Theory of Elasticity
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Lecture - 16
Constitute Relation – 1 (Contd.)

Welcome, this is the last lecture for the module 3. So, here we are discussing anisotropic elasticity. So, in the previous two lecture we first discuss the how to find out the number of independent constant for a general anisotropic material. Then what we did is essentially we derive the Hook's Law in a Voigt notation from the general for the anisotropic material.

And then we found out that it is a 21 independent constants out of 81 only 21 is independent constant using major and minor symmetries. Then what we did is essentially we found out the strain energy functions for general anisotropic material and we point out that that the strain energy function is invariant under coordinate transformation, so since it is a scalar.

So, using this relation essentially we found out the monoclinic material constitutive equation of a monoclinic material. Also we have used the indicial notation for the transformation of the elastic constants to find out the transformed constants of the elastic components, elastic material constants for the monoclinic material. So, and also we have discussed the stress strain, comparing the stress strain also we can find out this elastic constant for the monoclinic material. Now, in this lecture we will basically find out the how to would come up with the orthotropic material constant, and what does this orthotrophy means.

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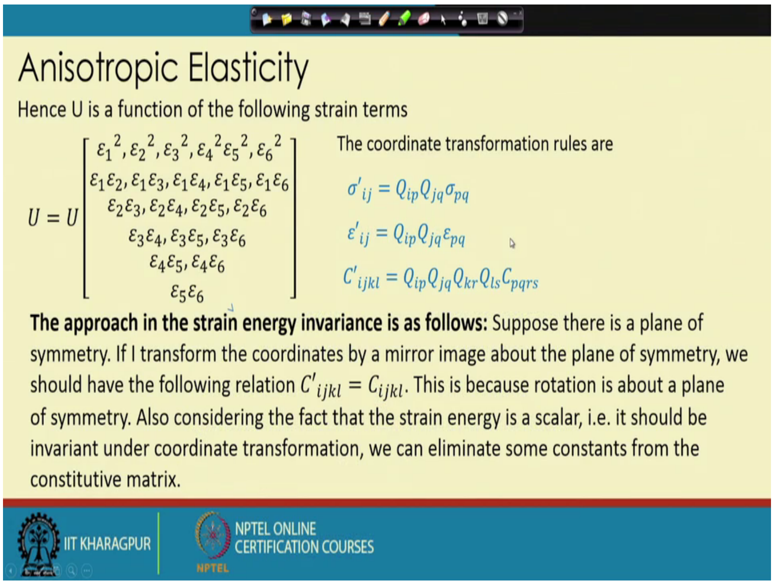

So, essentially we will quickly go through the process in the last class we have discuss. So, strain energy function is a this function, and this is the transformation coordinate transformation rules for the this thing.

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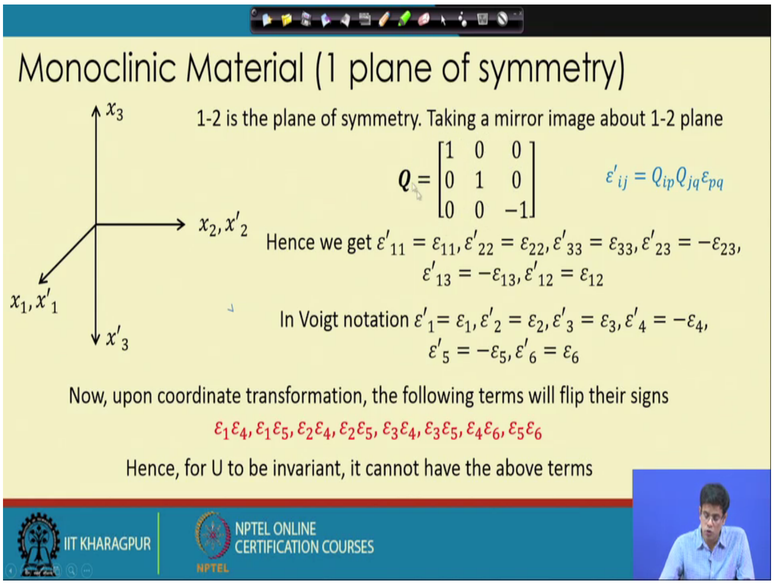

So, what we did essentially a for the monoclinic material, we used only x 1, x 2 plane or 12 plane is plane of symmetry. So, reflection about x 3 axis gives me the coordinate rotation matrix of this form, then using this relation we find out the strains which will change in sign.

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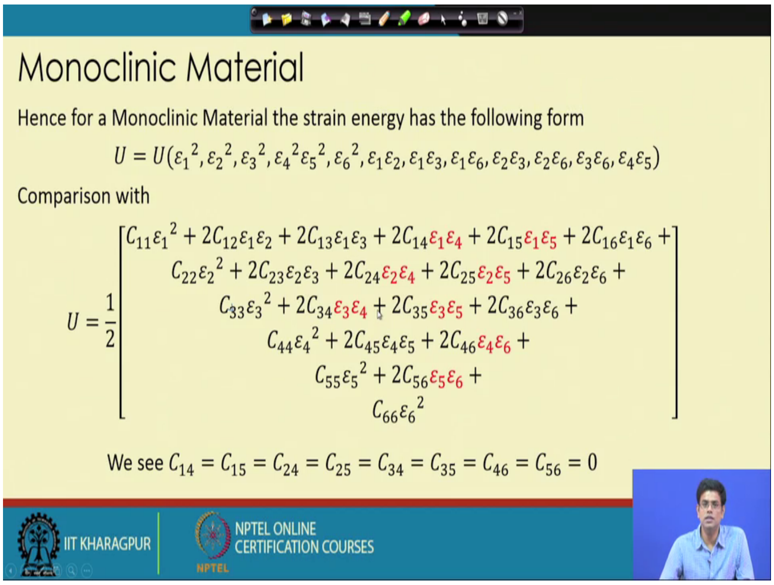

And then from that we from the general triclinic material anisotropic triclinic or anisotropic material we have observed that these quantities the rate quantities will flip the signs. So, which cannot be, which will change the strain energy and since strain energy cannot be change then this material constant has to be 0 so that it will not change.

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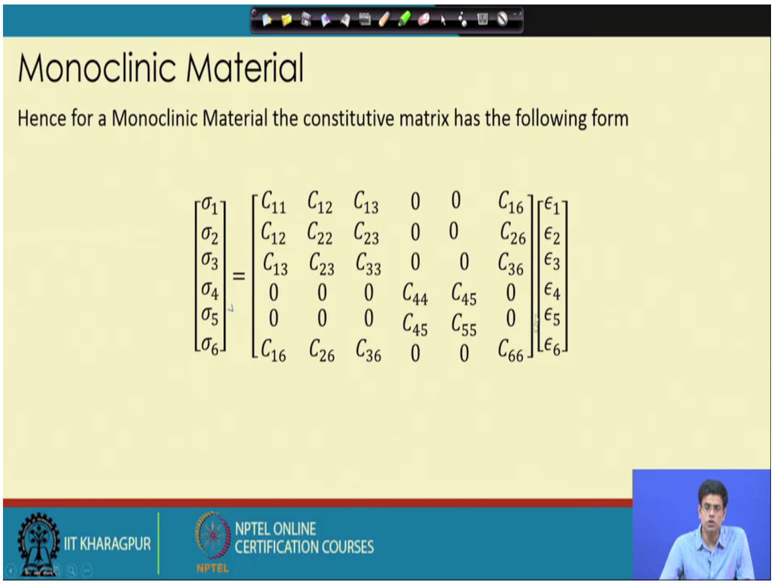

So, we find out finally, the constitutive matrix for a monoclinic material.

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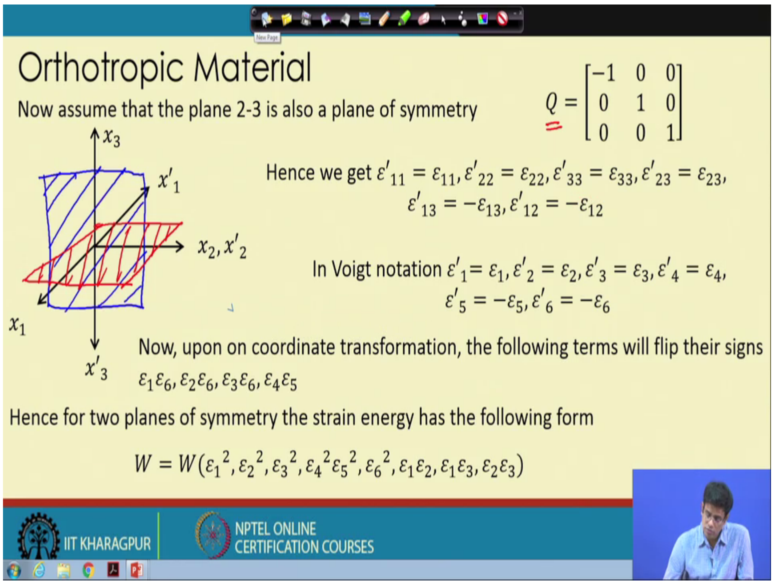

Now, here in the previous case we have assumed that this 1-2 plane is symmetric plane. So, in addition to that if we assume that 2-3 plane is also plane of symmetry that means, if I just draw another plane from here which is 2-3 plane. So, this plane is actually also the symmetric plane, in addition to our already the 3 1-2 plane symmetry is already there. So, if I now draw it in this form. So, this is my 1-2 plane symmetry for the monoclinic material and then again also we assume the 2-3 plane is symmetry.

Then my rotation matrix can be written in this form, then again with the similar concept we can find out the, what is the stress strain, what is the strains. So, essentially we can write this strain matrix which is essentially a epsilon dash is essentially epsilon 11, epsilon 12, which will be minus epsilon 13 which will be minus epsilon 23, epsilon 22, epsilon 12 minus minus epsilon 13, epsilon 23, and epsilon 33. So, this is my strain matrix.

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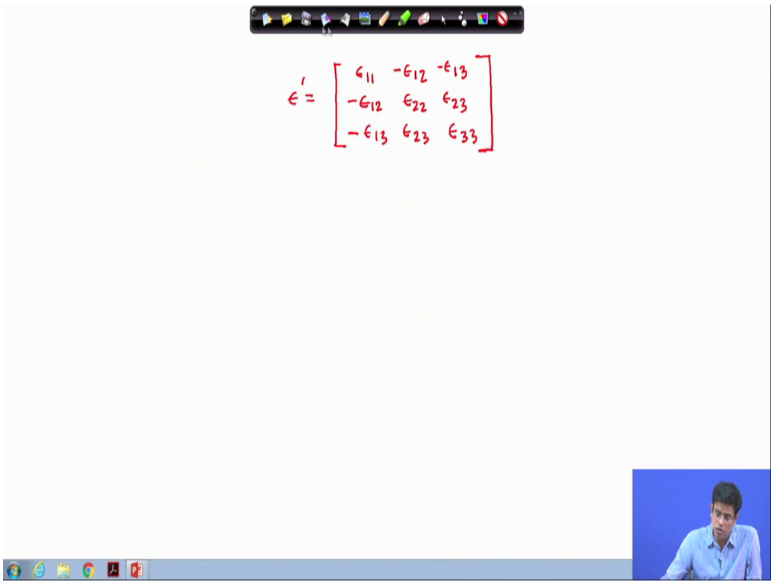

So, once this strain matrix is known to me then we can just simply convert it to the previous case, that is the these stain matrix. So, we will convert it the Voigt notation. So, epsilon dash is epsilon 1, epsilon 2 dash is epsilon 2, epsilon 3 dash is epsilon 3 and so on. So, now if you do this then we can see from here that epsilon 16, epsilon 26, epsilon 36, epsilon 45 these quantities will lead to 0 or coefficient associated to these quantities will be 0 essentially because these quantities will change the sign.

Now, this 2 understand this we have what we did is essentially we took the previous case the monoclinic material which is having one plane of symmetry that is x 1, x 2 plane is symmetry. Now then we are again saying that 2-3 plane is also symmetric. So, 2-3 plane is also plane of symmetry. So, reflection about x 1 is symmetry. So, that means, x 2, x 3 plane is a symmetry plane, it has it at the 2 plane of symmetry. So, previous case we have seen what are the components of the what are the components of the or what are the material constants which will be 0 in the previous case which is essentially C 14, C 15, and so on this components is 0. Now, on top of that we are this is due to the one plane of symmetry. Now, on top of that again we are assuming that 2 3 plane is also symmetry plane of symmetry.

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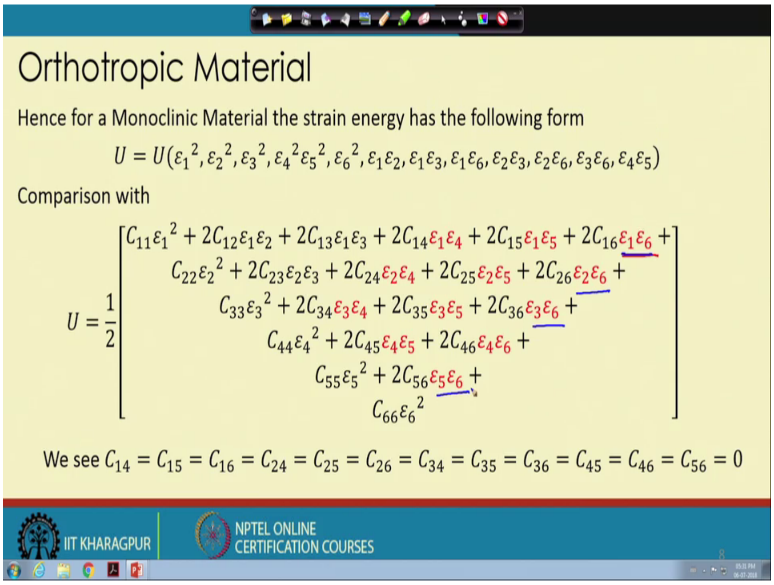

So, then finally, we considered the general anisotropic material for which these due to the 2-3 plane of symmetry these quantities will again come into or these quantities again will change the sign. So, epsilon 16, epsilon 24, these no epsilon 26, epsilon 36 these will change the sign. So, and epsilon 56 will also change the sign. So, these this again this was there for the previous case.

Now, this we again to do this again we have to again use the ingredients of the strain energy function so which will again leave it to me that C 14, C 15, C 16, C 24 and so on this will be 0. So, finally, if I use this condition then it will be my, this case my constitutive matrix will look like this.

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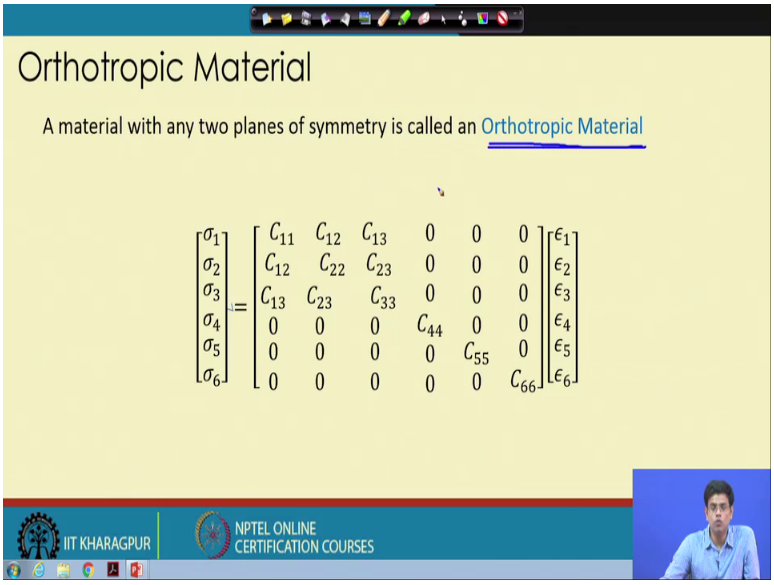

So, which will have; if you have remember. So, we had this only these quantities are there are in the monoclinic material now extra these quantities will be 0 due to the two plane of symmetry and then these quantities all was also there so this will be again 0. So, this gives me the full orthotropic material matrix or the. So, a material with any two planes or symmetry is orthotropic material. Actually if two mutually perpendicular planes are symmetric then third plane is also plane of symmetry.

So, that means, if we say that orthotropic material is mutually to mutually orthogonal plane of symmetries are orthotropic material 3 mutually plane of symmetries are orthotropic material. Now, these can be done through this the previous procedure where we have seen that this using this C ijkl formula we can also do that, this formula. We can also use this formula to do the, to find out the changes.

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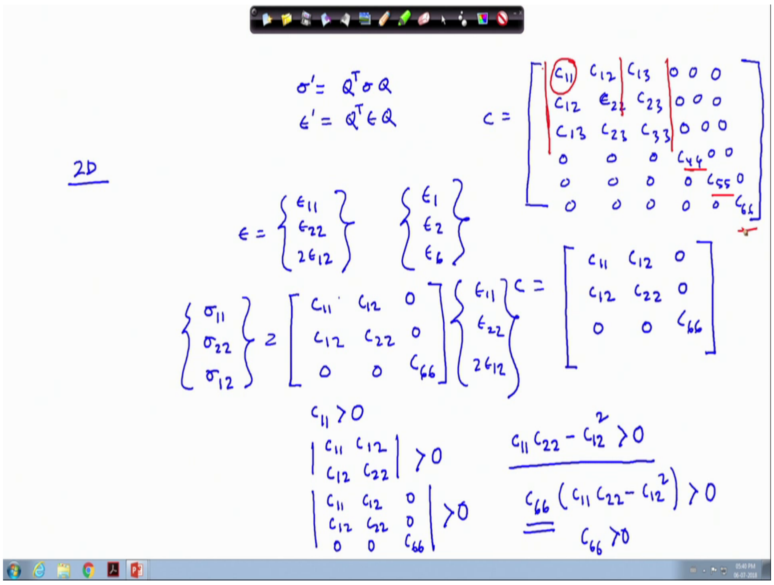

Now, also we can use the stress transformation formulas. For instance the in the last class we have discussed that sigma transpose equals to sigma Q transpose or Q sigma into Q and epsilon transpose equals to Q transpose epsilon Q. If you compare this stress strains then also we can come up with the orthotropic in case of an orthotropic material what will be the constitutive constants.

So, in case of a 3D orthotropic material what we have see a C matrix looks this or the constitutive matrix looks like this C 11, C 12, C 13, 0, 0, 0. So, C 12, C 22, C 23, 0, 0, 0; and C 13, C 23, C 33, 0, 0, 0 and then all 0s C 44, 0, 0; then all 0s 0, C 55, 0; 0, 0, 0, 0 and then C 66. Now, this is for 3D material.

Now, for a 2D material what will be the case? Now, for a 2D material naturally the strain components will be, essentially the strain components will be like this for 2D case for 2D case 2D orthotropic case the strain components will be epsilon 11, epsilon 22, and 2 epsilon 12. So, these gives me in a Voigt notation epsilon 1, epsilon 2, epsilon 6 right with the previous assumption because we have a assume that 12 is 1 and 12 is 6.

So, now my for a 2D constitutive matrix will come like this C 11, C 12, 0; C 12, C 22, 0; 0, 0, C 66. So, this is 2D constitutive matrix so, finally, the stress strain relation will be your sigma 11, sigma 22, sigma 12 will be C 11, C 12, 0, C 12, C 22, 0, 0, 0, C 66. Now, this will be your epsilon 11, epsilon 22, 2 epsilon 12.

Now, you see now there is another condition, that condition says that this constitutive matrix what should be the value of this material parameters. For instance can it be C 11, can it be any value C 12, can it be any value C 13, can it be any value. Actually what are the possible values of these constants are also very important because for instance all of you must have heard that Poisson's ratio if you remember the from the strength of material or solid mechanics concept that Poisson's ratio the range of the passions ratio is 0.5 to minus 0.1.

So, this why this range is given I think we will discuss it a later, but here what do you want to point out that these matrix the component of these matrices this C and for 2D case. For instance, this, this matrices what is the condition for which the component values what should be the values of these C ij's. So, that this matrix is representing a real anisotropic material. So, the basic condition is that that these matrix has to be the positive definite matrix.

So, what is positive definite matrix? So, the positive definite matrix means here is essentially all principle minors are determinant of all principle minors should be 0. So, for instance I will explain with for this 2D case. For instance this is the first minor C 11. So, C 11 has to be greater than 0, right.

So, now determinant of this C 11 second principle minor that is C 12 and this C 12, C 22 this has to be greater than 0, right. And then the third one that is the C 11, C 12, and 0; C 12, C 22, 0; and C 0, 0, C 66 this has to determinant of this has to be 0.

So, now see this gives us a condition on which the C 11 and C 22 and C 13 and it is inter relation how it is related, it is these condition gives us. For instance C 11 cannot be negative. So, it has to be greater than 0. For the second condition what we see that that C 11 into C 22 minus C 12 square has to be greater than 0. This imposes another restriction, right. So, this quantity has to be greater than 0.

Now, similarly this if you look carefully that C 66 into C 11, C 22 minus C 12 square this has to be greater than 0. So, this also this C 66 has to be greater than 0. So, this is how these for 3D case also we can represent its C 11 C; in that case we can also represent this all principle minors for instance if I write it this has to be greater than 0. Then again this condition, this condition will evolve and then this condition which is actually the determinant of 3 cross symmetrics this will be 0 then multiplied with this. So, all these quantities has to be 0.

So that means, the even though we know the material properties, the material properties of a constitutive tensor that means, constitutive we know how it looks like the constitutive matrix is, but we do it is we have to be very careful how this material parameters can be taken. So, this is for instance in this orthotropic case how we cannot take this arbitrary value of the material parameters. So, we cannot take C 44 with 0.

So, the basic premises of these is that the strain the constitutive matrix has to be the positive definite otherwise what will happen is that the strain energy will not be positive definite or strain energy or the strain energy cannot be greater than 0. So, this will restrict the constitutive matrix to be symmetry is coming from the major and minor symmetry, and positive definite coming from the existence of a strain energy and which is greater than 0. So, this, this actually restricts the component of the elasticity matrix.

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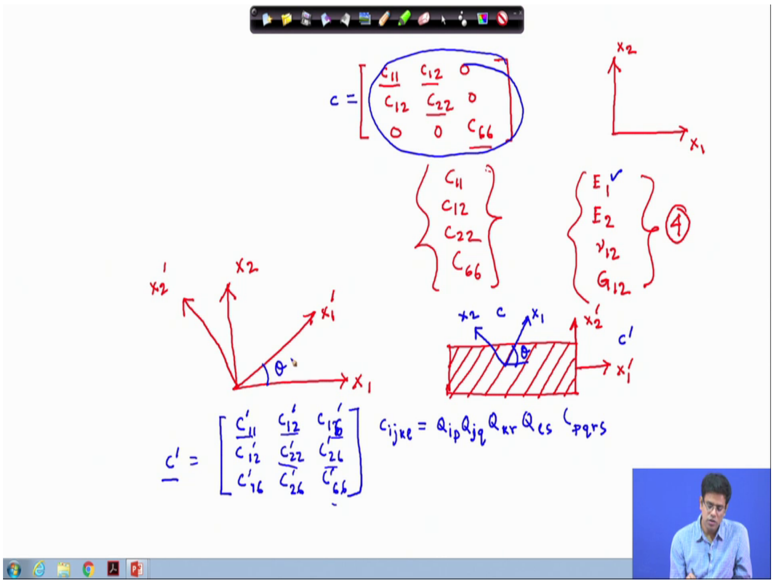

Now, again another thing I want to point out here is that, suppose we were discussing about 2D material where the 2D orthotropic material where will have C 11, C 12, 0 and C 12, C 22 and 0; 0, 0, C 66, right.

Now, if you remember first class we have discussed this is C 11 can be also represented in terms of engineering constants, for instance if it is a 2D orthotropic material. So, this is x 1 and x 2 component, x 2; x 1 and x 2 axis. So, C 11 is along the modulus along x 1 direction and C 12 is a 12 direction, and then C 22 into 2 direction.

So, now this C 66 is the shear direction shear part. Now, these C 11 can also be represented in terms of engineering constant. For instance that we will discuss separately how to represent it in the engineering constants, but basically what we have seen earlier is that I am repeating it here again that we have defined the anisotropic material or the orthotropic material here that along 1 direction it is E 1, along 2 direction it is E 2 along and the Poisson's ratio is nu 12 and G 12 is my shear modulus.

You see there are 4 independent, 4 independent material constant and from these matrix also we observed that this is a 4 independent material constant which are actually C 11, C 12, C 22, and C 66. So, these all 4 constants are equivalent to this constant. So, I this constants can be represented in terms of E 1, E 2, E 3, nu 12 and G 12. Again there is another case which we will also discuss probably in subsequent lectures that these material maybe in a orthotropic in a particular direction.

So, that is x 1, x 2 display in this direction it is orthotropic. Now, if it if I rotate this is called the material axis, but the body axis the a structural axis may not be the same as the material axis. For instance the structural axis could be in the x 2 dash x 1 dash direction x 1 dash direction. So, this is my body axis or the structural axis, but my material direction is like this. For instance, this case will appear in case of a woven composites where we will have the material axis is along this direction and the structural axis is along that different direction.

So, this is my structural axis x 2 x 1 dash and x 2 dash this is my structural axis, but my material axis is in this direction x 1, x 2. That means, when I have the material property the Young's modulus is along E 1 is this E 1 not in this direction. My body axis or the structural axis is along this direction, but my material axis is along this direction. So, when to when we solve a particular problem we have to solve it in the body axis or in x 2, x 1 dash or the prime direction. So, we have to convert this constitutive tensor to the again to the body axis.

So, again the rotation matrix comes and so the the formula that again C ijkl we can write it Q i p Q i, Q j Q and Q k r Q c s, C pqrs. So, this rotation this if you if you see this rotation is along theta. So, here Q will be different so cos theta sin theta minus sin theta cos theta one for 2D case. So, this has to be properly transform this rotation matrix has to be properly this constitutive matrix or the c matrix has to be transformed to the structural axis.

Now, due to this transformation it will look like. So, if this is the theta this angle is theta so, this C in the prime coordinate system may not look like the same form. So, C 1 11, C 12 dash, C 13 dash, C or sorry 16 dash, C 12 dash, C 22 dash, C 26 dash, and C 16 dash, C dash 26 and C dash 66. So, this is this prime coordinate system this is the this is the in x 1, x 2 coordinate system it is C but in prime coordinate systems it is C dash right. So, because you have to solve the problem in the structural axis.

Now, this does not represent orthotropic material right. So, and apparently it means it may looks like that these relventory represent the orthotropic material but it is actually orthotropic material it is coming from the rotation of these matrix, so these matrix. So, when we see it is an orthotropic material we are talking about the material axis, we are not talking about the structural axis. This distinction has to be remembered carefully because we may need to transform the material in the structural axis in that case the form of the orthotropic form may not be preserved.

So, these are the even though it looks in C that 1 2 3 4 5 6, there are 6 independent components in this form, but actually there are 5 independent components. The 5 independent components come from here that C 11, C 12, C 22, C 66 and the theta. So, the rotation as well as the material constant in the material axis.

So, essentially is rotated orthotrophy or special class of orthotrophy, so where we will essentially rotate the material axis to match with the structural axis. So, this is the one point, one has to remember when applying for a composite material. We will discuss this composite material in a detail form in the subsequent classes.

So, here I stop today in the next module. We will again discuss some parts of the anisotropic, and then basically will discuss transverse isotropy and then isotropy, and then some strain energy, form of strain energy and then will go for the composite analysis.

Thank you.