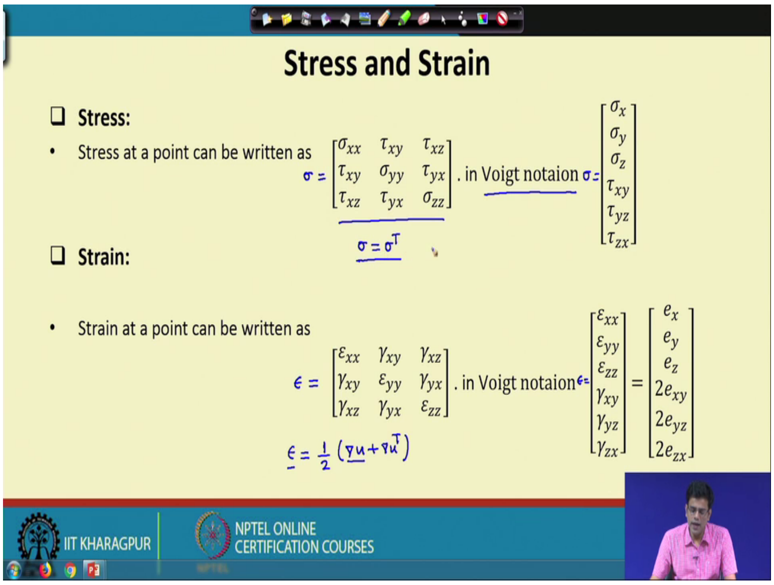
Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 12
Constitutive Relation – I

Welcome. This is the third module of Theory of Elasticity. So, in this module will basically learn what is constitutive equation or constitutive relation of a material. So, not only in this module, in the next module also will discuss constitutive relation.

(Refer Slide Time: 00:48)



Now, to start with in the previous module you have learned these quantities for instance stress, strain and specifically the quasi stress, and traction, and strain displacement relation everything you have learned in the previous module.

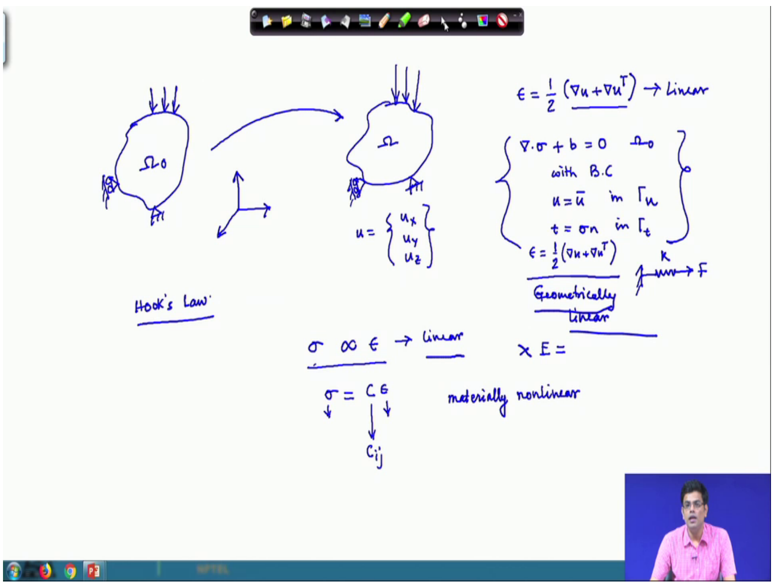
Now, probably also you have learned that stress in this is a tensorial representation. So, sigma we represented in this way and then in Voigt notation or in a vector form, we represent in this way, that strain similarly also we represent in a tensorial notation epsilon and then in Voigt notation in a vector form. Remember one thing that this is a 2 E xy is actually the shear strain and the strain tensor strain. So, these quantities we have learned in the previous module.

Now, it is also important to note here that this stress matrix is symmetric matrix. So, here this is sigma is actually sigma transpose. So, similarly strain matrix is also as a symmetric matrix. Now, how this stress matrix or strain matrix become symmetry will learn after this 2 module in specifically the stress become symmetry it to the balance of angular momentum. So, we will learn these things in a 5th module.

Right now, we know the stress, strain and probably also we have learned that strain displacement relation which is half of del u plus del u transpose. So, probably this thing you know now, and this del u is the displacement gradient and then half of plus del u transpose and half of this is known as the strain.

So, this is linear strain. What do I mean by linear strain? Because this relation is essentially linear. So, relation between the displacement and strain is linear. So, this is small strain or linear strain theory and so we do not comment about anything on the stress. So now what it comes after this is the balance law. So, anybody which is essentially deforming suppose there is a body in the body is deforming from, initial configuration which is omega 0 to another configuration which is omega, omega.

(Refer Slide Time: 03:58)



So, this is a body. So, you know what is the concept of traction. So, here some due to load is given, some force is given, some boundary condition is given and something is given to resist the motion some points. So, this body deforms finally, to a current configuration and with the action of this stress strains traction force.

Now, this deformation is actually what do you measure about this deformation. So, essentially we measure or we are interested in knowing, what is the deformation that means, what is the displacement. In truly true sense it is u x, u y and u z say. So, we are interested in knowing the displacement vector. So, if I considered this is a 3 dimensional plane; so, x, y and z coordinate system. So this is a 3 dimensional plane. Now, this we are interested in finding out this displacement.

Now, to find this displacement what we have done is initially we relate this displacement with another quantity known as strain, and this strain components strain components are given or we know the strain components. Now, similar to that we also define stress components. So, now, important thing is that how this stress and strain is related will knowing this module.

Now, before that I want to emphasize this that after knowledge of strain, displacement and knowledge of stress we can formulate the governing equation or the equilibrium equation of the body. So, that body goes from undeform state to the deform state, and which is essentially the force balance equation. So, we can write its deform a, deform a differential equation which is of this form. We have not seen this earlier but just to mentioned this that this del dot sigma is plus B equals to 0.

Now, this is over the domain omega 0 and with proper boundary condition, with boundary condition. So, boundary condition could be a traction boundary condition, if you have. If you remember that boundary condition can be done in terms of some displacement or also known as deeply boundary and traction or the sigma in the traction boundary. So, this with this boundary condition and this is a static problem. So, there is no inertia term. So, the difference between static and dynamic problem is the inertia term. So, this is a static problem and B is the body force. So, this will discuss in the 5th chapter, or 5th module.

Now, you see we are interested in finding displacements, but our governing equations are in terms of stresses. So, we need to know how stress is related to the displacement. So, that means, we know strain how displacement is related with the strain and this equation is linear as I said earlier because linear means the relation this these are all linear terms. So, this note this is linear strain tensor are we know this linear strain tensor and now, how to relate displacement with the stress we need to relate stress with the strain probably.

So, probably you have heard these Hooke’s law or stress is proportional to strain the linear elastic case, where essentially few have probably this kind of spring force, if you apply and if there is a stiffness k. So, this force is proportional to delta the displacement and then F equals to k delta, right. This probably we have learned from your basic mechanics concepts.

Now, if you look back or if we generalized this think probably you have also learned that stress is proportional to strain in 2 dimension or 3 dimension. So, now, the essence of this discussion is that we need to relate stress and strain. Now, here if I define strain as above which is half of del u plus del u transpose. Now, this implies that this problem is a linear strain problem or small’s deformation problem or small displacement problem. This kind of problem is geometrically linear that means, the strain displacement relation is linear. So, we say this is geometrically linear problem.

Now, certainly when this strain and displacement relation is not non-linear in is not linear. So, it can be geometrically non-linear also for instance will define green Lagrange strain is something the later part of this course. So, geometrically non-linear problem this is strain in which the strain displacement relation is non-linear. So, this is one possibility. So, we are not discussing this kind of problem here where there is a geometrically, non-geometrically non-linearity present.

Now, when we start relating strain with the stress, so we need to assume or we need to know how material behave. So, essentially if I load steel beam modified load a, concrete beam modified load you wooden beam material will deform a in a different way. So, to identify or to know how material will be behave we need to know how stress and strain relates for this material. For instance we assume linear stress strain relation or Hooke’s law, where we see that we see that stress is proportional to strain.

So that means, when we say sigma is proportional to epsilon right, and the proportionality constant also you know for one dimensional case it is Young’s modulus for 3 dimensional case appropriate definition of Poisson’s ratios or the ratio between transverse strain and the longitudinal strain. So, these things we know from our basic knowledge of solid mechanics. Now, this relation when sigma is proportional to strain this relation is known as linear right, because it has a only constant coefficients. So, the proportionality constant becomes Young's modulus or in a multidimensional case it is known as the constitutive matrix.

So, essentially Voigt is the need for this constitutive relation is comes here. So, in our governing equation or what equation we want to solve that for a different material this constitutive equation will be different, right. So, the constant proportionality constant here, for instance if I write it in a just simple manner without knowledge of this tensors; so, this if I assume as a matrix or as assume as a vector and this assume as a vector. So, this becomes a matrix, so which is C ij, right.

So, this matrix is constant. So, there is no, it is not dependent on any other think. So, this is a constant matrix and this relation is known as the Hooke's law or the linear elastic material, the represents the material behavior of the structure or materialistic behavior of the structure is linear.

Now, certainly this is important where if you want to solve or find out the displacement because this is the connection of sigma with the epsilon or the sigma epsilon relation. So, we need to substitute these relation here and then solved for displacement. So, this is over all the process.

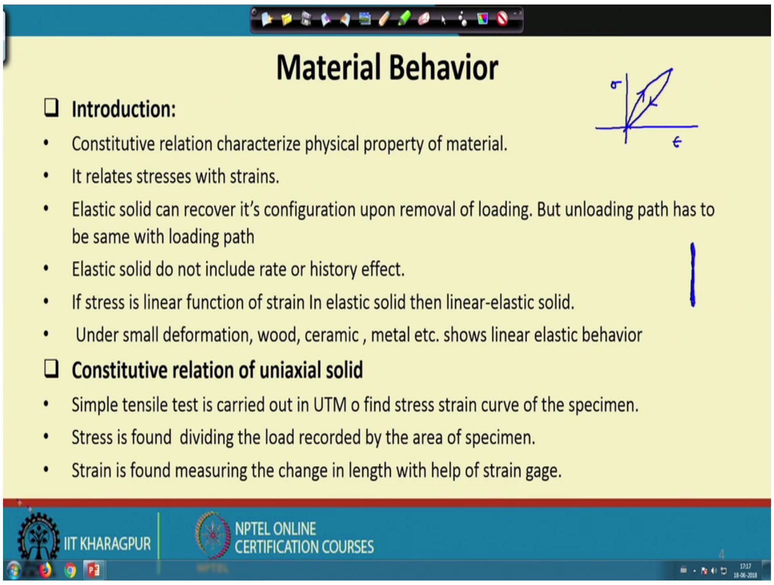
Now, another important thing is that this relation sigma is proportional to epsilon or stress is proportional to epsilon this we are assuming or we are discussing those material which follows this relation. So, this is certainly linear relation. So, that can be non-linear also that means, sigma may not be directly proportional to epsilon. So, there is a relation between sigma and epsilon which is non-linear in terms of strains. So, they are that kind of material or that kind of relation we will say this materially non-linear, so materially non-linear.

So, now, we learned 2 things one important thing is that geometrically linear problem and materially linear problem. So, we are discussing geometrically linear problem and materially linear problem. So, the non-linear when we separate it out from a linear problem nonlinearity can be of two type, geometrically linear problem or materially non-linear problem. Now, why do we need to know stress strain relation? Because we are interested in finding out u x, u y, u z are the displacements. So, this is the background on which we are starting.

So, in this module basically will study this relation or the stress strain relation for a linear elastic material. Now, the word linear elastic here is implied that it is geometrically linear problem as well as materially linear problem, right.

So now, if we if we proceed. So, naturally the stress strain relation will be different for a different material for instance if we are talking about steel, the stress strain relation will be different, if we are talking about concrete the stress strain relation will be different. So, here is essentially the material behavior comes into picture.

(Refer Slide Time: 16:37)



So, essentially constitutive relation characterizes the physical property of the material or how material behaves. So, essentially it relates stress versus strain.

So now, another important thing probably we have discussed in the first lecture of this or the introductory lecture of this course that elasticity. So, elastic solids can recover its configuration upon removal of loading, but unloading path has to be same with the loading path this also I have discussed with an example. That means, if loading and unloading path are different then material may not be elastic.

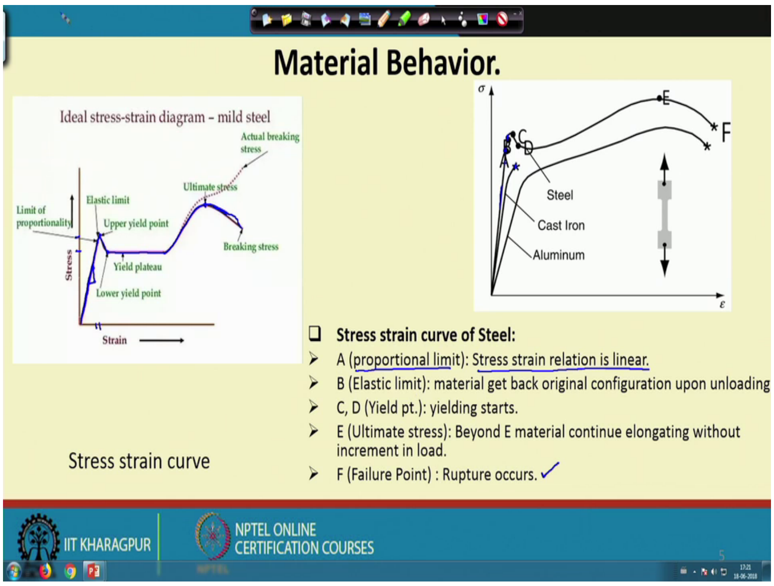
So, in case of this kind of material where loading and unloading path are different; so, it loads here and unloading path here, so, this is not an elastic material. So, this is a strain and this is stress. Now so, this kind of material we are not talking about. So, loading and unloading path has to be same, and this implied that the there is no change in the energy because if we know from our strength of material or solid mechanics concept that area under the stress strain curve is the strain energy, right. So, we will discuss this also in detail.

Now, elastic solids do not include rate or history effect. So, the path is important but the effect on this how this path is taken and what the memory effect or there is no memory effect on the elastic solid this is also well understood by us.

Now, is stress is linear function of strain, then it is linear elastic solid, similarly it is strain is linear function of displacement it is geometric linearity. So, under small deformation that means, when the deformation or the displacement of the body is small then wood ceramic metal etcetera shows some linear behavior.

Now, probably all of you have done some test on the uniaxial solids where we just carry out simple tension test. If you remember that one rod we test it in a laboratory, where we see cup and cone behavior of the rod and in a universal testing machine we obtain stress strain curve. So, essentially how stress is found out stress is found out by dividing the load recorded by the area of the specimen. So, this is essentially how we find out stress and strain is fine load by measuring the change in length. So, that is the basic definition and with the help of strain gages we found out find out strain.

(Refer Slide Time: 19:41)



Now, so this is this as a typically stress strain behavior or stress strain curve, for instance if I just see the mild steel mild steel having very distinctive material behavior. For instance this portion is essentially linear. So, now, from here this is a in a some nonlinearity of curve behavior. So, then it reaches some point and then suddenly it drops to another point.

Suddenly means that is small increasing strain from here from here to here, small increasing strain suddenly the stress decrement happens largely. Then it follows a straight path which is, then in then after that it again gain stress or it take stress and then it follows somewhere after it reaches the some maximum value and then also this is an ideal stress strain diagram.

So, a material may not follow this but this is a ideality. So, we will discuss what are these things. So, for instance different material behaves in a different way. For instance if we draw the steel it looks like this, if we do the cast iron it will be just after there is no distinctively point or something and then suddenly break if you press aluminum then it may looks like this. So, this is the specimen if we test it in the laboratory.

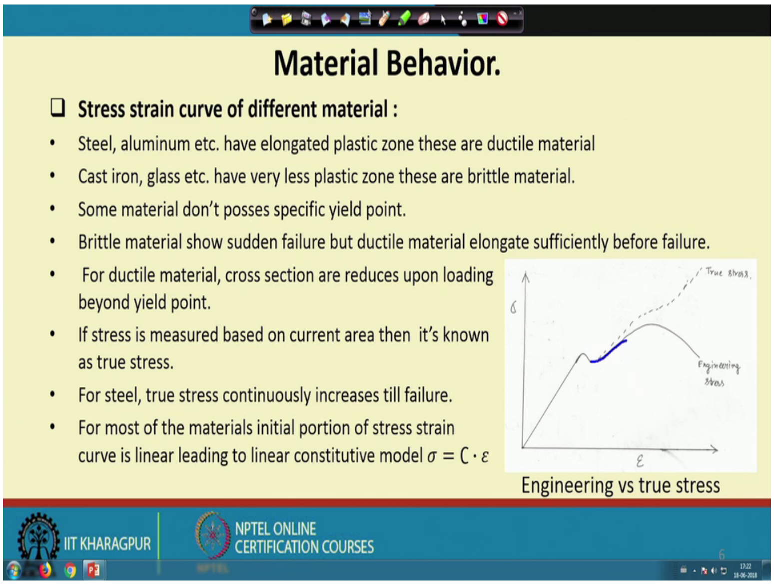
Now, so A is the proportional limit. So, here if we see that the proportional limit. What this proportional limit means? Proportional limit means that stress strain relation is linear. So that means, stress is proportional to strain and the slope of this also we know the Young’s modulus.

Now, B is the elastic limit. You see from a or specifically from point A to point B the elastic limit is point B so that means, from point A to point B, B material is not behaving linearly that means, the stress strain relation is not linear, so it maybe non-linear. Now, after this elastic limit that means, if we release at point B the material will if you release the load at point B. So, material will come back in the same paths. So, this elastic limit material will come back in the same path.

Now, C, D is yield point. So, for mild steel this is a ideal mild steel stress strain diagram, the point the top portion here is known as the yield point, the upper yield point. And point, then after certain increase of strain we observe that there is a sudden drop of stress. So, this is known as the lower yield point. And this is probably specific for the mild steel, and then we observe a yield plateau or where stress does not change much and then with the increase of strain.

So, after that it again take stress. So, it reaches to with the increase of strength, it increases to ultimate stress. Now, once it reaches to ultimate stress, then with the increase of stress making and all those things occurred and then it goes to failure. So, the failure or the rupture occurs. So, this is a overall behavior of a stress strain curve. Now, this probably we know from our laboratory experiment.

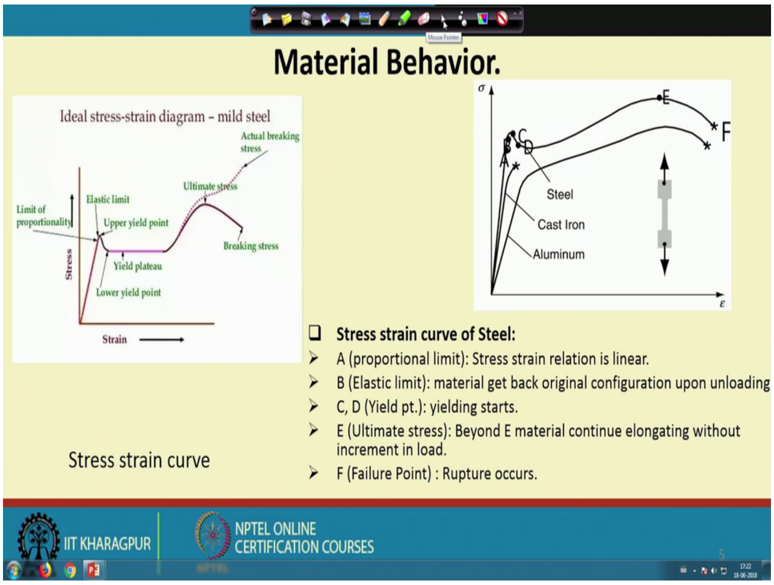
(Refer Slide Time: 23:48)



Now, so naturally steel and aluminum will behave in a different way. So, this yield plateau to or the some elongated plastic zone is known, if the material is having the elongated plastic zones so that means, if we just have this elongated plastic zones here.

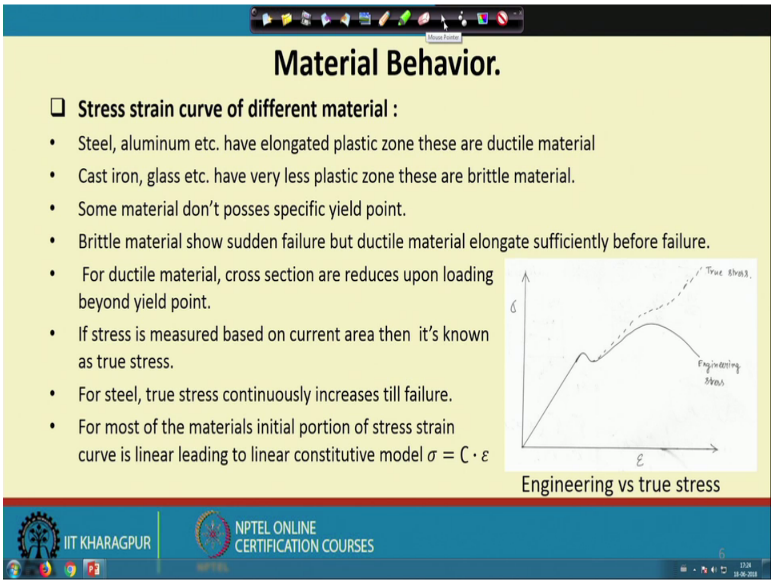
So, this is known as the yield, this the if the it shows a yield plastic zone a significant amount then this kind of material is known as ductile material. But on the other hand if you remember from the previous curve that we observe for a material which is essentially fails here for instance this is a cast iron. So, this material suddenly fails here. So, and this is known as this kind of material is known as brittle material.

(Refer Slide Time: 24:35)



For instances, the concrete if you draw the stress strain curve of concrete that will also look like this and so, this kind of material is known as the brittle material.

(Refer Slide Time: 25:05)

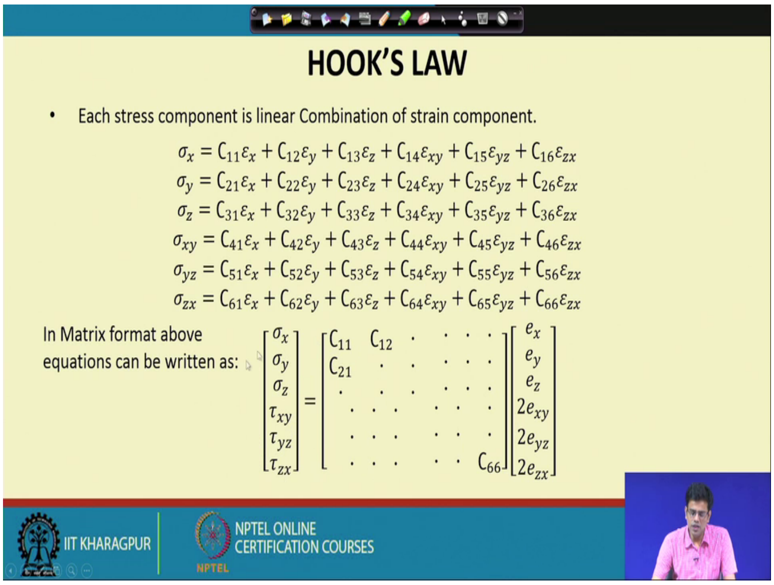


Now, glass also a material. So, there are very less plastic zone. And some material do not pause yield point, so that I have explained. So, that may not be, so in mild steel we observed distinct yield point. So, in case of a different other materials you may not observed yield point. So, brittle material shows sudden failure, but ductile material elongate sufficiently before failure this is well understood from our, if you remember we use concrete, so concrete is brittle material. So, it fail suddenly, so we that is why we do not prefer concrete in several structure.

So, for ductile material cross sections are reduced upon loading beyond this yield point. Now, there is also called something called true stress probably we know this is which is the stress we measured is the based on the current area. If we measure the load versus current area then we call it true stress. So, true stress will be if you plot the engineering stress and the true stress that curve will be different because from here actually the area is also changing. So, the amount of stress which is engineering stress which is always based on the undeformed area. So, is the area is changing. So, the quantity will be higher and the true stress will be this. So, engineering stress verses true strain curve.

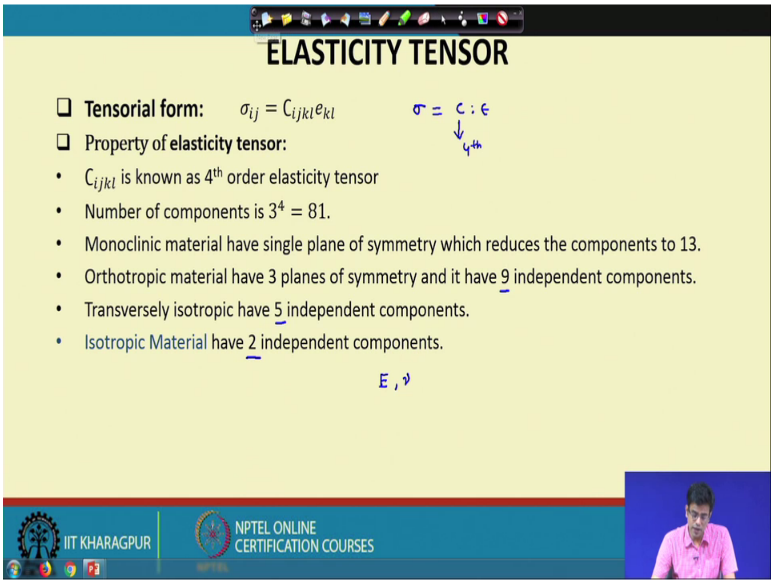
Now, for most of the material however, the initial stress strain curve though for concrete the initial stress strain curve is not very prominent. So, for most of the material initial portion of the stress strain curve say this for this material up to these. So, this stress strain curve is linear and the constitutive model for this is the Hooke's law which is sigma C epsilon. Now, this kind of material will be discussing here, mostly of linear these materials are called linear material.

(Refer Slide Time: 27:29)



Now, what will do here, will just learn something without proving anything or without knowing how it comes initially and then will prove each things separately. So, if we write the since we know that linear combination or the linear stress strain relation. So, naturally each stress will be related with the strain with a linear coefficient and which is C 11, C 12 and C up to C 66. So, in a matrix format this is the form. So, this relation is known as the Hooke's law for a general 3 dimensional body.

(Refer Slide Time: 28:11)



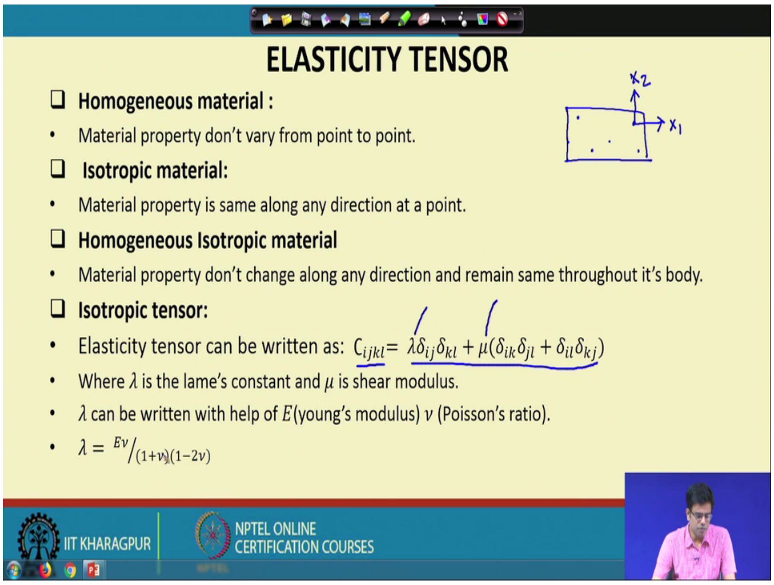
Now, what will do here is essentially ok. So, here in the tensorial form we can write it sigma i j or the indicial form we can write it in this manner and in a tensorial form we can write it in this forms C double colon epsilon. So, essentially C is a 4th order tensor, C is a 4th order tensor and sigma an epsilon is second order tensor. So, number of components in this 4th order tensor will obviously, 81 because 3 to the power 4 and.

So, as stress tensor it is 3 to the power 2, so 9 and from the symmetry we get the 6 independent stress components. So, based on the property of these coefficients or how these coefficients are arranged or based on the property of this matrix C, we can have different material. If the material poses single plane symmetry it is known as the monoclinic material. If at if it is having 3 plane of symmetry then we say it is a orthotropic material, and will see the 9 independent component.

Similarly for a transverse isotropic case, we will have 5 independent component, and for the isotropic case which we know from our solid mechanics or strength of material knowledge will have 2 independent components. And these two independent components are essentially Young's modulus or and Poisson's ratio or it can be shear modulus or first lame constant. So, these quantities essentially we know from our previous knowledge.

Now, what will do here is essentially will start with this and then will derive each of the material. For instance, will drive monoclinic, orthotropic, transverse isotropic material considering the different symmetry, but before that let us see how for an isotropic material how stress strain relates and what is the stress strain equation. Now, there will another important thing we have to remember here in material can be a homogenous material.

(Refer Slide Time: 30:39)



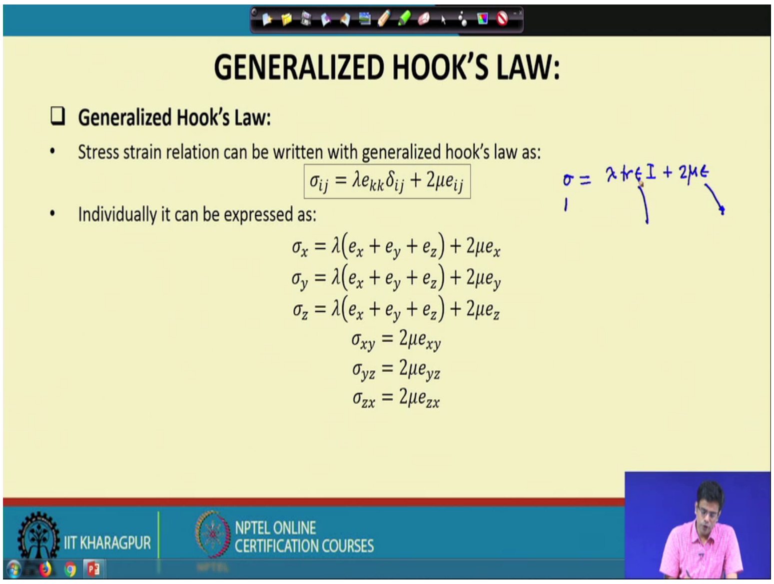
Probably this we have discussed in the first lecture that homogenous means, if there is a suppose there is a body. So, this body this body if I measure or if I find out the material property at this point, and if I find out at this point at this point if the material properties or the Young's modulus at all points are same then we see the material properties are homogenous.

Now,, but this is a 2 dimensional body. So, suppose at this point the axis is here. So, this is x 1 and x 2. So, if the material Young's modulus along x 1 and x 2 are same then we say it is a isotropic body or isotropic material. So, material properties are same along any direction. So, Poisson's ratio similar to Young's modulus, Poisson's ratio has to be same.

So, now, this is very important distinction between homogenous and isotropic material. So, it is very important to distinguish between heterogeneous heterogeneity. So, isotropic means, so material property in all direction will be same.

So now, if you remember the isotropic tensor or isotropic or the isotropic elasticity matrix, it is it looks like this. So, will derive this equation, but let us know this equation first. So, this is known as the first lame constant and then this is the second lame constant or the mu shear modulus. So, and how lambda is related to Young's modulus and Poisson's ratio this is the relation. So, we will learn this or will in the essentially prove this. So, the essence of this work this discussion is that isotopic modulus will assume this isotopic modulus will have 2 material constant which is E and nu Poisson's ratio, this is from our previous knowledge.

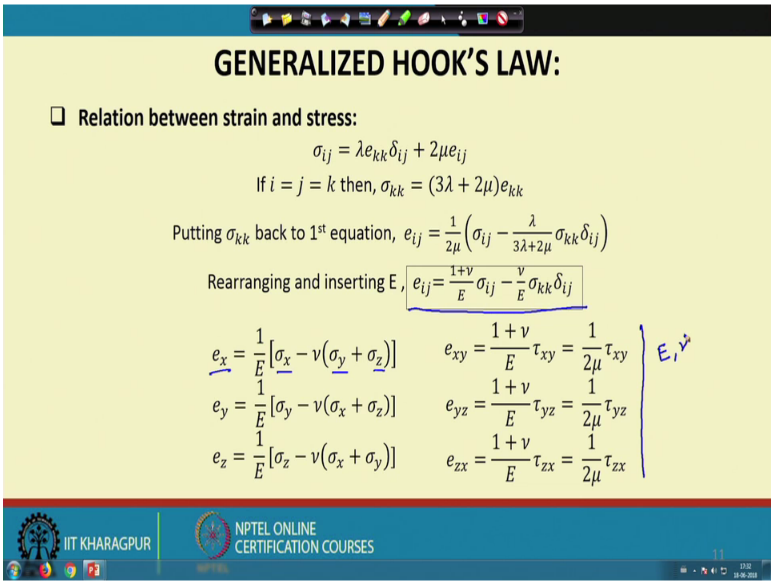
(Refer Slide Time: 32:46)



Now, if I just take this relation the generalized Hooke's law so which is sigma i j is lambda epsilon k; so, in a tensorial notation 2 mu epsilon E i j. So, if I write in tensor form or the in coordinate independent relation, so which is essentially lambda trace of E I plus 2 mu E or epsilon. So, this is a second order tensor, this is a second order tensor and this is a second order tensor. So, now, this relation if I write it in a Voigt notation which is sigma x, sigma y, sigma z so it looks like this. So, this is Hooke's law for a material.

Now, if we now, invert this relation and there is a reason why we are doing it here. So, if you invert this relation and will get another expression or which relate strain to stress.

(Refer Slide Time: 34:03)



So, if we invert after doing some calculation if we invert this relation. So, it looks like this. So, this is the essentially the strain how it relates with the stresses. So, if I write this in a form in a component form. So, if this will be sigma x, sigma y sigma z and so strain, so if I have stress then I can calculate the strength components.

So, the basic purpose of this in the next lecture will understand what this means. Actually what is the physical meaning of this Young's modulus, or the Poisson's ratio, or will define some quantities like bulk modulus. And these quantities what is the physical meaning, so material how it means to us. A material is given to you or, so you are as an engineer very interested in knowing what is its material property. So, that is our motivational studying this. And in the next lecture will essentially understand, what is the physical meaning of this isotropic material constants; E nu, right.

So, I stop here today. So, in the next class we will see the physical meaning of the isotropic elastic constants.

Thank you.