Numerical Methods in Civil Engineering
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Lecture - 5
Linear Systems-III

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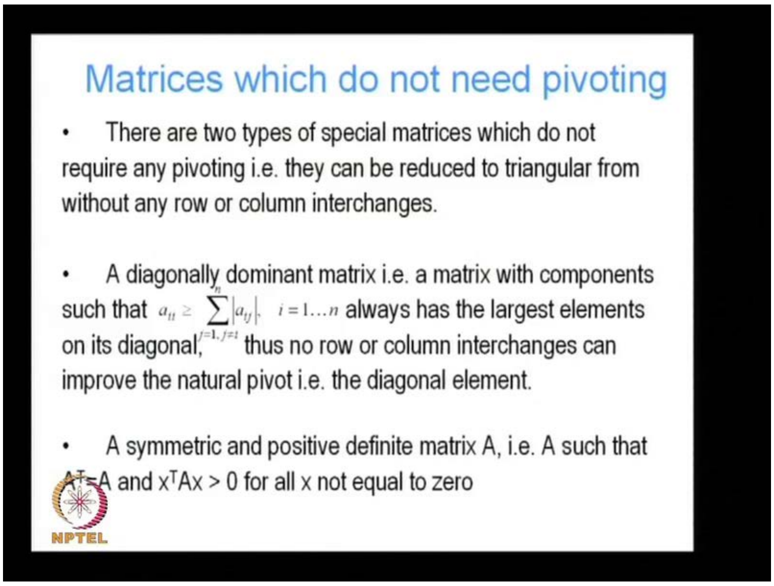

Methods in civil engineering we will continue our discussion on linear systems, which we talked about last times. Last time we ended our lecture talking about the need pivoting. We found that most matrices, if we have to perform Gaussian elimination on those matrices to solve the linear systems successfully they require pivoting. However, today we will concentrate on types of matrices, which do not need pivoting. There are two types of special matrices, which do not require any pivoting that is they can be reduced to triangular form without any row or column interchanges.

A diagonally dominant matrix that is a matrix with components a i i diagonal elements a i i such that each diagonal element is greater than the sum of the absolute magnitudes of the off diagonal elements is knowing as a diagonally dominant matrix. It always has the largest elements on its diagonal thus no row or column interchanges can improve the natural pivot that is the diagonal element. Since the diagonal element is the largest element in the row, it always is assured to be nonzero.

Another type of matrix, which does not need any pivoting is a symmetric positive definite matrix. A positive definite matrix is defined such that it is symmetric first of all a symmetric positive definite matrix by definition is symmetric, which means that a transpose is equal to A, and in addition x transpose A x is greater than zero for all x not equal to zero given any arbitrary vector x and given that that vector is not identically equal to zero. We can be assured that if it if we perform the operation x transpose A x. We are going to get a positive number for a positive definite matrix. So, for these two types of matrices we do not need any pivoting by performing Gaussian elimination.

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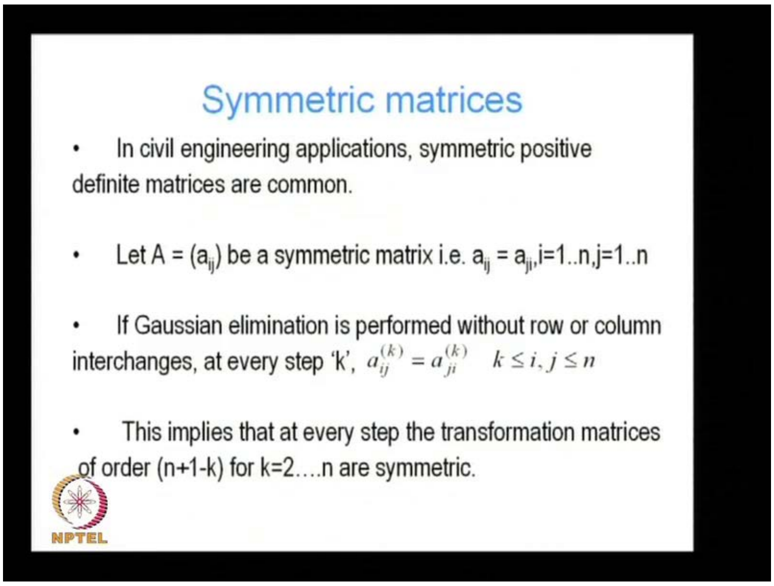

We will talk about positive definite matrices in lot of detail, because in many civil engineering applications we encounter positive definite symmetric, positive definite matrices. First let us talk in more detail about symmetric matrices we will come back to talk about positive definite matrices further on in the lecture. Let A is equal to a i j with components a i j be a symmetric matrix which; obviously, means that a i j is equal to a j i that is elements on either side of the principal diagonals are mirror images of each other that is, if we change the row and column we are going to get if we transpose the row and column we are going to get identical elements of the matrix. So, a i j is equal to a j i for all i equal to 1 to n, j equal to 1 to n.

If Gaussian elimination is performed without row or column interchanges at every step k we can be assured that the matrix is going to remain symmetric that is a i j after the k th step is going to be equal to a j i after the k th step. So, for a symmetric matrix, if we performed Gaussian elimination that is if we are able to perform Gaussian elimination without any row or column interchanges we can be assured that at every step the matrix is going to be symmetric. So, at every step the matrices of order n plus one minus k for k equal to two through n are going to be symmetric.

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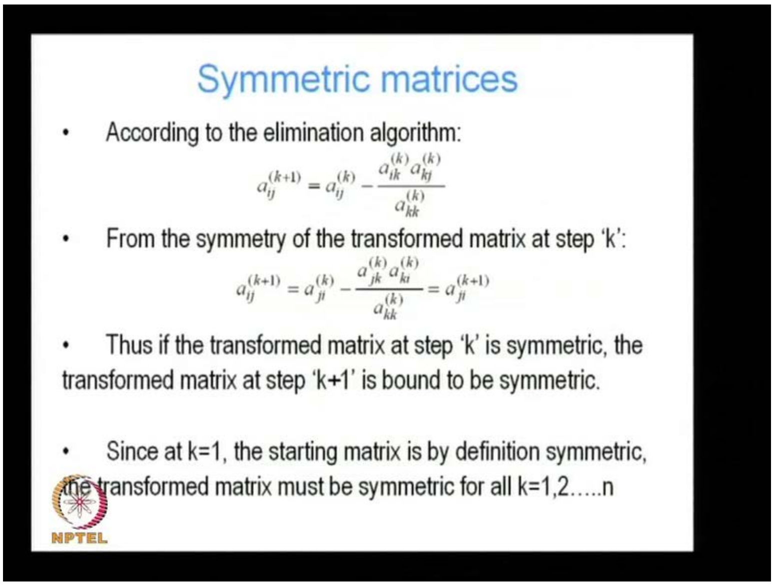

According to the elimination algorithm lets go back to the elimination algorithm, which we derived last time for Gaussian matrices for Gaussian elimination according to that elimination algorithm at any step k plus 1 the transformed element a i j k plus 1 is given in terms of the elements at step k according to the following relationship a i j at k plus 1 is equal to a i j at k minus a i k at k times a k j at k divided by a k k of k, we uncounted this equation this transformation relation in our last lecture.

From the symmetry of the transformed matrix at step k, we can write a i j at step k plus 1 is equal to a j i at step k. Please note the difference between these two equations in the first equation we had a i j k at the right on the right hand side, second equation in the we use symmetry to write a i j in terms of a j i. So, we replace a i j k by a j i k. Similarly, we replace a i k k by a k i k and a k j k by a j k k.

So, these two expressions are identical for symmetric matrices. So, this tells us that if the transformed matrix at step k is symmetric, the transformed matrix at step k plus 1 is also bound to be symmetric, because this is this expression is identically equal to a j i k plus 1 since a j i k plus 1 is equal to a i j k plus 1; that means, at step k plus 1 also the transformation matrix is symmetric. This is a j i k plus 1 by definition by definition at.

Since at k equal to 1 the starting matrix is by definition symmetric that is, because are initial matrix was symmetric. So, i starting matrix is by definition symmetric, the transformed matrix must be symmetric for all k equal to 1, 2 through n why is that, because i starting matrix is symmetric and we have assure that, if the if at step k my transformation is matrix is symmetric at step k plus 1, it is bound to be symmetric. So, at k equal to 1, if it is starts with symmetric matrix that is at k equal to 1, it is symmetric then it is obvious that at k equal 2, it is going to be symmetric at k equal to 3. It is going to be symmetric and it is going to be symmetric through all the steps of the Gaussian elimination.

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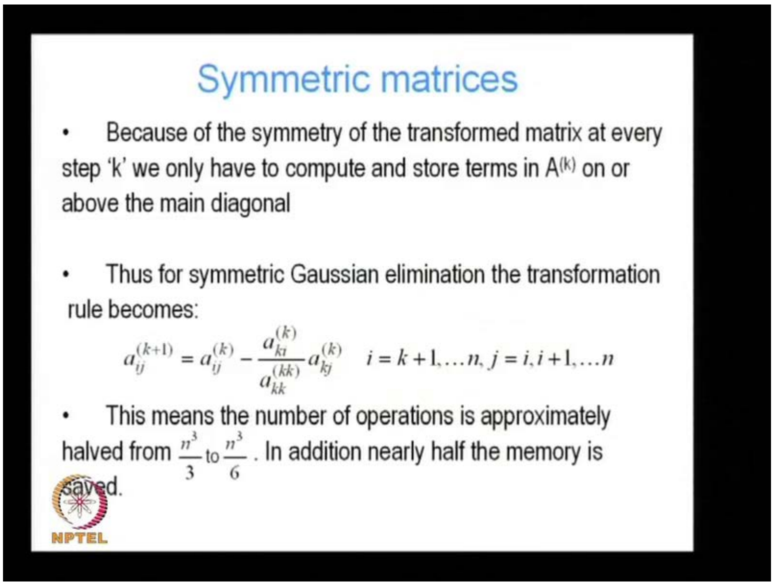

Because of the symmetry of the transformed matrix at every step k we only have to compute and store terms in a k on or above the main diagonal. The advantage of symmetry is this it reduces drastically the storage required. So, since the matrix is symmetric since at each stage the transformation matrix is symmetric, we will take advantage of symmetric and store only the terms above the main diagonal we will no longer the stored the whole matrix. So, this automatically reduces the storage requirement by 2.

Thus the symmetric Gaussian elimination the transformation rule becomes a i j k plus 1 is equal to a i j k minus a k i k a k j k divided by a k k k k as you can see from the previous slide this is basically identical to what we obtain in the previous slide. Except, you note that we are now taking i is equal to k plus 1 through n, and j is equal to i through n, which is basically we are doing the computations only for the one half of the matrix right. We have reduced to the number of operations. The number of operations is approximately half from n cube by 3 to n cube by 6 in addition nearly half the memory is save. So, we are performing the operations only on the elements, which are above the main diagonal and we have also storing only half the previous the storage is also half the previous storage.

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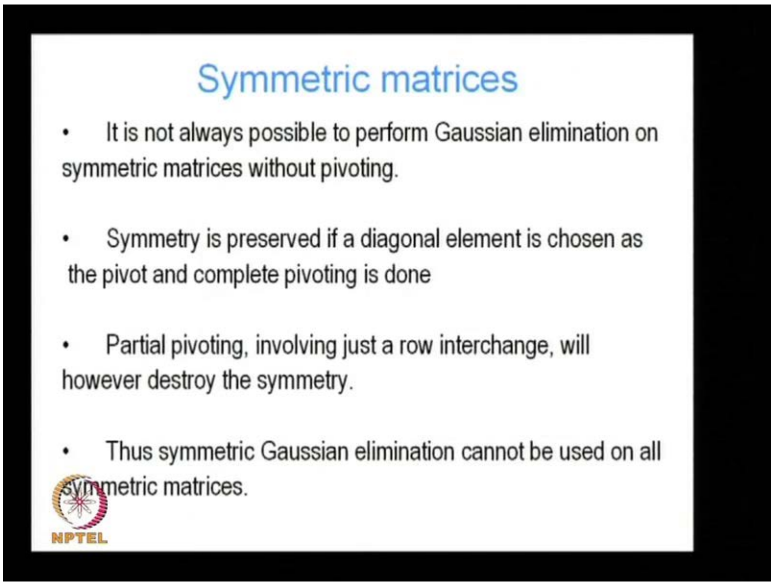

However, we should enter a void of caution here, it is not always possible to perform Gaussian elimination on symmetric matrices without pivoting, symmetry is preserved we note that symmetry is preserved if only if we perform the Gaussian elimination on a symmetric matrix without performing any row or column interchanges right. So, this is only possible, if we do pivoting on a diagonal element either this is possible only in 2 situations either we perform Gaussian elimination without performing any row or column interchanges or we took do pivoting on a diagonal element, and when we do pivoting on a diagonal element we do complete pivoting that is we interchange both the rows and the columns to preserve the symmetry of the matrix. So, either we do not do any pivoting at all or we do complete pivoting on the diagonal elements, if we do partial pivoting we just involves the row interchange we will destroy the symmetry of the matrix.

Thus symmetric Gaussian elimination cannot be used on all symmetric matrices, it is important to note that symmetric Gaussian elimination cannot be used on all symmetric matrices particularly when we encounter elements on the pivot, which are either zero or too small we have to go for pivoting, and in that case if we do partial pivoting we are going to destroy the symmetric only vocation when we can preserve the symmetry is if we do pivoting on a diagonal element, and then we do total pivoting that is we interchange the rows as well as the columns.

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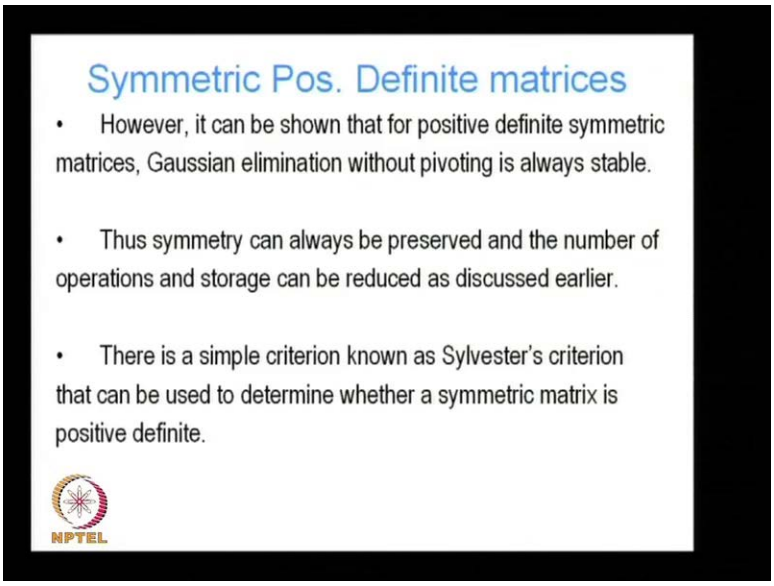

However, this is where the advantage of positive definite matrices come in, because it can be shown that for positive definite matrices Gaussian elimination without pivoting is always stable that is no need to do any pivoting for positive definite symmetric matrices, in that case symmetry can always be preserved and the number of operations, and storage can be halved as discussed earlier. There is a simple criterion known as Sylvester's criterion, which enables us to determine whether a matrix is positive to whether a symmetric matrix is positive definite or not.

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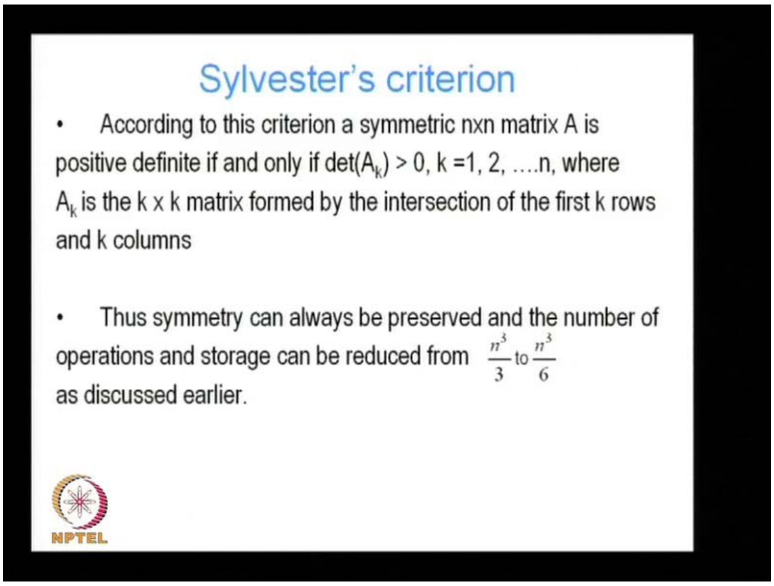

According to this criterion a symmetric n by n matrix, a symmetric matrix A with n rows and n columns is positive definite if and only if that is, it is necessary and sufficient condition that the determinant of a k must be greater than 0 for all k equal to1 to n, where k A k is the k by k matrix formed by the intersection of the first k rows and k columns. So, we basically look at series of sub matrices starting with k equal to 1, which is basically just the element a11 then with A k equal to 2 we are basically looking at the elements of the first row and the first column the square matrix formed by the L intersection of the first row and the first column, which is going to be a 2 by 2 matrix with elements a 11 a 12 a 21 a 22.

Similarly, for k equal to 3 we look at the square matrix formed by the intersection of the first 3 rows and 3 columns with elements a 11 a 12 a 13 a 21 a 22 a 23 a 31 a 32a 33 if all these sub matrices turn out to have positive determinants, if all those determinants are positive then the symmetric matrix is going to be positive definite, in that case symmetry will always be preserved by Gaussian elimination and the number of operations can be reduced from n cube by 3 to n cube by 6 as discussed as we discussed earlier. Similarly, the storage two can be halved.

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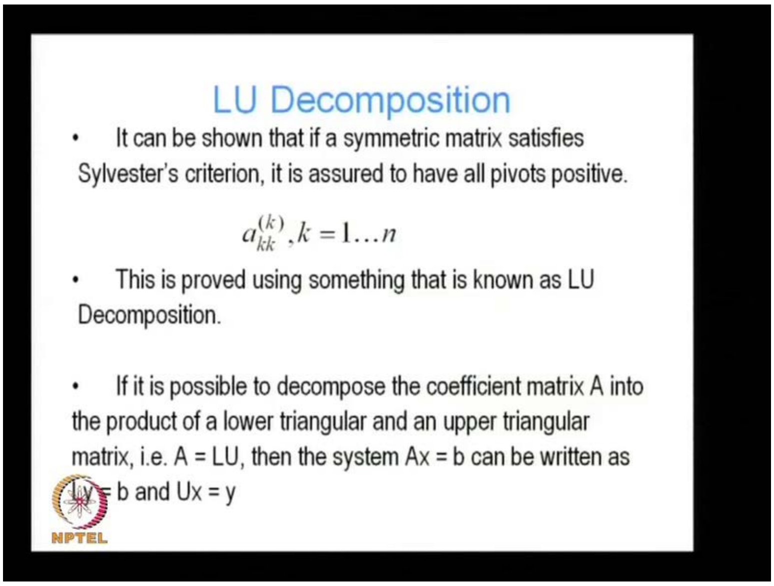

It can be shown that if a symmetric matrix satisfies Sylvester's criterion that is, it is positive definite it is assured to have all its pivots positive that is a k k k, k is equal to 1 through n is always going to be greater than 0 that is all these pivots a kk with superscripts k k equal to1 through n are going to be greater than 0. This can be shown by something known as L U decomposition, if it is possible to decompose the coefficient matrix a into the product of a lower triangular and an upper triangular matrix that is A is equal to L U recall, what is a lower triangular matrix? A lower triangular matrix has got elements, which are above the principal diagonal zero, it is got elements non zero elements only below or at the principal diagonal, where an upper triangular matrix is A matrix, which is got elements above the at or above the principal diagonal greater than 0 well all elements below the principal diagonal are zero.

So, if it is possible to decompose the coefficient matrix A into the product of a lower triangular and upper triangular matrix that is A is equal to L U then the system A x can be written as L y equal to b, where y is an intermediate variable, L y equal to b and U x equal to y. So, basically L a x equal to b is basically L u x equal to b we denote U x equal to y and we solve the system L y equal to b to determine y, and then we solve we again solve the system U x equal to y to determine x.

So, instead of solving A x equal to b at one go we are solving two triangular systems. First we are solving the system L y equal to b to determine y, and then we are solving the system U x equal to y to determine x, but each of these solution of this triangular systems as we have seen earlier is very simple, it just involves back or forward substitution.

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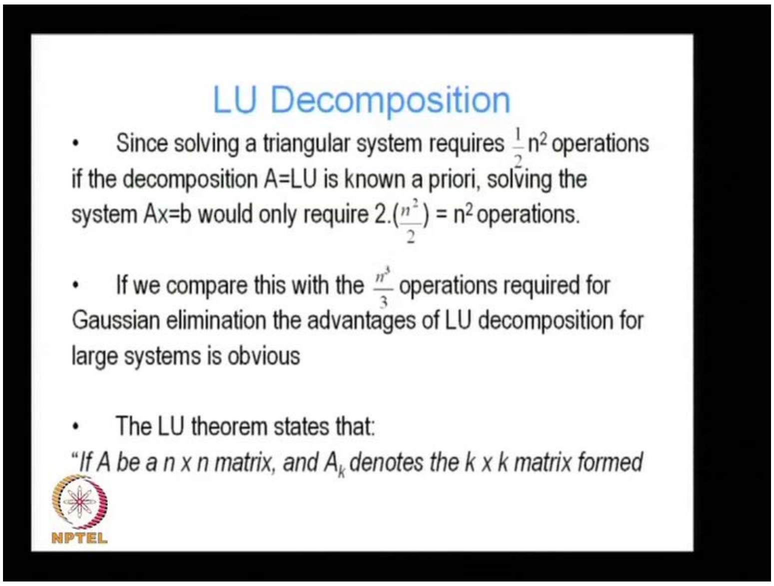

Since solving a triangular system requires half n square operations basically the number of operations in a typical back substitution operation, if the decomposition A is equal to L U is known a priori solving the system A x equal to b would only require two times n square by 2 is equal to n square operations.

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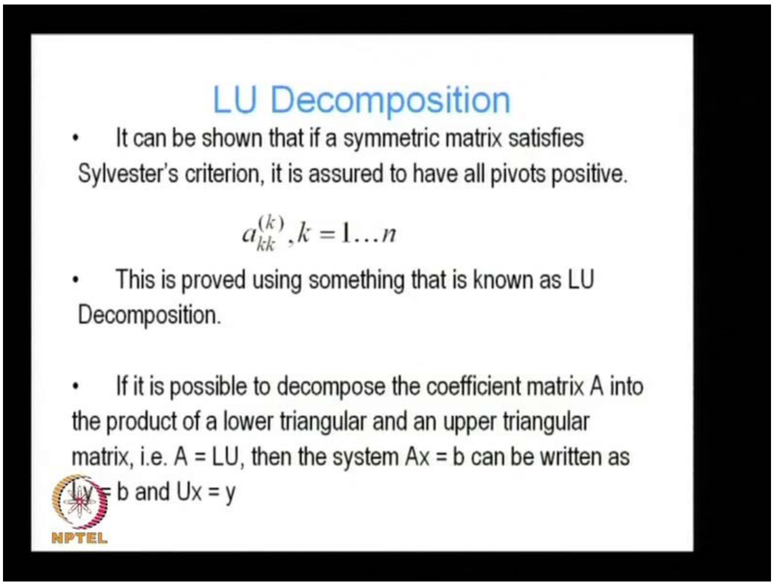

Basically we will solve the system A x equal to b by solving the system L y equal to b, which is a triangular system which would require n square by 2 operations similarly solving the system U x equal to y is again going to require another n square by 2 operations.

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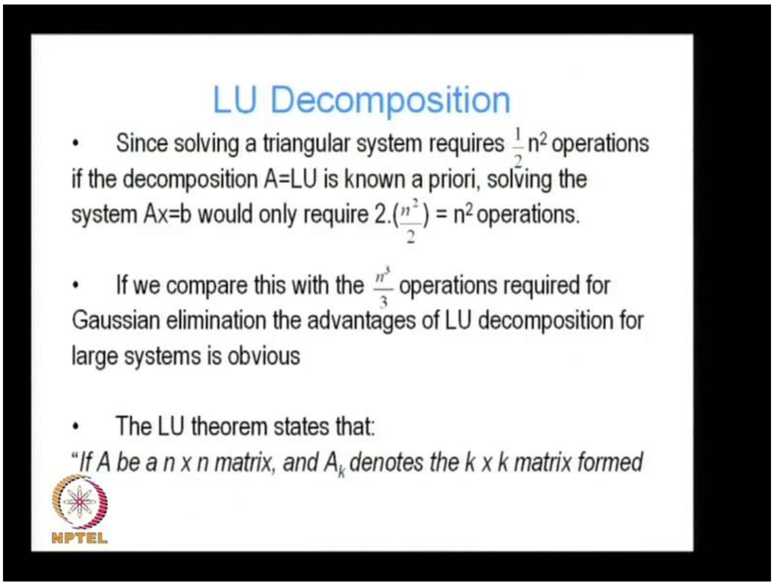

So, the total number of operations is equal to two times n square by 2 is equal to n square this of course, assumes that we know the decomposition the decomposition of a into L U is known a priori is known before hand.

If we compare the number of operations n square to the number of operations required for Gaussian elimination, which is n cube by 3, the advantages of L U decomposition for large systems is obvious. So, if we can perform the L U decomposition then we can solve the equation just by we can solve repeated systems A x of the type A x equal to b by just performing n square operations. The L U theorems states that, if A is an n by n matrix and A k denotes the k by k matrix formed by the intersection of the first k rows and k columns of A.

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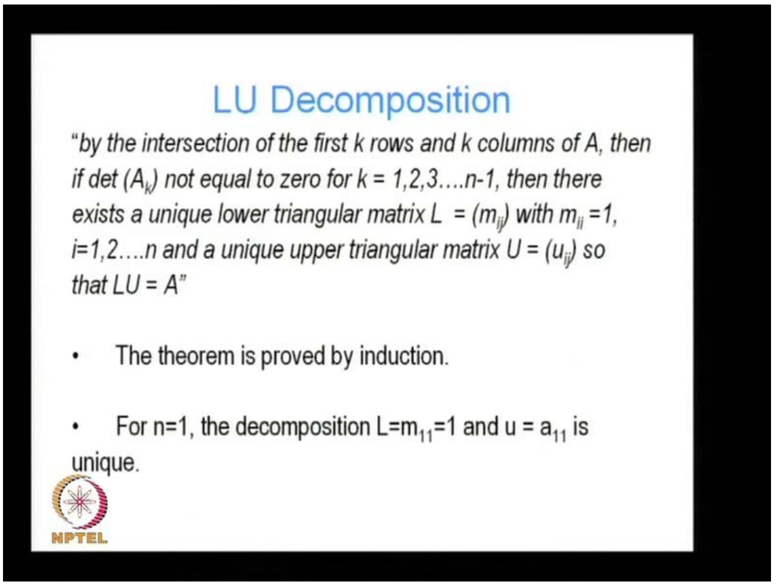

Then, if determinant of A k is not equal to 0 for k equal to 1 through n minus 1 then there exists a unique lower triangular matrix L with components denoted by m i j and the diagonal components m i i equal to 1 and unique upper triangular matrix U that components u i j. So, that L U is equal to A.

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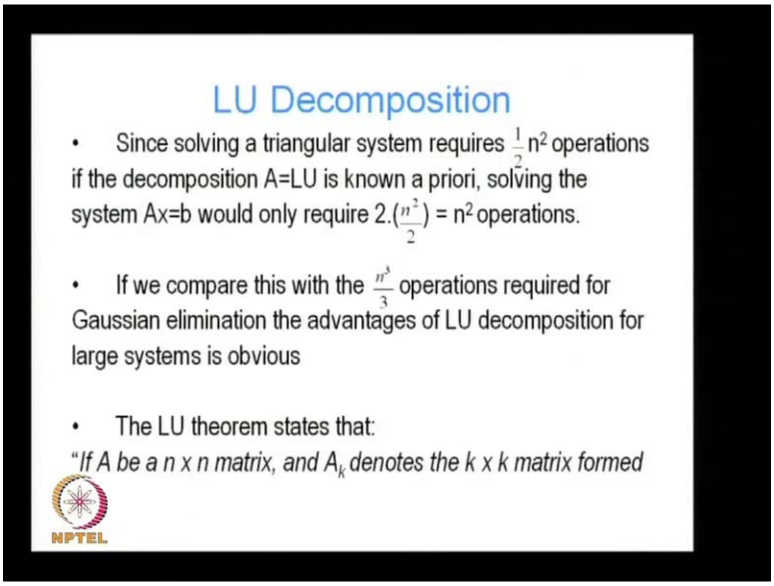

So, let us take a look at it again says, that if A is an n by n matrix and A k is the k by k sub matrix.

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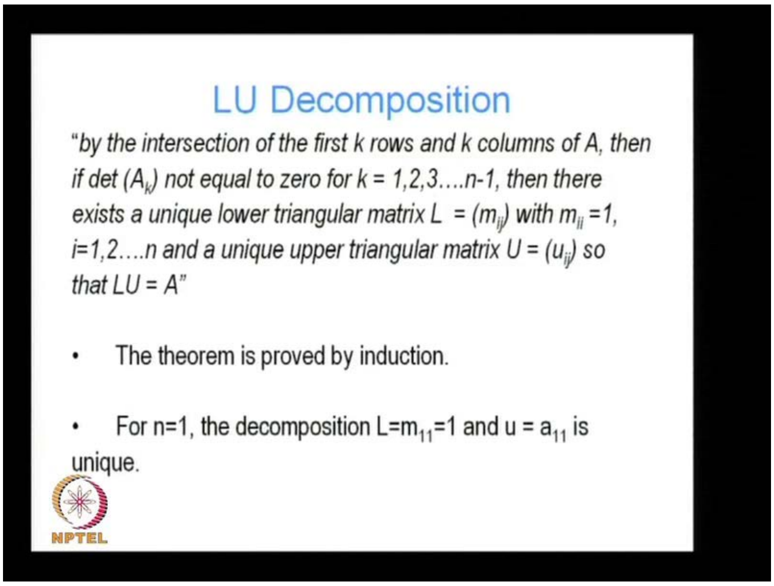

Formed by the intersection of the first k rows and k columns of A and if the determinant a k is not equal to 0 for k equal to 1 through n minus 1 then there exists a unique lower triangular matrix L with components m i j, m and with diagonal components m i i equal to 1 and a unique upper triangular matrix U with components u i j. So, that L U is equal to A. So, what this tells may is that if these conditions are satisfied that is, if all the A k is have positive determinant then there are unique matrices upper triangular and lower triangular matrices L U and L such that A can be written as L U equal to A.

How can we show this well this can be shown quite easily by induction suppose the matrix is our size one, which is basically it is a scalar in that case the decomposition is obvious, because we have assume that the diagonal elements of the lower triangular matrix L or 1 right. So, it is obvious that the upper triangular matrix is now just a 11 an; obviously, these decomposition is unique.

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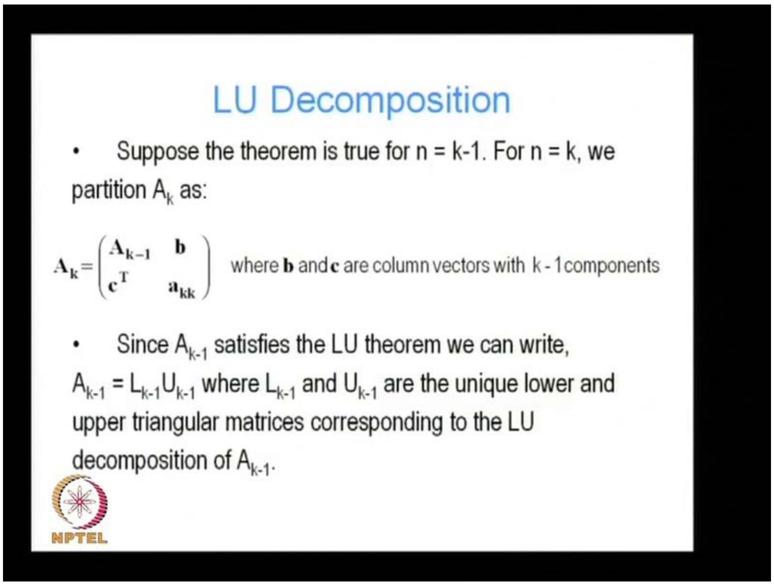

Now, suppose the theorem is true for n equal to K minus one for any k right. So, it is suppose that it is true for K minus one then for K for n equal to K we can partition A k into the following A k can be written as A k minus 1, which is the A k minus 1 denoting the part, which is formed by the intersection of the first k minus 1 rows and k minus 1 columns of my original matrix and the remaining part which is b to b is a b and c comprising column vectors with k minus 1 components. So, b is a column vector with a minus 1 components, c transpose is a column vector c transpose is actually a row vector with k minus 1 components, and then the diagonal term a k k.

Now, we know that since A k minus 1 satisfies the L U theorem we can write A k minus 1 is equal to L k minus 1 times product with U k minus 1, where L k minus 1 and U k minus 1 are the unique lower and upper triangular matrices corresponding to the L U decomposition of a k minus 1. Since A k minus 1 satisfies the L U theorem, we know basically that there is an L k minus 1 unique L k minus 1 and unique U k minus 1 such that A k minus 1 is equal to L k minus 1 product U k minus 1.

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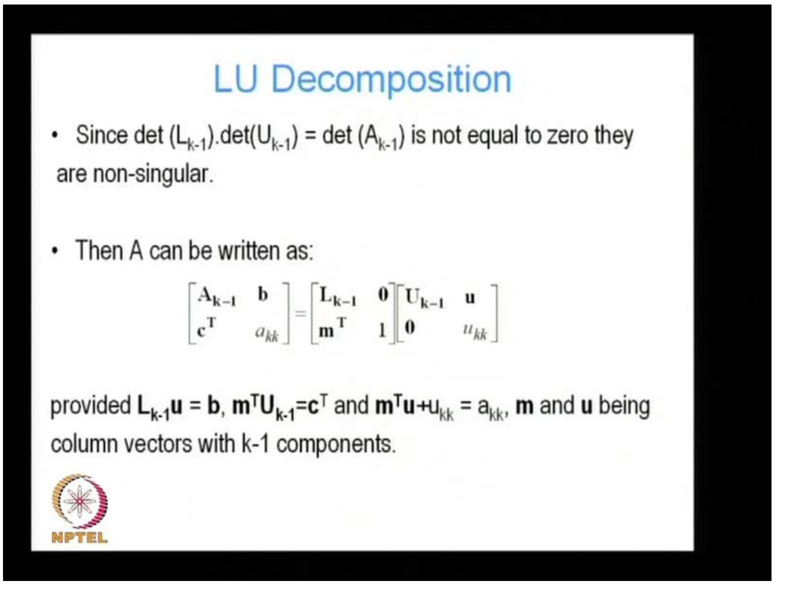

Since determinant of L k minus 1 times determinant of U k minus 1 is equal to determinant of A k minus 1, because the product of two matrices is equal to the product of the determinants and since we know the determinant of A k minus 1 is not equal to 0 it; obviously, means that both U k minus 1 and L k minus 1 have non zero determinants since the product of two of these two numbers determinant of L k minus 1 and determinant of U k minus 1 is greater than 0, it is obvious that none of this individual numbers can be 0. So, each of these matrices L k minus 1and U k minus 1 have positive determinant, which means they are non singular and they can be inverted.

Thus, then we can then if we can write A k minus 1 b c transpose a k k has L k minus 1 zero m transpose one times U k minus 1 u 0 u k k note the form of these matrices on the right hand side. The first matrix is of the form of a lower triangular matrix, because all the elements above the principal diagonal as 0 while the second element on the right hand side is in as an upper triangular matrix, because all the elements below the principal diagonal as 0.

So, we are saying that if A k minus 1 is has an LU decomposition then we can write A k in terms of the product of these two matrices provided that L k minus u is equal to b basically L k minus 1 operating on this row, L k minus 1 zero operating on this row u u k k. So, basically L k minus u is equal to b, m transpose U k minus 1 equal to c transpose. So, basically it is this row operating on this column gives me c transpose, and m transpose u plus u k k is equal to a k k. So, if these conditions are satisfied then we can write this matrix a k a k in terms of two a lower in terms of a lower triangular and a upper triangular matrix, but

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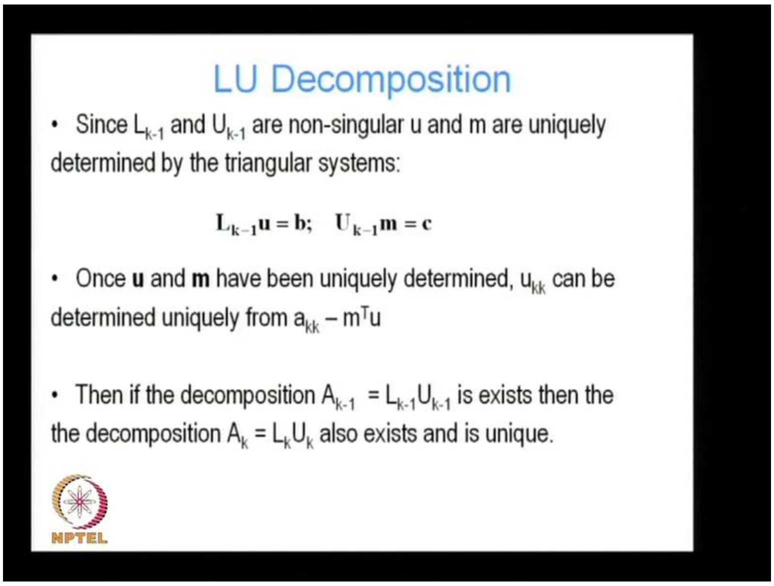

Since we now that L k minus one and U k minus 1 are non-singular; that means, that if we try to solve this equation L k minus u equal to b or for that matrix this equation m transpose U k minus 1 equal to c transpose, since L k minus 1 is non singular, U k minus 1 is non singular; that means, that we can always solve this system and get the appropriate values for m and u.

So, ones we have determined u and m uniquely u k k can be uniquely determine from a k k minus m transpose u, thus if the decomposition A k minus 1 is equal to L k minus 1 U k minus 1 exists then the decomposition A k is equal to L k U k also exists and is unique. So, what is the meaning of this we have seen that for a1 is just scalar right and that the sub matrix formed by the intersection of the first row and the first column and for that sub matrix we have shown that an LU decomposition trivially exists with the L component equal to 1 and the U component just equal to a 11 itself.

So, since the L U decomposition exists for k equal to 1 and we have shown that if the LU decomposition exists for k minus 1 it also exists for k. So, if it exists for k equal to 1, it exists for k equal to 2 since it exists for k equal to 2, it also exists for k equal to 3 and so on so forth. So, if the condition the determinant A k is greater than 0 is satisfied then the L u decomposition exists for any matrix a of size n by n.

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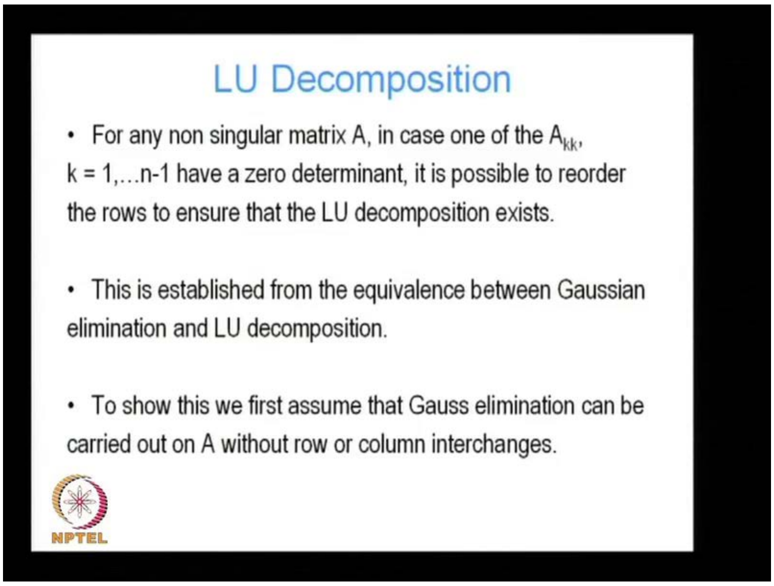

For any non-singular matrix k A in case one of the A k k, k equal to 1 through n minus, L U have a zero determinant, it is possible to reorder the rows to ensure that the LU decomposition exists right why because what we saying is that the LU decomposition is going to exists is each of if those sub matrices A k have a positive determinant now if it. So, happen that it 11 matrix A which is non singular, the non singular matrix A one of the k by k sub matrices suppose of some order have a zero determinant then it is possible to reorder the matrix to reorder the rows to ensure that k by k sub matrix no longer has a zero determinant that is it has a positive determinant. So, if they can basically by interchanging the rows we can ensure that for any non-singular matrix, all the sub matrices A k k equal to 1 through n minus 1 have a non zero determinant, which meanings that the L U decomposition is going to exists for that matrix and each of these L and U matrices are going to be unique.

How can we show this well this can be shown from the equivalence we can show that there is an equivalence between Gaussian elimination and L U decomposition that is, if we perform Gaussian elimination on a matrix without any row or column interchanges we can we are going to get very closed to the L U decomposition bearing a miner modification. So, basically what we are saying is that if we take a matrix a n by n and suppose we can perform Gaussian elimination on that matrix without performing any row or column interchanges in that case it is possible to recover the L U decomposition of a with just a miner modification.

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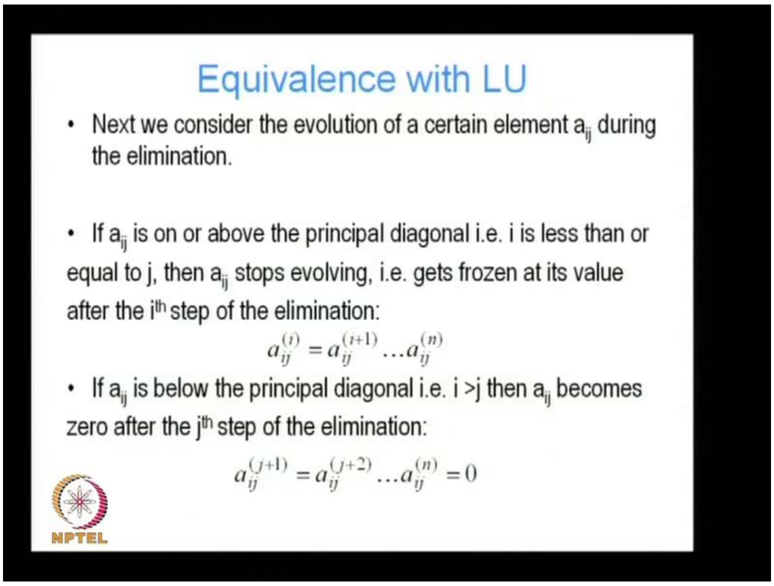

we want to show the equivalence of L U decomposition with Gaussian elimination in order to do this we consider the evolution of a certain element a i j in my original matrix during Gaussian elimination. if a i j is on or above the principal diagonal that is i is less than or equal to j that is the column index is always higher than the row index in that case a i j stops evolving that is it gets frozen at its value after the i th step of the elimination, because after the i th step that element is it has reached its final value, it does not change any further.

So, a i j i is equal to a i j i plus 1 through a i j n, where the superscript now denotes the various steps in the Gaussian elimination. Similarly, if a i j is below the principal diagonal that is if i is greater than j that is its row index is greater than the value of the column index then a i j become 0 after the j th step of the elimination, because recall at the end of the Gaussian elimination what are we going to get an upper triangular matrix, which means that the elements below the principal diagonal are all going to be 0 and any i a i j reaches its final value 0 value after the j th step of the elimination. So, a i j j plus one is equal to 0 and it is going to remain 0 for the j plus 2 through j plus 3 through n th step.

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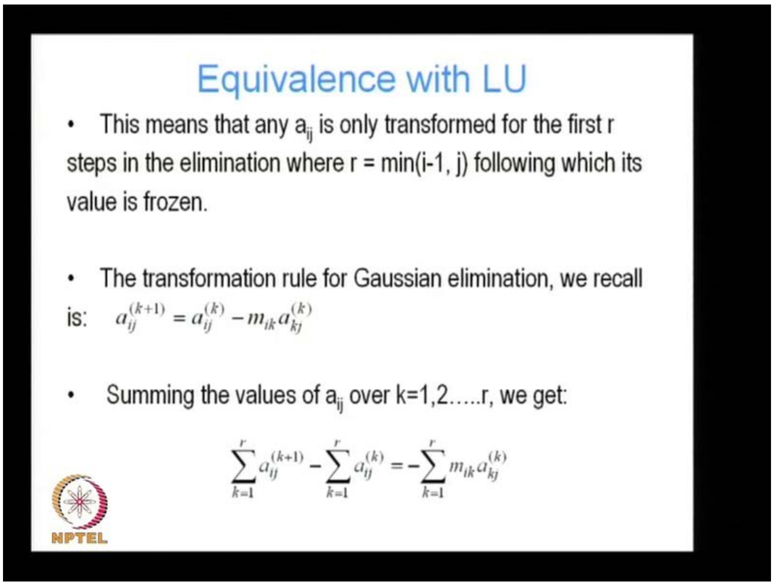

This means that any a i j is only transformed for the first r steps in the elimination, where r is the smaller of i minus 1 and j i being the row index, and j being the column index following, which its value is frozen. So, basically we are combining these two statements that if a i j is on or above the principal diagonal, it is stops evolving after the i th step of the elimination, and if a i j is below the principal diagonal it stops evolving after the j th step of the elimination and noting that i is the row index and j is the column index we are combining these two statements together and making this additional this new statement that a i j only changes for the first r steps in the elimination, where r is the smaller of i minus one and j following, which its value is frozen.

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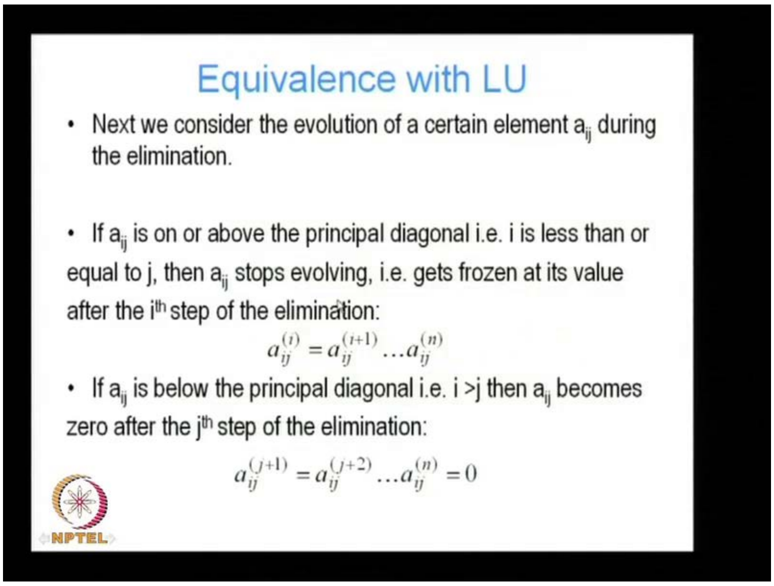

Let us go back to the transformation rule for Gaussian elimination, which is let us recall a i j k plus 1 is equal to a i j minus the multiplied m i k times a k j k with a multiplied m i k if we recall is nothing, but a k i divided by a k k.

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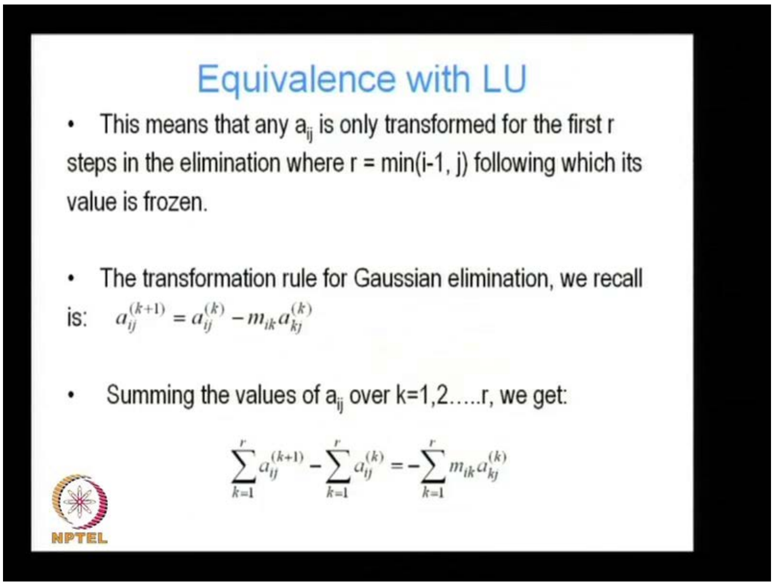

So, let us look at this equation and we sum the values over k equal to 1 through r. So, apply a summation on both sides for k equal to 1 through r, and then we. So, basically we are just taking this equation and summing it for k equal to 1 to r to get this expression nothing changes we have just added a summation sign in front and we have performing the summation for k equal to 1 to r why are we performing summations for up to r only, because we know that after r when k reaches, when k goes beyond r the elements do not change at all. So, we are performing the summation for the first r steps of the Gaussian elimination.

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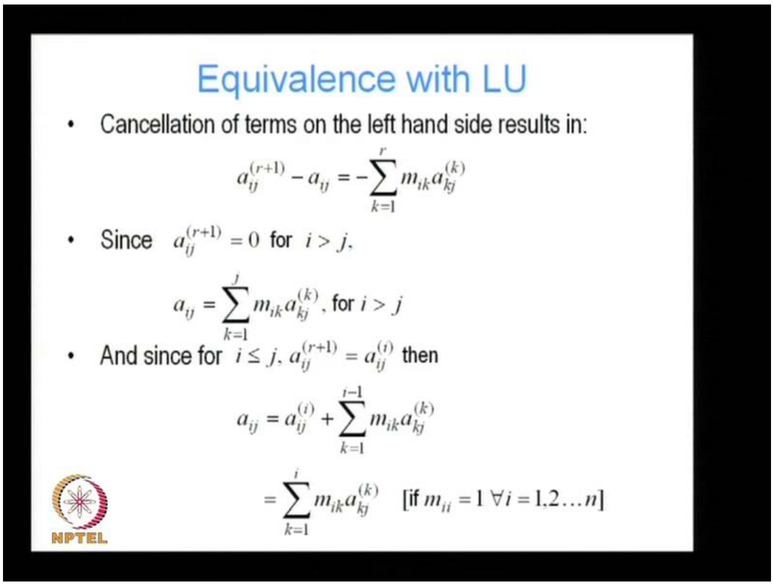

If we cancel the terms on the left hand side how can we cancel the terms, if we look at the left hand side you can see that this minus this right. So, all the terms will cancel out except the r contribution from this term a i j r plus 1 and the k equal to 1 contribution for this term, which is going to be a i j 1 otherwise for instance for k equal to 1 here we are going to get a i j 2 and this is going to cancel out from the a i j 2 contribution from the second term. So, only terms which are going to survive are going to be a i j r plus 1 and a i j. So, that gives me my left hand side the right hand side remains identical, which is equal to minus k equal to minus summation k equal to 1 through r m i k a j k.

Since a i j r plus 1 is equal to 0 for i greater than j recall that all the terms, which are below the principal diagonal are going to become zero at the end of Gaussian elimination. So, when r is when for r plus 1 a i j r plus 1 is going to be 0 for i greater than j. So, we can write we can get rid of the a i j r plus 1 on the left hand side in the first equation, and we can write this equation as a i j is equal to summation of over k equal to 1 2 j of m i k a k j k for i greater than j again let us recall that for i lesser than or equal to j that is for elements above the principal diagonal a i j r plus 1 is equal to a i j i that is it gets frozen at its value at a i j i right.

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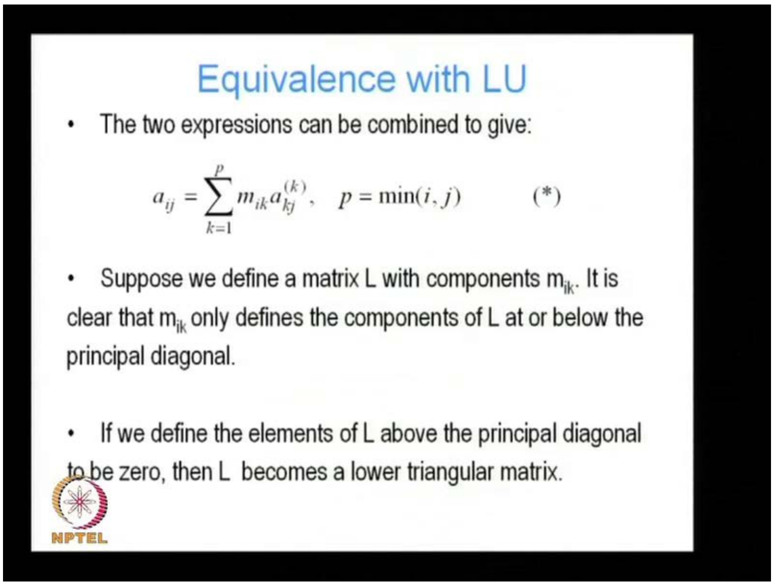

Then we can write a i j is equal to a i j i plus sigma m i k a k j k k is equal to one through i minus 1, which we can combine these two terms together and write it as sigma k equal to 1 through i m i k a k j k, if we assume that for a i i m i i is equal to 1. So, we are putting these two terms together and writing it within the summation, and extending the summation index from i minus 1 to i. So, we can combine these two expressions we can combine this expression and that expression right we can combine this expression, and that expression to get this expression a i j is equal to sigma k is equal to one to p m i k a k j k, where p is the smaller of i and j. So, p is equal to j in this case right and p is equal to i in this case. So, we add we are combining these two expressions together to get this expression.

From this expression, if we look at the contributions of m i k it is clear that m i k is we are only using the values of m i k that are, if we consider m i k to be representing a matrix m, m i k being the components of a matrix n, it is clear from this expression that we are using only the components of m, which are at or below the principal diagonal why because m i k, k is summed from k goes from one to p and p is again the minimum of i and j. So, j cannot exceed i right. So, it is p is equal to minimum of i and j. So, that the second index here k, which is denoting the column index cannot be more than the row index so; that means, that since the column index cannot be more than the row index; that means, we are only considering the elements of m i k, which are at or below the principal diagonal. So, if we take these elements m i k and then on that elements above the principal diagonal we make those elements zero then this matrix m becomes a lower triangular matrix and we can denoted as L, which is the notation for a lower triangular matrix.

Similarly, it is clear that a k j define the elements at and above the principal diagonal lysed that let us consider the term a k j and it is clear that k cannot be k is being summed from one to p and k cannot be greater than, p cannot be greater than j right. So, basically the first index the row index cannot be greater than the column index; that means, that meet we are only concerned with that a k j, which are at or above the principal diagonal. So, again if we assume that the elements of a k j below the principal diagonal are 0 in that case we can write a k j k as a upper triangular matrix.

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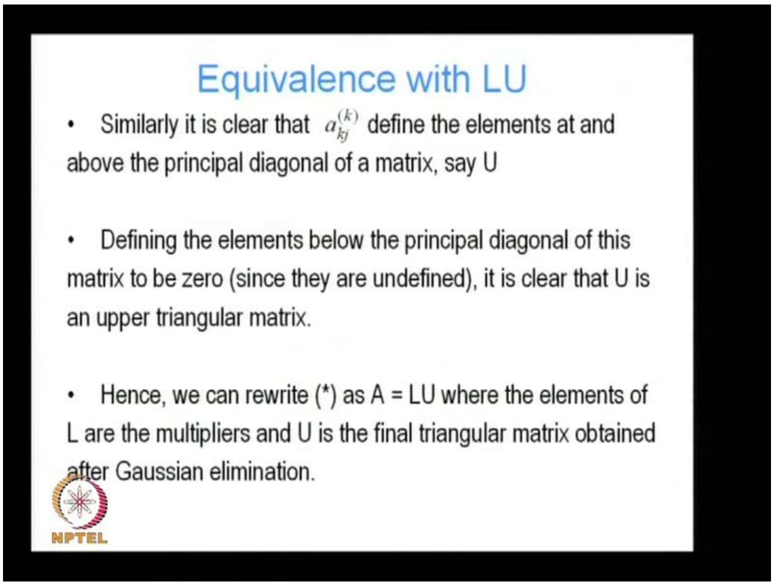

Similarly, it is clear that a k j k define the elements at and above the principal diagonals of a matrix, and if we defined the elements below the principal diagonal of this matrix to be zero, it is clear that that U, which where we have used due to denote the matrix comprising the components a k j u is an upper triangular matrix hence we can rewrite our previous equation a i j equal to summation k equal to one to p m i k a k j k as the product of a lower triangular and an upper triangular matrix, and since recall the this expression that we have obtained from usual from Gaussian elimination we did not do anything special to get here right. So, this is what we get after guassian elimination and we can see that is can be written as in terms of the product of an upper triangular, and lower triangular matrix provided that in the Gaussian elimination in the normal Gaussian elimination we just to out the upper triangular part, but now in the lower part below the principal diagonal, which individual Gaussian elimination is zero.

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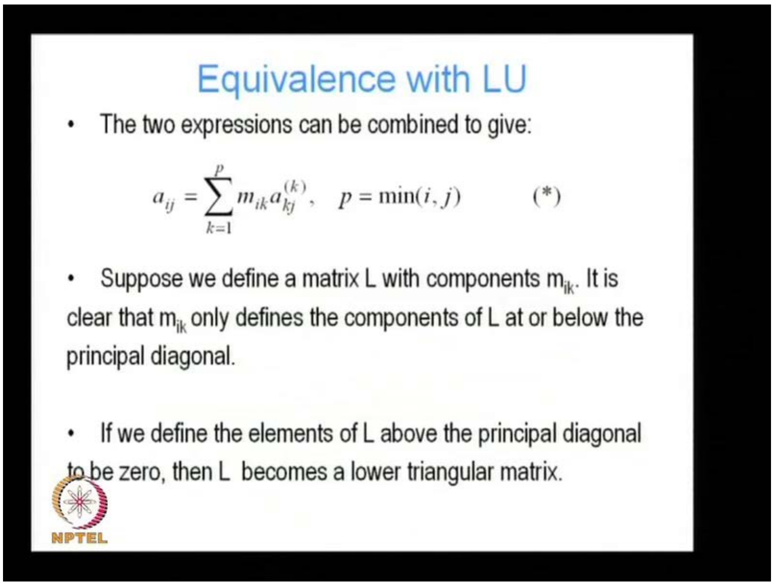

If we instead of making at zero we store these multipliers m i k then Gaussian elimination we are going to recover our L U decomposition bearing the trivial diagonal elements of L, which are always equal to 1.

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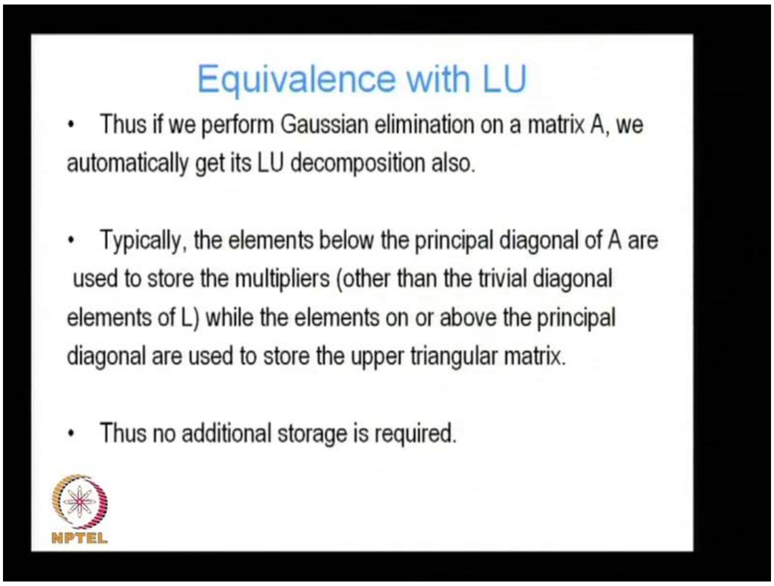

Thus, if we perform Gaussian elimination on a matrix A, we automatically get its L U decomposition also typically the elements below the principal diagonal of A are used to store the multipliers other than the trivial diagonal elements of L while the elements on or above the principal diagonal are used to store the usual upper triangular matrix of Gaussian obtain that the end of Gaussian elimination thus L U decomposition does not required any additional storage. So, basically if you over write are original matrix after storing the upper triangular elements above or below the above the principal at above the principal diagonal and used the elements below the principal diagonal to store the multipliers.

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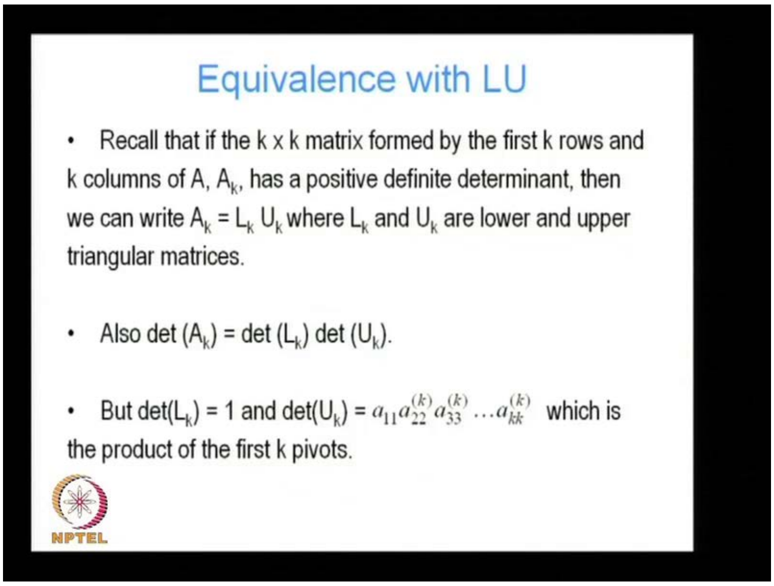

We are automatically going to get our L U decomposition recall that if the k by k matrix formed by the first k rows and k columns of A, A k has a positive determinant then we can write A k is equal to L k U k, where L k and U k are lower and upper triangular matrix also let us remember the determinant of A k is equal to determinant of L k times, determinant of U k the product of the determinants is equal to the determinant of the products.

But we have seen that the determinant of the L k is equal to 1 why is that, because all the diagonal elements of L are identically equal to 1 we have seen in a previous lecture that the determinant of a triangular matrix is nothing, but the product of its diagonal elements. Since all the diagonal elements are one determinant of L k is equal to 1 and determinant of U k is again the product of its diagonal elements and its diagonal elements a 11 a 22 k a 33 k etcetera are nothing, but the pivots in the Gaussian elimination. So, the determinant of U k is nothing that the, but the product of the first k pivots.

.So, Gaussian elimination can be carried out without row or column interchanges, if each of those pivots is different from zero. So, basically each of these pivots is different from zero then determinant of U k is going to be non zero always determinant of A k is refer for is also going to be always non zero, because determinant of L k is always going to be 1 and determinant of U k is always going to be positive. Since determinant of A k is always going to be positive what does that mean; that means, this matrix is going to be positive definite, and since it is positive definite it is for all if this determinant is not is greater than 0 for all case in that matrix is positive definite, and in that case we can perform Gaussian elimination without the need to perform any pivoting.

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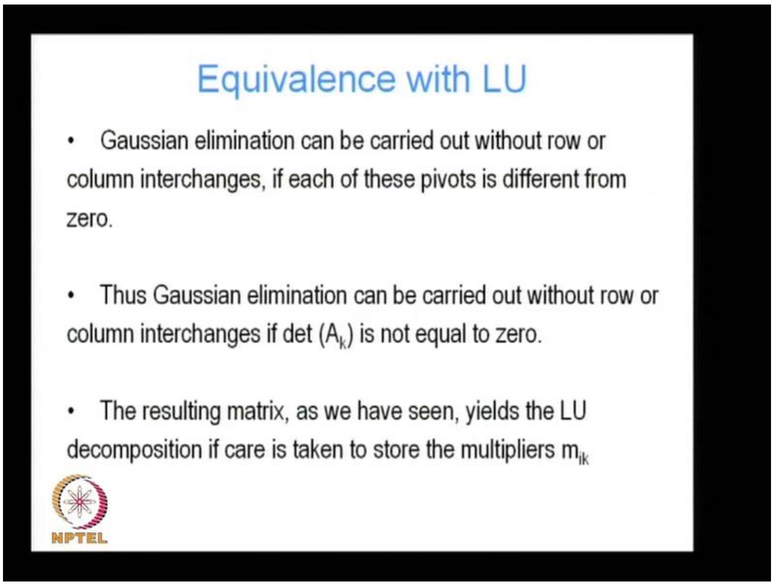

So, Gaussian elimination can be carried on carried out without row or column interchanges, if each of these pivot elements is different from zero each of these pivot elements different from zero means U k as positive determinant, L k by definition has positive determinant, the A k always has positive determinant thus Gaussian elimination can be carried out without row or column interchanges, if determinant of A k is not equal to 0 resultant matrix as we have seen yields the L U decomposition, if care is taken to store the multipliers m i k below the principal diagonal.

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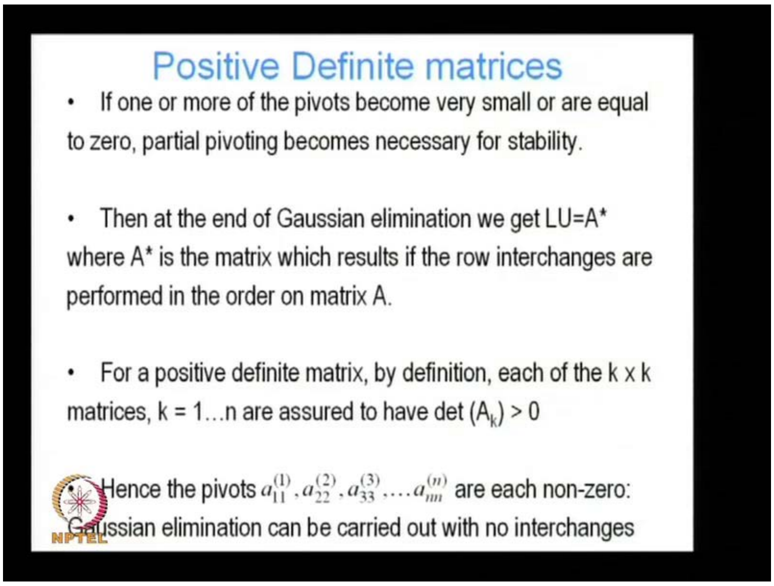

If one or more of the pivots become very small or are equal to 0 partial pivoting becomes necessary for stability we have seen that right, if any of the pivots is not required only of each of the these pivots is greater than 0; however, if one of the pivots becomes very small or actually becomes 0 then we have to do pivoting right. It also means that the matrix is no longer positive definite right, then at the end of if we if it. So, happens that the matrix is not positive definite. And we have to perform pivoting then at the end of Gaussian elimination, if we take care to store the multipliers below the principal diagonal we are still going to get an L U decomposition, but that L U decomposition that we are going to get is not going to be the L U decomposition of the original matrix A. It is going to be the L U decomposition of another matrix A star, where A star has been obtained by performing the same row interchanges performed on A.

So, let me repeat. So, in while we performing Gaussian elimination on A. We found that in order to do in order to make maintain this stability of this solution we have to do sum pivoting. So, we entertain certain rows of A right, and then at the end of Gaussian elimination we to care to store the multiplies below the principal diagonals. So, we got a L U decomposition, but the L U decomposition we got is not the L U decomposition of my original matrix A. It is the L U decomposition of another matrix A star, which is not which we which can be obtained by performing the same row interchanges, which we performed in the course of Gaussian elimination on my original matrix A.

However, for a positive definite matrix by definition each of my k by k sub matrices k through n are assured to have positive determinant therefore, the pivots a 11 a 22 a 33 a n n are each non-zero and therefore, Gaussian elimination can be perform with no interchanges what does it what does that mean; that means, that if we if while performing Gaussian elimination, we take care to store the multipliers at locations below the principal diagonal then at the end of Gaussian elimination are going to get and L U decomposition, and that L U decomposition is going to be the L U decomposition of my original matrix A.

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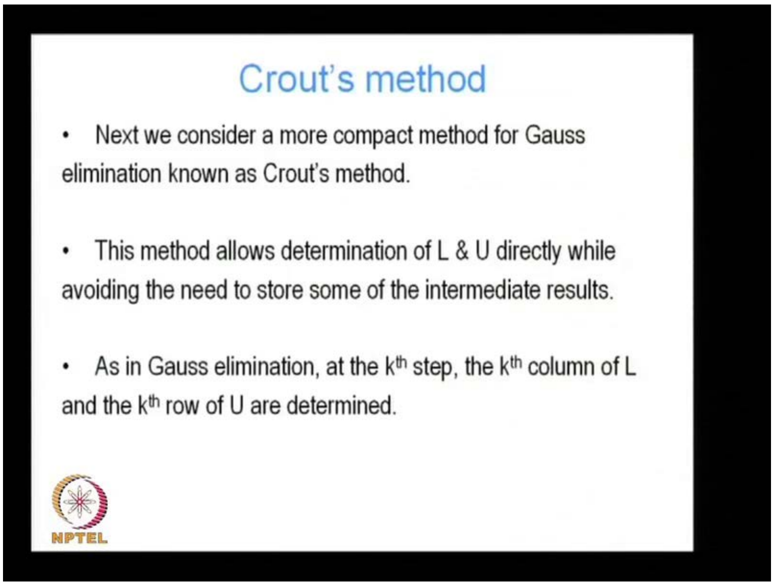

So, next we compare we consider a more compact method for Gaussian elimination this is known as crout's method. This method allows determination of L and U directly while avoiding the need to store some of the intermediate results as in the gauss elimination at the k th step. We compute the k th column of L and the k th row of U.

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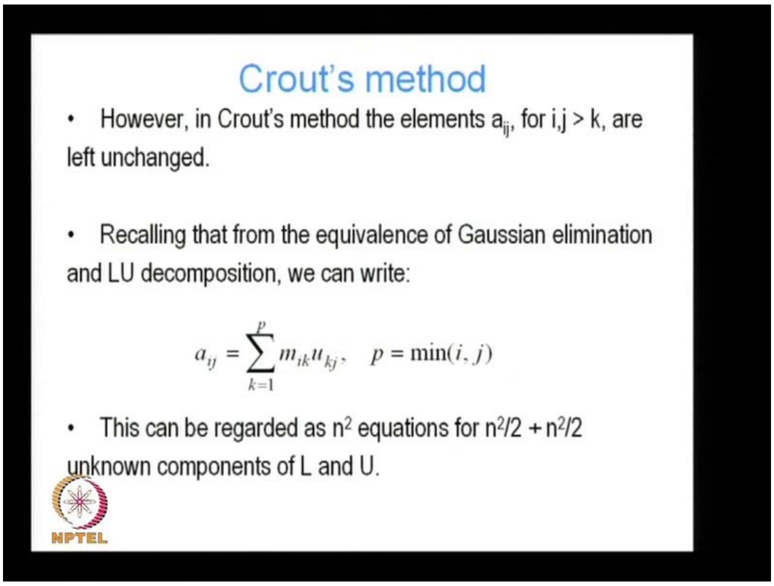

However, in crout's method the elements a i j for i j greater than k are left unchanged how would does that happen while to see how that happens let us recall our equation for Gaussian elimination in terms of L U components, which we have obtain just little while earlier right, and which gave as basically the equivalence between Gaussian elimination and L U decomposition that is recalled that equation which says a i j is equal to product of m i k u k j summed over k is equal to 1 to p, where p is the smaller of i and j. So, it is smaller of the column, if the row and the column index of a i j.

We can think of this equation as n square equations, because i is going to vary from 1 to n, j is going to vary from 1 to n. So, they are going to be n square terms rights. So, this is basically we can think of this as n square equations for the unknowns m and u and we remember that m and u are the non zero components of lower and upper triangular matrices. So, m and u have each n square by 2 terms. So, m has n square by 2 terms, u has n square by 2 terms. So, the total number of unknown is n square by 2 plus n square by 2. So, we can think of this as n square equations for n square unknowns, where by unknowns are m and u, which are the components of the lower and upper triangular matrix. So, we can think of this equation as except of n square equations for determining the unknown coefficience unknown components of my lower and upper triangular matrix.

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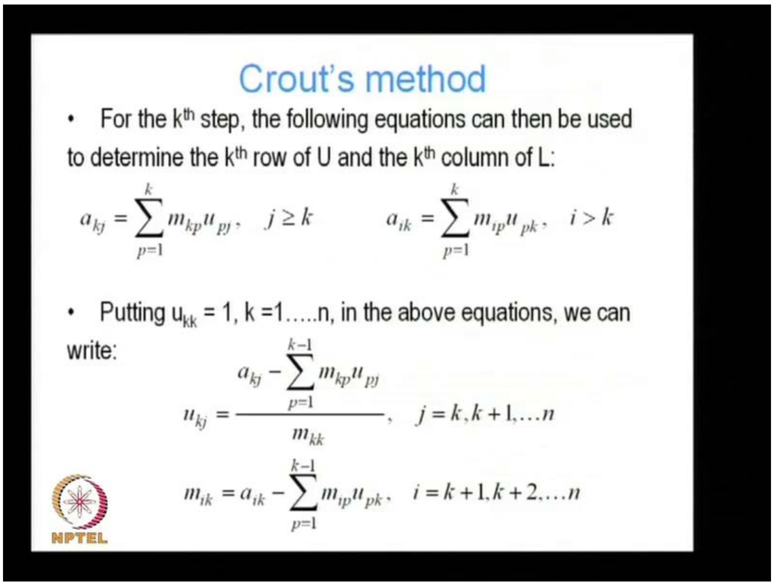

For the k th step the following equations can then be used to determine the k th row of U and the k th column of L. So, basically I am rewriting those same equations, but once I am assuming that j is greater than or equal to k that is I am trying to find a k j such that j is greater than or equal to k. So, basically I am finding the components, which are above the principal diagonal right. Since j is greater than k the column index is more than the row index. So, I am looking at terms above the principal diagonal and that i can find by using my this equation, but now the summation has changed in the limit of the summation has changed from p to k why is that, because of this condition the j must be greater than or equal to k, and this condition that p is the minimum of i and j.

Similarly, a i k, I can obtain from the same from the previous equation, but now I am considering only the I am trying to find the k th columns of L hence because of that i is greater than k i is greater than k. So, I am using the same equation as before except that the limit of this sum is now k right, the limit of the sum is now k simply, because i is greater than k and let us recall from our previous equation p is the smaller of i and j and p is the minimum of i and j, and since i is greater than k, p is equal to k right.

So, we can use we can. So, now, splitting the previous equation into two parts and we will continue next time with this discussion and show that how it is using this equations.

It is possible to write u k k u k j and m i k in terms of the previous. So, we can these equations have this from the crout's method is useful, because you can see these u k j and m i k. It compute u k j and m i k, and only taking summing terms up to k minus 1 right. So, I am only using up known term that is terms, which are known up to the k minus 1 th step we will continue our discussion with crout's of crout's method in our next lecture.

Thank you very much.