to look at this result right. Unless we understand how we have this result it is not possible to establish the equivalence. So, this is the result which says that if I have a function f, f which is a function of for single independent variable right. And if I integrate it n times if I integrate the first time I f is f is a function, if I treat f as a function of x 1 right and I integrate it within the limits a to x 2, right.

So, then if I integrate f x 1 within the limits a to x 2, what am I going to get I am going to get a function of x 2 right I am going to get a function of x 2. And then I integrate that function of x 2 again within the limits a and x 3. So, I am going to get a function of x 3 and similarly if I continue to integrate this n times I get a certain function which is going to be my final integration limit is x. So, finally, I end up with a function of x, right.

So, this function of a single variable I am integrating it n times with variable limits right, with my variable limits and I end up with a certain function. And if I do this it shows it can be shown that doing this integration n times instead of this, if I performed this integration I would get the same result. If I took my function f, I multiplied it by x minus psi to the power n minus 1 integrated it with respect to psi within the limits a to x remember my final limits are a to x on the left hand side as well.

So, if I do that that is equivalent to integrating it n times, this is this is a very important result and we will use this result in showing the equivalence between differential, and integral equations. How can we prove this? Well it can be done in the following manner, to do that I define this function I n of x which is the basically the right hand side of this equation. I call I n of x where n is greater than or equal to one and a is a constant, a is a constant x is most definitely not a constant right.

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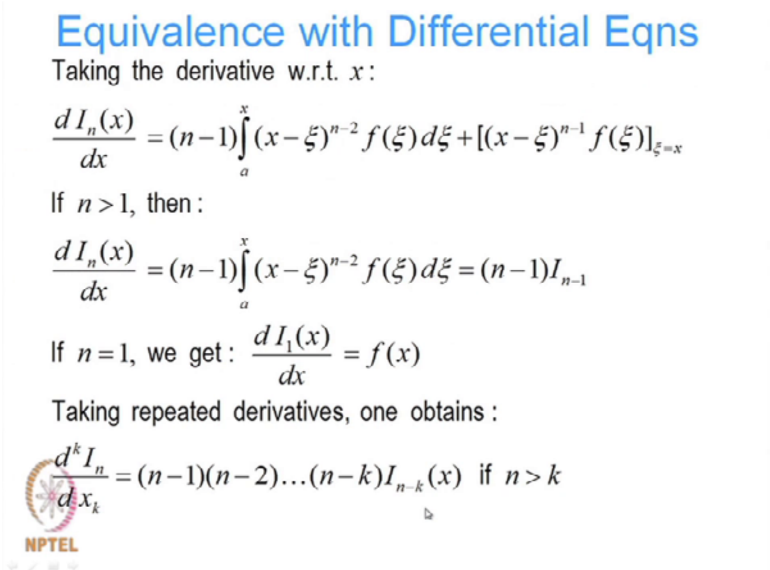

So, if I definite it like this and then if I take the derivative with respect to x, what do I get? If I take d I n d x I am going to get n minus 1 x minus xi n minus 2 right f psi d psi and in addition I will have this term, why will I have this term? Because my integral has a variable limit has a variable upper limit. So, integral if you recall from your integral calculus that if I take the derivative of an integral and the integral has limits has bounds which are not constant, I have additional contributions. And in this case this is going to be d x right it will involve the derivative of the limit and times the function evaluated at this limit minus if this limit where also not a constant, I would have to evaluate the derivative here right, I need to take the derivative of this times the function evaluated at this point.

So, in this case only the upper bound is non-constant. So, this is going to give me if I take the derivative of x that is going to give me 1 right and then I will have to evaluate this function at that value. So, x minus psi n minus 1 f psi I will have to evaluate it at psi is equal to x, the derivative of x giving me 1 right and this is the part where I just brought the by brought the derivative inside the integral and that gave me this contribution.

So, I have these two terms, and if n is greater than one you can see what is going to happen well at psi is equal to x this term is going to go to 0. This term is going to go to 0 and I will be left with n minus 1 a integral a to x x minus psi n minus 2 f psi d psi. And if we look at it closely this is nothing but the n minus 1 times if we look at that term, what is that term is equal to I n minus 1 because we defined our I n to be this right x minus psi to the power n minus 1. So, I n minus 1 is integral a to x x minus psi to the power n minus 2 f psi d psi.

So, this term is going to be n minus 1 I n minus 1 right. So, so this is what we get and in case if n is. So, this was when n was greater than 1, if n is equal to 1, what is going to happen? In that case n is equal to 1. So, this term becomes one. So, this term goes to 0 0 to the power anything is equal to one. So, and then we have f of x. So, this term is going to be f of x right, this term is equal to going to be f of x and that term is going to contribute 0 because n is equal to 1. So, this term becomes 0 right.

So, if n is equal to 1 we get d I 1 x d x is equal to f of x and taking repeated derivatives in this manner you can look at this. So, if I take repeated derivates of I n of x, what am I going to I got the first derivative if I took the first derivative I had n minus 1, if I take repeated derivatives like that k derivatives. In fact, here I have shown right then I get n minus 1 n minus 2 through n minus k I n minus k x right if n greater than k right. So, this is what we get by taking repeated derivatives. And if k is equal to n minus 1 if k is equal to n minus 1, so d n minus 1 I n d x n minus 1.

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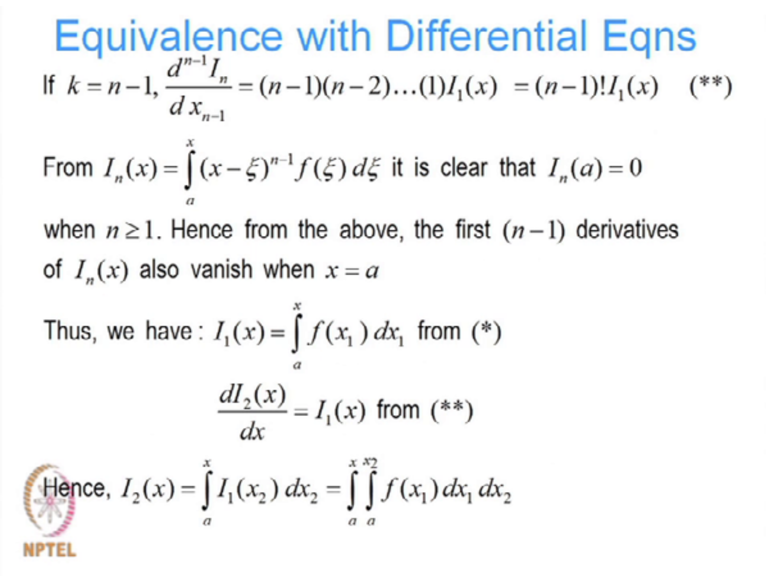

And here what we are going to get on the right hand side is n minus 1 n minus 2 through one times eventually I have I 1 of x which is going to give me n minus factorial of n minus 1 I 1 of x. So, this has this nice property this integral I n has got this nice property that by taking its derivatives can be expressed in terms of a can be expressed recursively right. Now, from I n equal to from the definition of I n of x it is clear that if at x is equal to a, this thing is going to be 0 for all when n is greater than or equal to 1. So, x is equal to a the integral limits become a and a. So, it is going to become 0.

So, since this since the first since the derivatives can be expressed in terms of the lower I n, so d n minus 1 I n d x n minus 1 for instance can be represented by in terms of I 1 and we have as we have seen before. So, d k I n d x k can be represented in terms of I n minus k. So, that means, that the derivatives all of the derivatives at x is equal to a must also be equal to 0. Hence from the above the first n minus 1 derivatives of I n x also vanish when x is equal to a is that clear I do not think. So, because I n of x has got bounds x and a right when x is equal to a this thing is equal to 0.

And I have seen that I can express the derivatives in terms of I I's right in terms of I's with smaller subscripts right smaller subscript values right, but these are equal to 0 at x is equal to a so; that means, these derivatives must also be equal to 0 at x is equal to a. So, we can see that the first n minus 1 derivatives of I n x also vanish, when x is equal to a. So, we already saw this I 1 of x is equal to y we saw this d I 1 d d I 1 x d x is equal to f of x. So, we can write I 1 of x is equal to integral of f of x 1 d x within the limits a to x right and we saw that d I 2 d x can be written in terms of this. So, d I 2 d x. So, n is equal to 2. So, this becomes one and this becomes one as well right.

So, d I 2 d x is equal to I 1 n from that. So, we can right I 2 x is equal to integral of I 1 x which means, but I 1 x is this. So, basically it is a double integral, it is a double integral of f of x 1 d x 1. So, first I integrate it with respect to a and x 2 where x 2 is an intermediate variable and auxiliary variable. And then finally, I integrate it between a and x to get I 2 of x right. So, I 2 of x is equal to that.

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Equivalence with Differential Eqns

In the general case, from (**) it is also clear that:

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_n} \dots \int_a^{x_3} \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \dots dx_{n-1}\, dx_n$$

which proves (+)

Using the above results equivalence can be proved between

$$\frac{d^2 y}{dx^2} + C(x)\frac{dy}{dx} + D(x)y = f(x)$$

with prescribed initial conditions $y(a) = y_0, y'(a) = y'(0)$ and its corresponding integral equation.

Integration between (a,x) with integration variable x_1 yields:

$$\left.\frac{dy}{dx}\right|_a^x = -\int_a^x C(x_1)y'(x_1)\,dx_1 - \int_a^x D(x_1)y(x_1)\,dx_1 + \int_a^x f(x_1)\,dx_1$$

So, and in the general case by doing this repeatedly it can be shown that I 1 I n of x is actually equal to this. So, I 2 of x is equal to that right I 2 of x is integral of f of x two times right twice integrating it twice. Similarly if I do that n times I am going to get back I n of x right I n of x which is what is wanted to prove right because I n of x after all, remember is this I n of x is equal to that and we have shown that I n of x is equal to this; that means, this is equal to that right.

So, integrating a function repeatedly n times within variable with one limit variable, one limit variable. And then integrating it again with respect to that variable and doing this repeatedly n times is equivalent to taking that function multiplying it with this, x minus psi n minus 1 integrating it between its final limits and dividing it by this factorial n minus 1. So, this is the result which we are going to use in order to show equivalence between differential equations and integral equations, but it is important to understand why it happens. So, it is sort of useful.

So, now, that we have shown this we will use this result to show that this differential equation, this differential equation and we think of this as an initial value problem right this difference. So, we can think if we if it is a little confusing, if we look at x we normally think of x as a as a spatial variable in civil engineering on mechanics applications. So, I am we are we are we might think that this is a boundary value problem we tend to think of this as a boundary value problem, but if you think of x as t right this becomes an initial value problem, it depends on time and we have to prescribe initial conditions.

And the initial conditions we prescribe are on y on the unknown variable unknown function y at a is equal to 0. So, a is the starting point of my analysis a at x is equal to a that is my initial starting point of my analysis at y is equal to a, y is equal to y 0 at sorry at x is equal to a y is equal to y 0 at x is equal to a y prime is equal to y prime that is a typo it is y prime with a subscript 0 right, y prime 0. And we will see that this differential equation by using the result which we have just proved, it is possible to convert that differential equation and those initial conditions into an integral equation.

How are we going to do this? Well we will integrate this differential equation naturally right since we are interested in integral equation and we have a differential equation, the first thing that we do is try to integrate right. So, we will integrate that differential equation within the limits a to x with integration variable x 1 right, with integration variable x 1.

So, this first term I get d y d x the first term I get d y d x within the limits a to x and that is equal to the integral of this c x d y d x. So, I put it on the other side. So, I have minus a to x c x 1 y prime x 1 d x 1 and I have this term. So, I integrate again d x 1 y x 1 again within the limits a to x. And finally, I have this term f(x) which I again integrate within the limits a to x with my intermediate variable with my integration variable x 1. And then what am I going to do I am going to try some integration by parts. So, let us see. So, d y d x I am evaluating it within the limits x and a.

So, I have y prime of x sorry I have y prime of x minus y prime of a. So, d y d x at x and at a. So, y prime of x minus y prime of a is equal to minus c x 1 y prime x 1. So, I have just rewritten it basically this equation I have just rewritten it. So, minus c x 1 y prime x 1 minus d x 1 y y x 1 and plus f x 1. So, this is what I have and then I will integrate by parts on this term. So, I have minus c x 1 y of x 1 evaluated between a and x minus integral of. So, this is this is the this is the other part form the integration by parts. So, c prime of x 1 I take the derivative and y of x 1. So, this is of this is the contribution this and that are the contribution from integrating that term by parts.

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Equivalence with Differential Eqns

Hence, $y'(x) - y'(a) = -\int_a^x C(x_1)y'(x_1)\,dx_1 - \int_a^x D(x_1)y(x_1)\,dx_1$

$+\int_a^x f(x_1)\,dx_1 \quad = -C(x_1)y(x_1)\big|_a^x - \int_a^x [D(x_1) - C'(x_1)]y(x_1)\,dx_1$

$+\int_a^x f(x_1)\,dx_1$ on integrating by parts

Thus, $y'(x) = -C(x)y(x) - \int_a^x [D(x_1) - C'(x_1)]y(x_1)\,dx_1 + \int_a^x f(x_1)\,dx_1$

$+C(a)y(a) + y'(a).$

Integrating again between (a,x) we get:

$$y(x) - y(a) = -\int_a^x C(x_1)y(x_1)\,dx_1 - \int_a^x \int_a^{x_2} [D(x_1) - C'(x_1)]y(x_1)\,dx_1\,dx_2$$

$$+\int_a^x \int_a^{x_2} f(x_1)\,dx_1\,dx_2 + (C(a)y_0 + y_0')(x-a)$$

And then I have d x 1 y x 1 right I have d x 1 y x 1 and integral f(x) 1 d x 1 why do not I integrate that term by parts? Well I do not need to because that term only involves y right that term only involves y and y is within the integral. So, that seems to fit my the form of my integral equation right because we remember that in the integral equation I have the unknown function within an integral I have the unknown function outside, but what I do not have are any derivatives of my unknown function y unknown function y.

So, I had to integrate this term by parts, I have to integrate that term by parts because this term involves a derivative of y, but this term does not involve any derivative of y. So, I do not need to integrate that by integrate it at all right. So, we get this on integrating by parts and therefore, again rearranging terms now. So, I have y prime of x is equal to minus. So, I am evaluating this at the at the limit. So, I get C of x y of x plus C of a y of a. So, this is just evaluating it here and then I have this term right there and then I have integral f of x 1 and I have brought this y prime a to the right hand side right from the left hand side I have brought it to the right hand side.

But again so, you can see that there is this there is the surviving derivative of y prime x I want to get rid of the derivatives right. So, I want to convert it to an integral equation. So, I integrate it one more time right I integrate it one more time and I integrate it again between the limits x and a. So, I have y x minus y a and then I have the integral of that term and this term now becomes a double integral right d x minus c prime x 1, now becomes a double integral.

And then I have a double integral in f of x as well right I have a double integral in f of x as well. And the integration of that term gives me c a y a plus y prime of a, but y a is equal to y 0 we saw earlier right y a is equal to y 0 y prime a is equal to y prime of 0 y prime 0. So, c a y a plus y prime a, I am writing as c a y 0 plus y, y prime 0 and since I am integrating with respect to x and I am integrating within the limits x and a, I will have this x minus a appearing here. So, this is what I get finally, and then let us see what we can do.

So, now, we have this appears fine right it is there is one there is y there is y and it is lying within an integral, but here we have double integrals right we have double integrals a to x 2 and then again from a to x. Similarly here I have a double integral between a to x 2 and a to x, this is exactly of the form that we just saw a few minutes few slides ago, it is of this form. I have an integral double instead of n integrals I have two integrals there right. So, I know that I can convert it to a single integral using this relationship, I can convert it into a single integral using this relationship which is what I am going to do.

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Equivalence with Differential Eqns

Using (+) on the right hand side of the above equation:

$$y(x) = -\int_a^x C(\xi)y(\xi)\,d\xi - \int_a^x (x-\xi)[D(\xi)-C'(\xi)]y(\xi)\,d\xi$$

$$+\int_a^x (x-\xi)f(\xi)\,d\xi + [C(a)y_0 - y'(0)](x-a) + y_0$$

Denoting $(\xi - x)[D(\xi)-C'(\xi)] - C(\xi) = K(x,\xi)$ and

$$\int_a^x (x-\xi)f(\xi)\,d\xi + [C(a)y_0 - y'(0)](x-a) + y_0 = F(x)$$

the above equation can be rewritten as:

$$y(x) = \int_a^x K(x,\xi)y(\xi)\,d\xi + F(x) \quad (++)$$

It is clear that (++) is a Volterra equation of the second kind with kernel $K(x,\xi)$ a linear function of the independent variable x

So, this if I use that relationship if I use that relationship that term gives me x minus psi d psi minus c prime psi y psi d psi look at this I have d x 1 minus c prime x 1 y of x 1 d x 1 d x 2 right. I can convert it into a single integral because by multiplying by x minus psi within the integral right, I have to multiply by only x minus psi because it is a double integral.

So, x minus psi d psi c prime psi y psi d psi right that is what I get here right, and similarly this integral I can convert it into a single integral. So, again I will have an x minus psi appearing. So, I will have x minus psi f psi d psi and of course, this part remains unchanged that part remains unchanged, now or we in the in the appropriate form it looks like we are because look at this. So, if I denote.

So, here are two integrals where y is within the integral right. So, certainly if I can add these together if I and the integral limits are also the same. So, if I add these two together whatever is in front of my y that is going to be my kernel right, that is going to be my kernel that is going to be my kernel because that is I have to bring it to the standard form of the of the Volterra integral equation. So, if I denote this thing to be k of x psi and this whole thing I denote this is going to be sorry this whole thing is a function of x right, this whole thing is a function of x because here I have x right and this thing if I integrate it out, this is also going to be a function of x.

So, this is a function of x and this thing if I denote by f of x, I can write this equation in that form, which is basically my homogeneous Volterra equation, it is my homogeneous Volterra equation and this is homogeneous Volterra equation of the second kind is a homogeneous Volterra equation of the second kind because there is nothing there is no alpha, alpha is equal to one right. So, it is clear that this is a Volterra equation of the second kind with kernel k of x psi a linear function of the independent variable x you can see that the kernel is a liner function of x right. X is a it is a linear function of the independent variable x.

So, that is how we could convert a second order a second order differential equation second order ordinary differential equation, linear differential equation with two initial conditions into a integral equation. So, implicit in the transformation is the assumption that the coefficients D C C prime as well as the function f for integrable because you can see that we have integrated them repeatedly right we have integrated f, we have integrated c prime we have integrated d. So, those are integrable.

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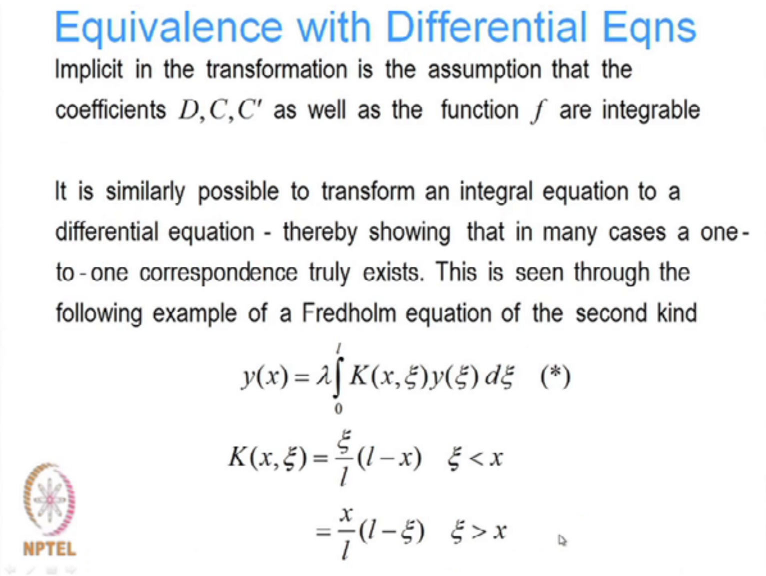

So, that is what we have assumed, it is similarly possible to go the other way it is possible to transform an integral equation to a differential equation, how can we do that? Well let us see and this time the first time we converted the differential equation into a Volterra integral equation now we will start with a Fredholm equation, and see what differential equation results from that Fredholm equation, right.

So, again we are going to consider a Fredholm equation of the second kind, but this Fredholm equation, you can see this is this is a Fredholm equation of the second kind that is true. But the kernel the kernel is a is a the kernel function is different in different intervals right. So, when x is greater than psi this kernel is given by this function say, and when x is less than psi the kernel is given by that function. Is the kernel continuous?

Well you can see if you look at it closely you will see that when x is equal to at x is equal to psi the kernel is actually continuous right, but. So, this is a Fredholm equation with a kernel and the kernel has got has got two representations in two different intervals well which is not a problem right because this is an integral. And so that function it is the certain function it a certain in a certain region and then it is a different function in a different region right that is always possible, and is also continuous right there is no discontinuity at x is equal to psi which is perfectly, fine.

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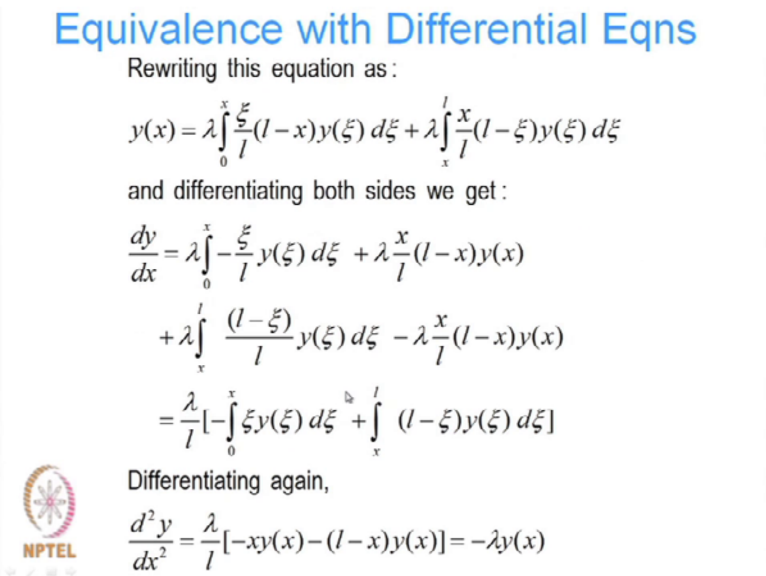

So, we rewrite this equation in this form. So, basically I am breaking it up into 2. So, I my original integral limits were between 0 and l. So, I am going to break it up into two regions between 0 to x and x to l right. And I since my integration variable is psi then I integrate between 0 to x psi is always less than x right psi is always less than x and when I integrate between x to l psi is always greater than x, so in this region which my psi is always less than x. So, my kernel is going to be this one right, my kernel is going to be that one while in this region psi is always greater than x. So, my kernel is going to be this one right this is going to be my kernel.

So, now I have y of x in this form where you can see that the kernel is different in this two regions between 0 to x, it is this kernel between x to l it is that kernel and we then we differentiate. So, it is now then we were trying to go from differential equations to integral equations, we were integrating now we have to do the opposite, we have to take derivatives right we have to take derivatives. So, differentiating both sides I have d y d x is equal to lambda 0 to x, so minus psi l.

So, again you can see in order to take the derivative of this thing, it is I have to take account for the fact that the limits are not constants right the limits are not constant. So, first fist term will be just the plain derivative right, so derivative of psi by l, l minus x. So, that is going to give me minus psi by l y psi d psi then plus I have the derivative of x right d x. So, that is going to give me one and then I have to evaluate this function at x.

So, that is going to give me x by l l minus x y of x right I do not have to attempt for this part this lower limit because the lower limit is constant, right.

And then I have this thing, I have that thing which is going to give me, which is going to give me this term right this term. So, again if I take the take the bring the derivative inside I have l minus psi by l y psi right and now in this case the lower limit is a non-constant. So, I have minus sign right minus derivative of this which again gives me one derivative with respect to it gives me 1 x which gives me one and then I evaluate this function at psi is equal to x. So, I sorry I evaluate this function at psi is equal to x which gives me x by l l minus x y of x, right.

So, this is this is what I get and then you can see that these two terms cancel each other. So, I am left with something like that which is basically I have brought these two integrals together, I have added them and then I take the derivative of this again and I am left with lambda by l. So, again now you can see that if I take the derivative of this inside there is no dependence on x right. So, inside it gives me 0, right.

So, the only part which is going to give me a contribution is because of the non-constant limit. So, derivative of that is going to give me 1 and then I have evaluate this at psi is equal to x. So, minus x y of x minus again this term is independent of x. So, derivative of that is going to give me 0 and then, but the lower bound is still dependent on x. So, again I take the derivative of that times the function evaluated at psi is equal to x. So, that gives me l minus x y of x and that gives me minus lambda y of x. So, you can see that finally, we have arrived at a second order differential equation, from our initial integral equation from our initial integral equation we have arrived at a second order differential equation like this.

And this yields the equivalent differential equation and the boundary conditions, what are the boundary conditions? We can with the boundary conditions by setting x is equal to 0 and x is equal to l in star. Remember that our integral equation for the integral equation we did not have to specify any boundary conditions, but how do we get the boundary condition. But for the differential equation I do need boundary conditions. So, how do I do that? I set x is equal to 0 and x is equal to l in this right, when x is equal to 0 what do I have when x is equal to 0. That means, that the kernel is of this form right because my integration limits are between 0 to l and psi must be that x is equal to 0 then psi has to be greater than x because psi goes from 0 to l right.

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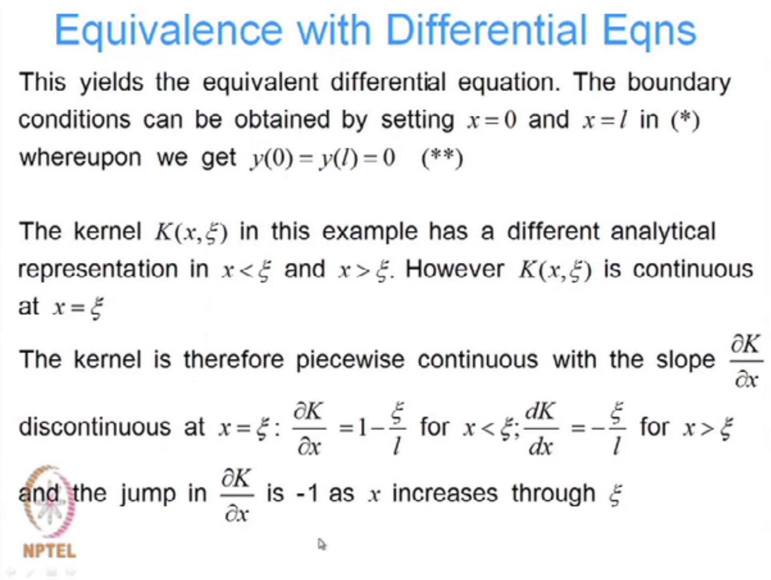

So, when x is equal to 0 psi must be. So, this is my kernel and look at what the kernel will become when x is equal to 0 it is going to become 0, right. So, then I have 0 inside the integral so; that means, at y of 0 this is going to be 0. Similarly when x is equal to l this is going to govern right x is equal to l this is because x is always going to be greater than psi. So, in that case what does the kernel become? The kernel again becomes I put x is equal to l here the kernel also again becomes 0. So, again. So, at y is equal to l this thing again become 0 and then I get I recover my boundary conditions for my differential equation.

The boundary conditions can be obtained by setting x is equal to 0 and x is equal to l in my integral equation, taking care that I choose the appropriate branch of the kernel appropriate to x is equal to 0 or appropriate to x is equal to l. And once I do that I get recover the boundary conditions. As I mention the kernel in this example has a different has two branches right it has it has a different analytical representation, when x is less than psi and x is greater than psi.

However, as I also mentioned if you look at it is continuous at x is equal to psi. So, no problems at all right; however, there is a discontinuity in the slope; obviously, right. So, it is it is represented by a certain function in a certain range and in the adjacent range, it is represented by different function it in this case it has a different, it has a discontinuity in the slope if we evaluate del k del x, this is equal to 1 minus psi of l for x less than psi d k d x is equal to minus psi of l for x is psi. And you can see that there is a jump there is a jump in the derivative and the jump is equal to minus 1 as x increases. So, at x is equal to psi there is a jump right there is a jump in the derivative.

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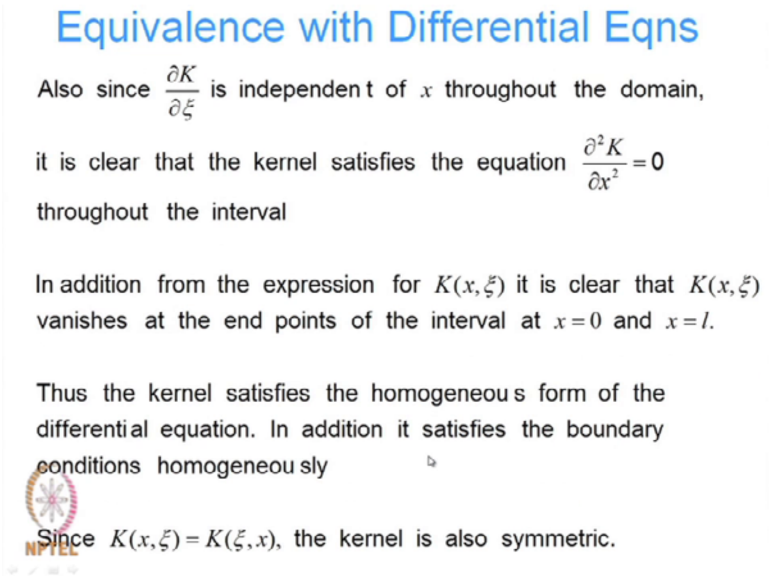

Also since del as you can see in both these ranges d k d x is independent of x right there is no dependence on x. So, that means that the second derivative of the kernel is 0 right is independent of x throughout the time, if I take the second derivative of the kernel with respect to x, that is going to be 0 throughout the interval. Again from the expression for k x psi it is it is clear that k x psi vanishes at the end points of the intervals at x is equal to 0 and x is equal to l which we just saw a while back right. So, it is it is x is equal to 0 x is equal to l it is 0. So, the kernel satisfies the homogeneous form of the differential equation right.

So, this is this was my differential equation the kernel satisfies this differential equation, but with the 0 on the right hand side. It satisfies the homogeneous form of the differential equation, in addition it satisfies the boundary conditions homogeneously. And in addition it has this very, very important property that it is symmetric in x and the psi. And we will talk about the properties of symmetric kernels in the next class.

Thank you.