Theory of Elasticity

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Lecture – 11
Concept of Stress and Strain (Contd.)

Hello everyone, welcome to the last lecture of this week. What we discuss in this week is we try to understand the Concept of Stress and Strain, definition of Stress and Strain and then, of course, with the assumption that is a small deformation, small deformation theory and then, we also studied how to represent Stress and Strain and different operations on Stress and Strain.

What is the purpose of this week is to just to have a summary of what we have discussed and some important thing that will be using in the subsequent weeks will discuss that today. So, it is essentially the closure of this week Concept of Stress and Strain.

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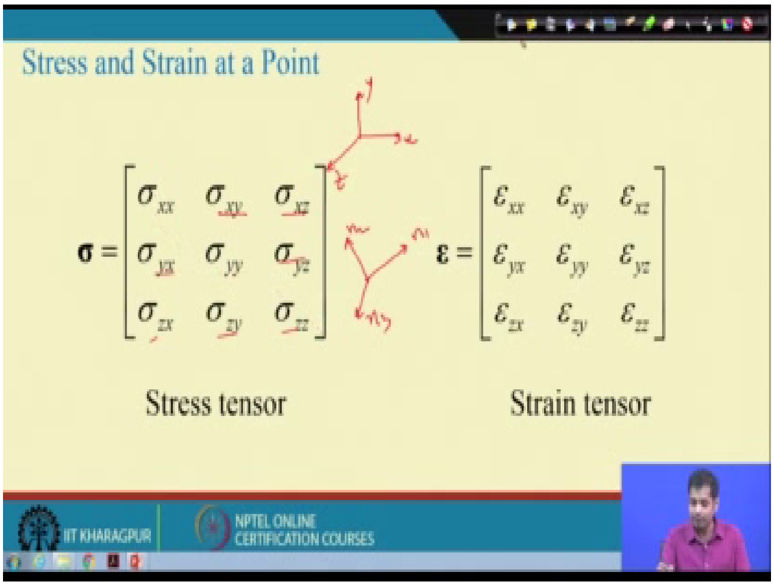

You see so we define Stress and Strain as a tensor. Stress is a point to strain at a point is a tensor like this and then, these tensors are a symmetric tensor second order tensor, symmetric tensor and similarly for a strain as well.

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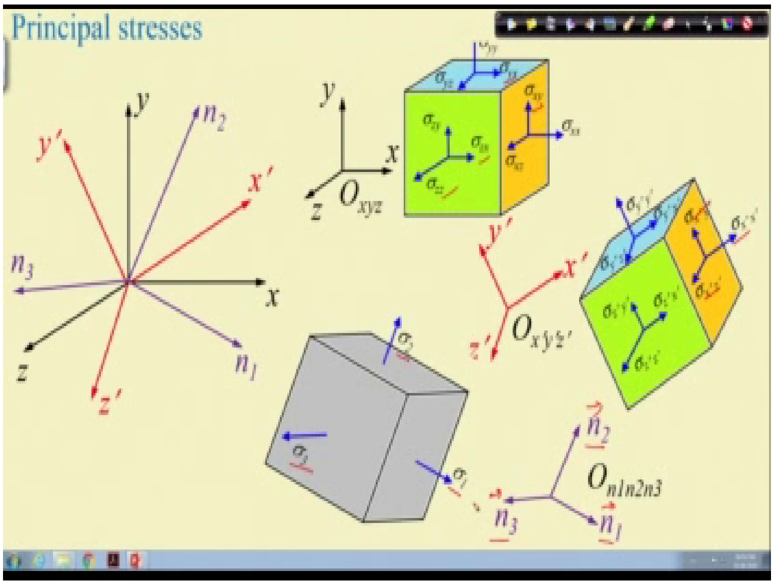

Now, then we also discuss now let us try to understand, what is Principal stresses?

We also discussed that since this tensor can be ah, this tensor can be the second order tensor; always when we say that it is a stress is represented like these or strain represented like this always the coordinate associated with this, always we represent these stress and strain with respect to some coordinate system right.

For instance, in this case if it is with respect to x y z given coordinate system, then the representation is like this. But then, if we change the coordinate system, then if we change a coordinate system say x dash y dash z dash, this with respect to that same coordinate system we can have the representation of the same stress like this right.

Now, you see now suppose, you think of a situation that you have a state of stress at a point which is represented like this with respect to a given coordinate system. Then, you are slowly changing the coordinate system ok. Now when you slowly changing the change the coordinate system, then what happens is the stresses; the individual components they also change right.

So, all the normal stresses all the shear stresses, they also change either they increase or the decrease; but they change. Now if we keep on change the this coordinate system, then you will come to a point when or a if you come across a coordinate system or the or the rotated coordinate system for which there is no shear strength on all these three planes.

For instance, if we have any arbitrary coordinate system, we are rotate we rotate the coordinate system and then, we keep on rotating the coordinate system and when the your coordinate system becomes n 1 n 2 n 3; how to determine n 1 n 2 n 3 and what are this sigma and sigma 2 that will that will come shortly.

But, let us try to understand, what is principle stress through again geometry? So, physically what is happening there? Now, will come to a point where on these 3 planes, there is no shear stress; all these shear stresses that we had here these shear stresses, all these shear stresses all these shear stresses they vanish. Only then, we are left with normal stress in 3 normal directions and this is sigma 1 sigma 2 and then, say sigma 3. This and then suppose that directions are n 1 n 2 and n 3, all our vector ok; n 1 n 2 n 3 are the directions.

Now, these sigma 1, sigma 2, sigma 3 are called principal stresses and I believe that you have already learned it in Solid Mechanics Course.

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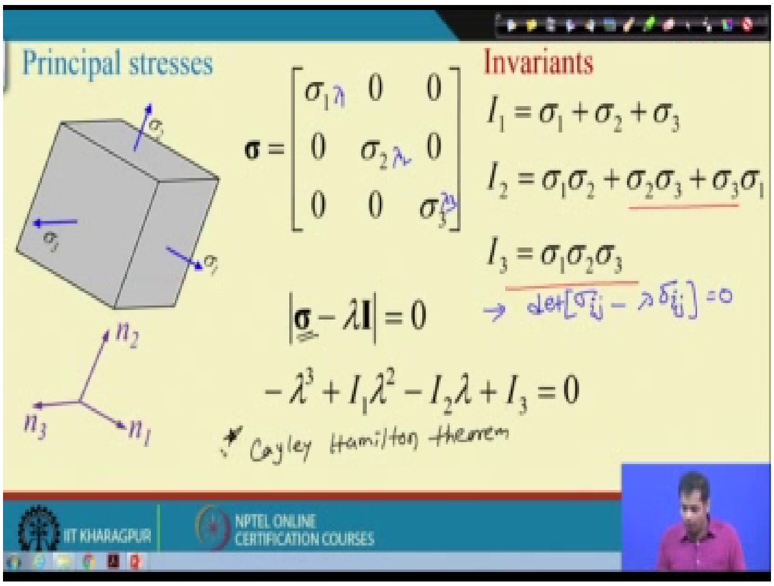

Now, so, when we have as I said this is the representation the if we represent stress like this, for instance if we represent stress like this, this is with respect to a given coordinate system right. A given coordinate system, this is x, then y and z ok.

So, now if we change the coordinate system, if you have a new coordinate system which becomes say it is n 1, n 2 and n 3 something like this and with respect to this coordinate system, there are no shear stresses only the normal stresses, then what will happen is all these diagonal terms all this diagonal term they become they become 0. And this strain the strain come this stress tensor essentially become only a diagonal matrix, where this is sigma 1 sigma 2 and sigma 3 ok. That is the so that is the the strain when you have.

So, this is the stress tensor with respect to the principal plane n 1 and n 2, n 3 with respect to principal plane. Now you recall the invariant. Invariant they do not they do not depend on the coordinate axis is chosen and there are 3 invariants for a for stress tensor the second order stress tensor. The first one is the stress of the matrix.

since it is invariant. So, whether you calculate this stress with respect to x y z coordinate system or n 1 n 2 n 3 coordinate system, they remain same the invariant with respect to the with respect to principle coordinate system is sigma 1 sigma 2 sigma 3. And similarly, these third invariant, second invariant which is the sum of cofactors this is sigma this is this and a third invariant is sigma 1 sigma 2 sigma 3, the diagonal term.

Now, the question is how to determine these principles says if you recall the principle stress can be obtained by the eigen value analysis of this matrix. And what is this eigen value analysis? So, eigen value analysis is that if we have this sigma any matrix and eigen values can be determined by lambda into I; lambda I is the identity matrix, the determinant of this will be equal to 0 right.

Now, this can be written in indicial form as a indicial form if I have to write it, then this will be something this can be written indicial form the determinant of say sigma i j minus lambda delta i j that is equal to 0, that is the same thing it is in indicial form.

Now, when you express this, then what happen? Then, you get an equation like this and or before that you can you that equation can be expressed as like this in terms of this invariant and we solve this equation, you have 3 roots; lambda 1 lambda 2 lambda 3 and these 3 roots are essentially your 3 principal axis; the lambda 1 lambda 2 and lambda 3.

Now, the eigenvalues give you the principal stresses. Similarly you can have the eigenvectors associated with each eigenvalues. Eigenvectors are essentially corresponding n 1, n 2, n 3 ok. So, this we already know. Now at this point so this is called Characteristic equation. This equation is called Characteristic equation.

What is that equation written? Now just to for the completeness of the discussion, there is a theorem called use different color Cayley Hamilton Theorem which says that if this is the characteristic equation that the sigma itself satisfied, if you substitute sigma in case of lambda there, this sigma also satisfy this equation ok.

We do not need it right now or probably in this course, but just for the completeness you can you can look into it and then, see what this theorem exactly says ok.

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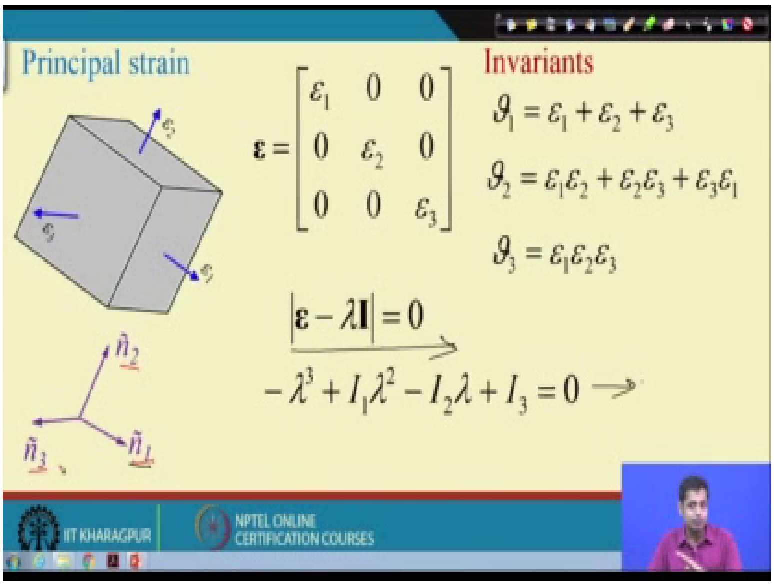

Now, let us similar exercise we can do it for we can do it first strain. So again, we have we have a principal strain in 3 directions; the epsilon 1, epsilon 2, epsilon 3 corresponding invariant, the similar way we can define invariance first strain with respect to any coordinate system you define this train your invariants remain unchanged.

And if those invariants are same; the first invariance is stress of this and the second invariance is the sum of cofactor and the third one is the determinant. And similarly these epsilon 1, epsilon 2, epsilon 3 can be obtained by this by the eigen value analysis and this is the corresponding characteristic equation ok.

Now, n 1 you please note that in these directions the eigenvalues are epsilon 1 epsilon 2 epsilon 3 and the corresponding eigenvectors gives you n 1 tilde n 2 tilde and n 3 tilde, where these are the principal directions. Now you see, now we have we have stress tensor; we have strength tensor; we know what are the principal stresses; how to determine principal stresses; we do not know how we know how to determine principle strains; how, what are the principal principle directions?

Now, the question comes here is will not discuss will not even try to answer this question right now, you have to wait for a couple of weeks to come to that point. The question is suppose I know this state of stress at a point and at a given point I know the state of stress and also the state of strain right.

Now, I determine the principal stresses with this eigen value analysis, with these eigenvalues and then I determine the principal strain and then, I also determine the direction of principal stresses and the direction of principal strain. Now, the question is whether these 2 directions; the principal the direction of principle stress and the direction of principle strain, are they same or different? You can write this question when you put star mark will not answer this question right now.

Let us spend a couple of weeks, try to understand what are the different kinds of material what is and what are the stress strain relation and then, will try to understand whether under what circumstances first of all whether this whether they are same or not and if they are same, then if they are not same; then, under what circumstances there they can be made same. We will discuss that in subsequent weeks ok.

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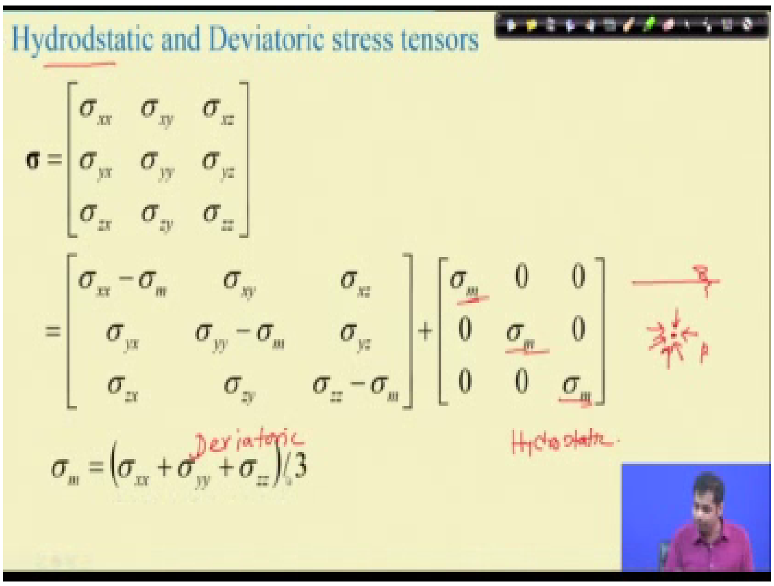

Now, so, this is another important thing. Suppose this is this is a stress tensor, these test tensor can be divided into is generally it is divided into 2 part you see. One is this and another one is this, if you add them you get this. This part is called Hydrostatic part. Hydrostatic part hydro static part or Spherical part sometime also and this is called Deviatoric part.

Hydro static part, why it is Hydrostatic part? You see there all the components all the normal stresses as same which is very similar to if you an object in underwater and then, if we see the pressure; pressure in all the directions are same right, which does not have any specific directions. Pressure is same in all direction.

Though all direction this is same. The sigma is essentially the pressure some name called Mean stress. So, that is why it is called Hydrostatic part and this is called Deviatoric part ok. Now, sigma m is can be written as is generally written as this sigma x plus sigma y plus sigma z 3; you look at one thing is very interesting here.

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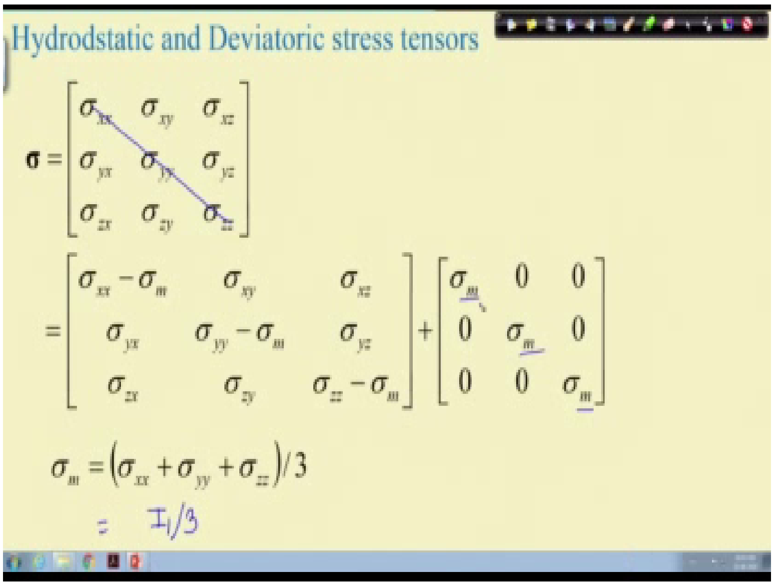

What is sigma x plus sigma y plus sigma z? This is the stress of this matrix means this is I 1. So, I 1 by 3 sigma n is equal to I 1 by 3. You see we just now, we discussed, what is the physical interpretation of I 1? I 1 is essentially give you the pressure, but you see always I 1 may not be always sigma m may not be sigma x is plus sigma y plus sigma z.

But yes, for this course yes it is fine, but we will see there are probably not in the elasticity, but elasticity within the scope of this course, but there are many materials where the pressure in order to calculate pressure, you need a set of different set of equation. Those equations are called equation of state, where the pressure depends on the energy. It may depend on temperature. It is may depend on other; it is may depends on density.

So, that is a different thing, equation of set will not going to discuss that is not within the scope of this course ok.

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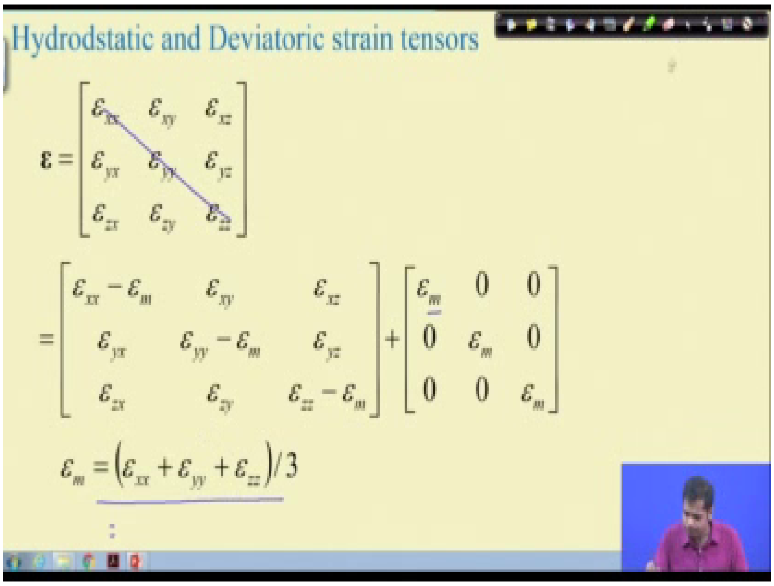

So, similar exercise you can do it for strain. Strain also can be divided into 2 parts; one is again Hydrostatic part, one is again Deviatoric part and then hydrostatic part is epsilon m is written as is again, you see this is essentially the first invariant of strain tensor which is the stress of this ok.

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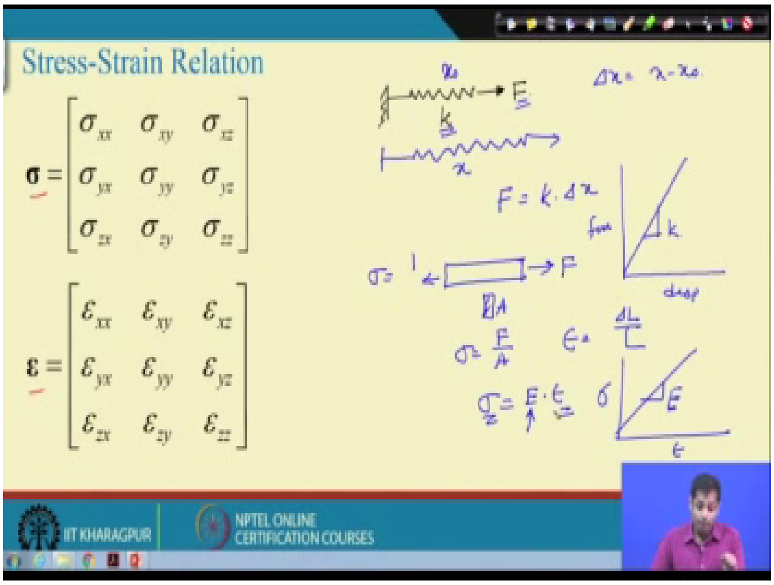

And then, so you see we are almost at the end of this lecture. So, what we where do we stand now just if a material is subjected to some kind of threat; then, as a response to that threat, the stress and strain are essentially 2 responses. The difference between is the unlike stress strain can be perceived, it can visualized through the senses that we have. But the 2 are the responses of material to a threat ok.

Now the next question is how to find out the relation between these two responses that how to find out the relation between this stress sigma and then, this stress sigma and then corresponding strain ok. Suppose, before that you recall take a problem of a spring, if you recall a spring which is fixed here and if we apply a force P or say yeah force F say F ok.

Now, suppose the spring constant is spring constant is this is F. Suppose spring constant is k and then, with this the deformed shape of this spring becomes after the spring undergoes deformation and this deform deformation is this ok. So, suppose if it is x 0, if it is x 0 and if it is x; then, delta x is equal to x minus x 0 ok.

Now, you recall or x minus x 0. So, recall that we in mechanics you study the relation between k F are essentially F is equal to k into delta x in this case or sometimes F is equal to k x or so we say ok. Now, if we plot it that force and displacement; if we plot in displacement here and then, force here, then this relation is linear and this slope is give so gives you k ok.

Now, and similarly I mean if you recall if you recall in your class 2 level when you studied probably the first time stress and strain, if stress is equal to if you define stress is equal to force by area, if you have a take a if you take a rod which is subjected to some force F and then, cross section a cross section is A. Then, sigma is force by area and epsilon is strain was delta L by L right, change in length by L.

And then, there you define the sigma is related to epsilon as sigma into E into epsilon or some time y into epsilon; where, y is the Young's E is the Young's modulus or elastic modulus and if you plot it, here it is epsilon and sigma, this is this ok. This is here, this is elastic modulus ok.

Now, this is the relation between sigma and epsilon ok. But now, in this case, we got we just do not have one component of strain and one component of stress, we have the 6 stress components and 6 strain components right. So, we have to find out relation between 6. Thus, so we have to find out relation between this tensor and this tensor.

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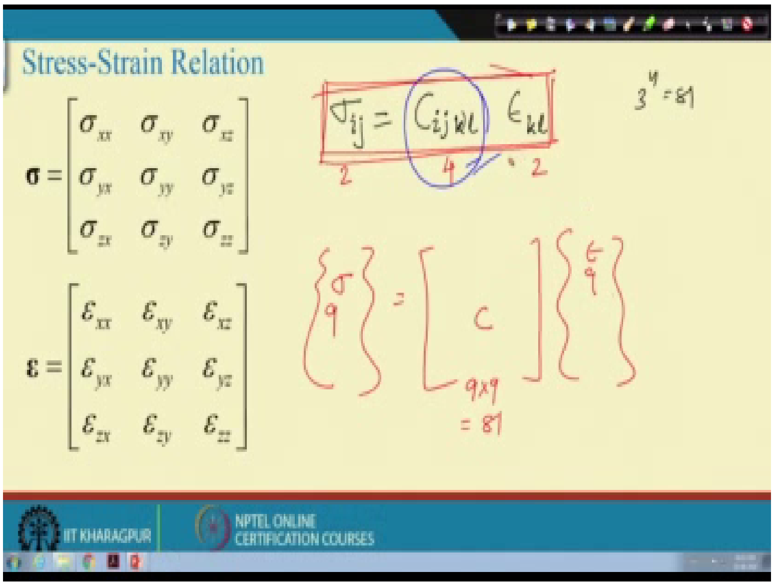

Now, if this is this tensor is sigma i j sigma ij and then, epsilon ij is the strain tensor ok. Then, we can define a for now sigma i j, so what is the order of this tensor? The sigma ij tensor order is 2; 2 here. It is a 2, order 2; it is also ordered 2, that is why we have 2 indices ok.

Now, we can define a fourth order tensor C i j or let us write it instead of ij let us write it k l k l ok, C i j kl ok. Now it is a fourth order tensor. So, what happens here? So, how many components you have? We have 3 to the power 4 means 81 components ok. Now, you see my k and l, they are repeated they are repeated index. Therefore, essentially they will vanish, there is actually dummy index. So, essentially you are left with only i j.

So, this is a relation a very general relation between stress and strain, where C is a 4 that the order is fourth order tensor. Now, this is in a vector in a tense in a tensor form, even if you write in a matrix form in a very general case, how many stress component we have? We have 9 stress components.

Why I am saying 9 because I do not want to put that symmetry right now because I want to start with a very general configuration, very general material ok. So, we have 9 strain components, then we have 9 strains, 9 stress components. So, we have 9 strain component and then, 9 stress component. This is sigma, this is epsilon and this is also related with something C, very similar to C.

So, here it is 9 here, it is right if you write in a matrix form. If you write sigma and epsilon m in a vector form, then how many what is the size of this matrix? 9 cross 9 which is 81 components ok. So, whether you do it in a tensorial notation or this form essentially you are do you are doing the same exercise.

But this is more general and we will try to stick to this, this representation. So, now here starts the motivation for the next 2 weeks. We have to find out what is the relation between stress and strain? And we have seen that if stress is this and strain is this, these can be you say this can be related through some through another tensor fourth order tensor.

See we do not know what is that fourth order tensor; we even do not know whether substance are exist or not. But, suppose if they exist and we also do not know what are the properties of that tensor, we do not know actually anything; let us if we can represent the stress and strain in this form, if we can; then, what is this?

How to find out this? See, next two weeks will spend to understand different kinds of material or rather different idealization of material and then, see what forms the Ct for different idealization ok. So, I stopped here today ah. So, next class is next class, will discuss is Constitutive Relation. This relation is essentially called the relation, just now we wrote here this relation is called Constitutive relation.

So, next class probably start with this expression sigma ij is equal to c ij kl epsilon k l and then, see how to find out this C and what are the properties that C may have. But stop it today; see you in the next week.

Thank you.