Theory of Elasticity
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Lecture - 17
Constitutive Relation – II

Welcome today will start the module 4 which is actually the constitutive relation to itself, so constitutive relation itself. So, in case of this module 4 we will be basically discussing an isotropic elasticity, and the constitutive behavior of composites.

So, in the last class we have derived the monoclinic and triclinic material what we have found is essentially from 81 elastic constants to 21 independent elastic constant for the triclinic material.

Now, for the monoclinic where one plane of reflection symmetry exist there we found it is 13 constant, 16 independent constant and we know how to derive those constitutive equation from our last lecture. Now, then what we did in where we have 3 planes of mutually orthogonal mutually perpendicular planes at this orthogonal planes of symmetry, reflection symmetry there we have derived the constitutive equation for the orthotropic material.

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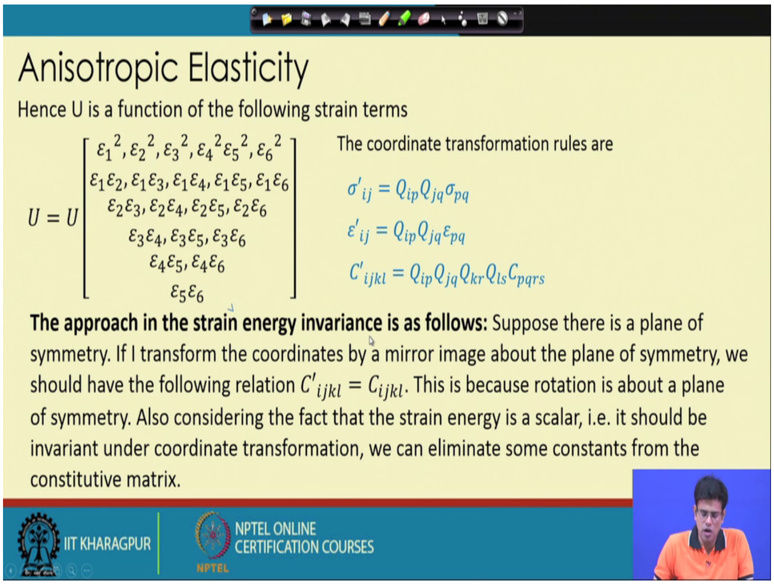

So, now here what we are doing essentially is we are using the invariance of the strain energy function. So, with the invariance of the strain energy function on the under reflection or rotation the strain energy function does not change. So, we use this thing to derive the specific constitutive matrix for the different class of material.

Now, anisotropic materials, now, this is from the last slide. So, the approach in the strain energy in variance is as follows. Suppose there is a plane of symmetry, if I transform the coordinates by mirror image and about in plane of symmetry we should have following relation that is as the strain transforms constitutive relation will also transform. So, there will be two different constitutive relation in two different coordinate axis, and this will be equal because rotation about a plane of symmetry and also considering the fact the strain energy is a scalar. So, it should be in variant under coordinate transformation.

So, by this we eliminated some constants from 21 independent constants to for the monoclinic material 13 independent constant and for the orthotropic material 9 independent constant. So, we will further reduced this number of independent constants from 9 to 5 for transverse isotropic material and finally, for the isotropic material it is 2.

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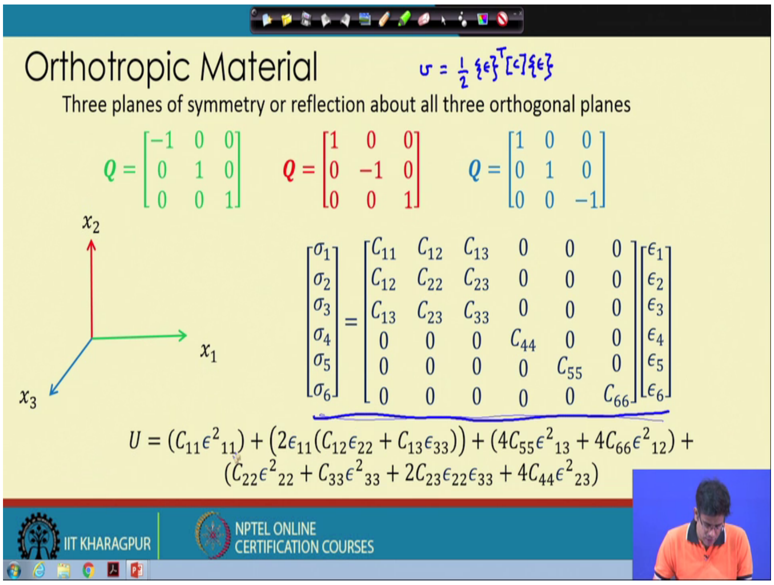

So, today we will see how this can be done in the framework that we have studied. Now, for a for an orthotropic material as I have told just now, is a 3 plane of symmetry or reflection about 3 orthogonal planes.

So, it is essentially x 1, x 2 plane, x 2, x 3 plane, and x 1, x 3 plane. So, now, there are 3 rotation and matrix we can derive. So, what we did in the last lecture is that we assumed the 2 plane of symmetry and then we derived it. So, we can be proved that if there is a two plane of reflection symmetry exist then third plane is also having that reflection symmetry. So, now, this is a constitutive relation that we have derived in the last class.

So, now, here the basic purpose is this giving this slide is basically when use this constitutive matrix in the subsequent development. Now, the strain energy density it is can also be written which we know from our previous module that strain energy density can be represented as half of u strain vector, it is a vector transpose C epsilon this form we I have seen in the Voigt notation. And then also in the general this thing that tensorial notations strain energy can also be represented.

And this is the a Voigt notation just in relation. So, now here we have used the strain tensorial component of the strain that is epsilon 11, epsilon 12 and etcetera. So, this is the strain energy function for the orthotropic material. Now, from here let us assume that material is orthotropic that is there is a reflection symmetry about 3 or the reflection about 3 orthogonal planes.

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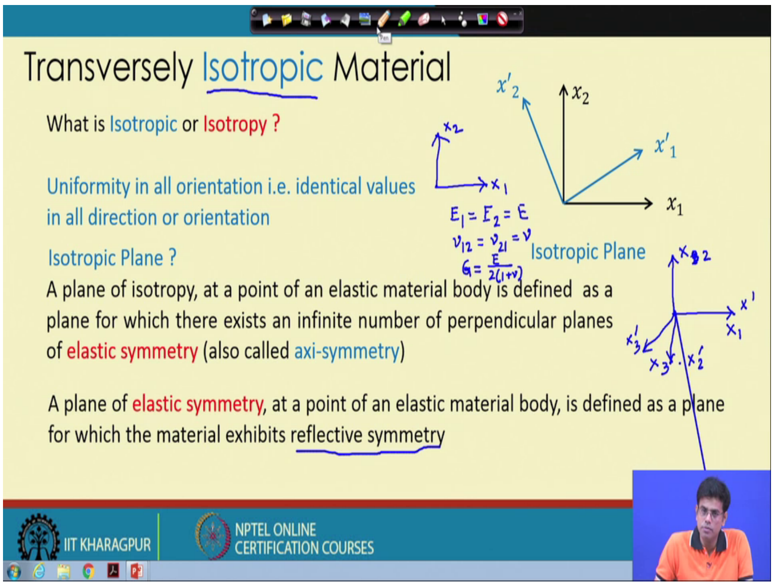

And then we reduce the basically the transverse isotropic material constants. So, first of all the word isotropic means what? The isotropy, so first we will discuss what is isotropic or isotropy essentially. So, isotropy means from our strength of material backgrounds we all know that is uniformity in all orientation and identical values in all direction or orientation whatever it is. So, we know that because isotropic case we have learned first which I have 2 material constant in terms of Young's modulus or Poisson, Young's modulus and Poisson's ratio or in terms of shear modulus and Young's modulus whatever or bulk modulus whatever you assume.

So, now, will also know that if there is a orthotropic plane, isotropic plane and then there is a x 1 and x 2 if that means, x 1 x 2 in this direction E 1 and E 2 will are same. So, we know this from our basic relation basic knowledge of isotropy. So, nu 12 is essentially nu 21 is essentially nu and shear modulus is they are not independent it is E by 2 into 1 plus nu. So, this we know from our basic strength of material knowledge.

So, now, and a plane of isotropy what is the plane of isotropy essentially. So, we understand from the definition what is plane of isotropy, but let us give a formal definition. So, a plane of isotropy at a point, a plane of isotropy at a point of a elastic material body is defined as a plane for which there exists a infinite number of perpendicular planes of elastic symmetry. So that means, if this I considered x 1, x 2 is a plane of symmetry or plane of isotropic then there is an infinite number of elastic symmetry exist around the x 3 axis or perpendicular axis. So, this is what plane of isotropic means.

So, now elastic symmetry at a point of an elastic material body is defined as a plane for which material exhibits reflective symmetry. Reflective symmetry means we know that what is reflective symmetry, if you remember for our monoclinic material where we just x 1, x 2 and x 3 what we did I am sorry x 3, x 2, and x 3. So, what we did basically we essentially transferred this to this x 2 minus, so x 2 dash. So, if this is x 1 and this is x 3 dash. So, basically x 1, x 2, x 1, x 3 plane is having the reflection symmetry. So, now, this is the this is known as the elastic symmetry. So, this is we know from our previous lecture.

Now, so what we know is essentially we know the isotropic means what does it mean for isotropic and what does it mean for an isotropic plane. Now, also we know that we have 3 reflection symmetry for or the reflection about the plane mutually orthogonal or mutually perpendicular plane for in orthotropic material. So, with this knowledge will proceed to derive the transverse isotropic material.

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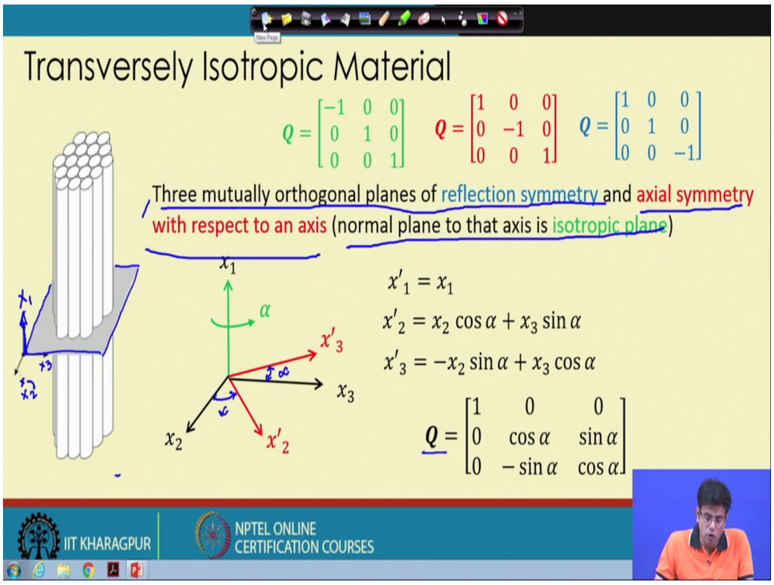

Now, what is transverse isotropic material the transverse, the word transverse means here actually here if you see this figure this figure tells you what is the transverse isotropic material. This is a fibered along this direction and these fibers is having a plane of symmetry about this hash line. So, about this hash domain this is a plane of symmetry. So, if you move along this, if you move this hash portion you will get the same properties. So, that is the idea.

So, now for an isotropic transversely isotropic material we are starting with the orthotropic material. So, orthotropic material we know that 3 reflection symmetry or the mutually orthogonal planes of having reflection symmetry. So, this condition is coming from the orthotropic condition and for which we know the constitutive matrix so and which is having 9 independent constants.

Now, in addition to that if I assume that there is an axial symmetry with respect to an axis. That means, if I about these axis about these axis there is an axial symmetry and normal plane to that axis is known as isotropic plane. So, that is clear from our definition, previous definition that about this axis there is a axial symmetry, and normal to this plane is an isotropic plane. Now, with this assumption what should be the constitutive relations.

So, first let us summarize what are the assumptions we are making here first. First we are making this 3 mutually orthogonal planes of reflection symmetry which is essentially the orthotropic material and the orthotropic material we know the constitutive equations second. Second is that axial symmetry with respect on axis. So, along this axis if say this is an x 1 axis, so about x 1 axis there is an axial symmetry so that means, that x 2 x 3 plane is essentially the x 2 x 3 plane is essentially the isotropic plane.

So, now, this with these 2 assumption if I rotate this body about x 1 with an alpha. So, if I rotate this body with an alpha, then I know what is a constitutive what is the coordinate transformation. So, the new coordinate system is x 1, x 2 prime or x 2 dash and x 3 dash. So, if I rotate this material about x 1 with an alpha, so x 1 will not change. So, x 1 prime will be x 1 and x 2 prime will be x 2 cos alpha x 3 sin alpha and x 3 prime will be minus x 2 sin alpha x 3 cos alpha which is which is coming from this rotation matrix Q. So, this we know from our basic knowledge. So, with this knowledge let us see what is the strain component we have.

So, now with this think let us convert the strains. So, strains converting the strains will know the strain formulas converting the strain formula. This is in a essentially tonsorial notation and this is the indicial notation.

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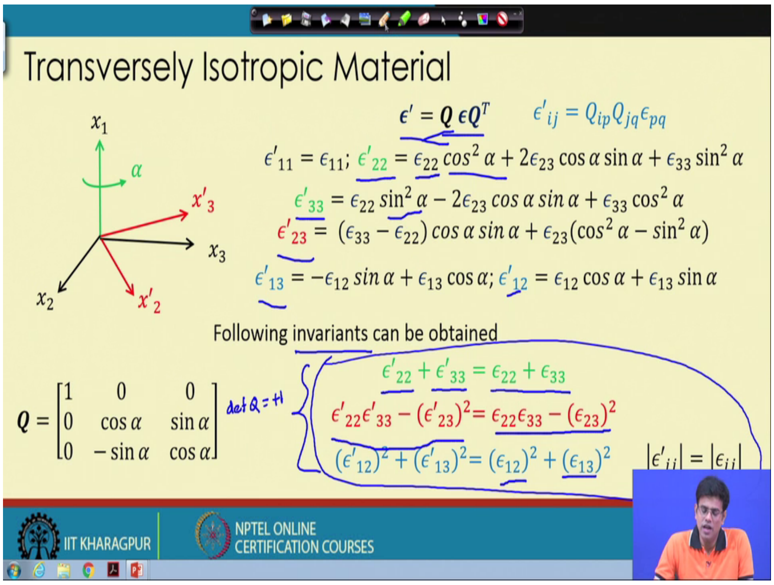

So, if you transform this. So, epsilon dash 11 is epsilon 11 and epsilon 22 is epsilon 22 cos 2 alpha and this cos and sin alpha are coming from the Q's. So, which was in the earlier in case of monoclinic and orthotropic material it was only 1; so, because there was a reflection symmetry. So, now here it is not a specific rotation, so any rotation is possible. So, alpha is essentially cos alpha and sin alpha. So, now, epsilon 2 prime, epsilon 3 prime, epsilon 23 prime in a all the components I have multiplied with this essentially Q is a matrix which is this. And epsilon we know epsilon is a second order tensor strain tensor which is having epsilon 11 and epsilon 22 and so on.

So, multiplying this I get this transformed strain components which is in the prime coordinate system and the prime coordinate system is x 1, x 2 prime, x 3 prime. Now, if you look carefully these 2 what is the relation of the strain components in though these two coordinate system. So, if you look carefully you will see that epsilon 22 prime and epsilon 33 prime will be summation of these 2 strain component will be equals to non prime component. So, this is an invariant within this about an axis rotation.

So, if you it can be derived very easily. If you some of epsilon prime 22 and epsilon prime 33 then these 2 quantities are essentially give me the sin squared alpha plus cos squared alpha is 1. So, it is giving epsilon 22 and then epsilon 23 components cancels and then epsilon 33 is again sin square alpha plus cos square alpha. Similarly epsilon 22 prime and epsilon 33 prime minus epsilon 23 prime square equals to non prime non prime coordinate system. So, this is also you will see that if you exercise this so, you will see this is a invariant quantity. Now, similarly epsilon 12 prime square and epsilon 13 prime square is epsilon 12 square and epsilon 13 square. So, this is also a invariant. So, this can be very easily seen.

So, now these quantities essentially does not change. So, what it means is essentially these quantities does not change if I rotate about an axis x 1 12 specific rotation alpha in rotation. So, now another quantity which is essentially does not change is the determinant of the strain tensor. Now, you know that determinant of Q is here plus 1 so this can be easily checked by this formula. So, determinant of epsilon prime is essentially equals to determinant of Q epsilon and Q transpose. So, the product up to matrixes we know the determinant is multiplicative.

So, determinant of Q epsilon is Q transpose is determinant Q, determinant epsilon, determinant Q transpose and Q transpose determinant is also plus 1. So, it gives the determinant of epsilon prime is equal to determinant of epsilon. So, this is also invariant quantity.

So, basic idea from this is that this does not change these quantities. So, if I given alpha rotation so, this does not change this these quantities does not change even if I rotate the axis. So, this is the basic idea. So, now, if these quantities does not change then my strain energy also should not change by rotation of this. So, naturally my strain energy will be function of these quantities. So, this is very obvious because strain energy is a scalar and it will be invariant so obviously, by this strain energy function will be based on this. So, will use these are will exploit this fact to divide the constitutive equation for the transverse isotropic material.

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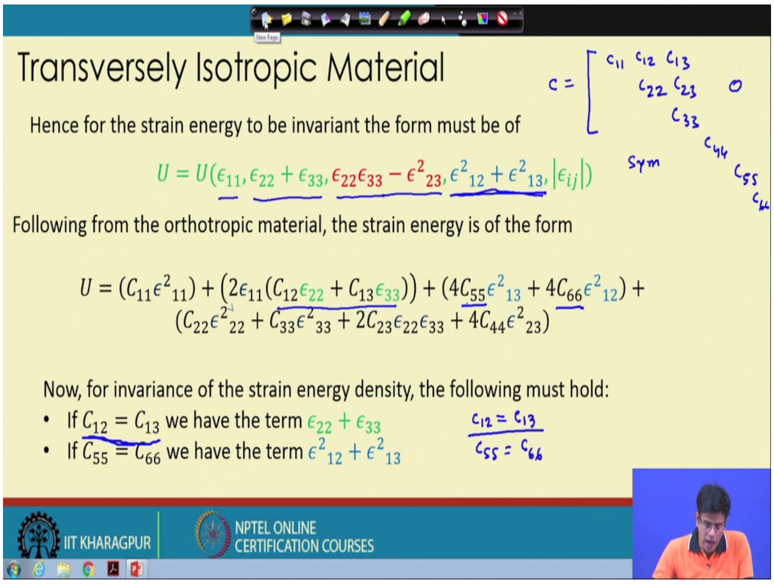

So, let us see. Now, so, hence strain energy function to be invariant of the form must be on this form.

So, now you see it will be epsilon 11, so clearly epsilon 11 is also does not change during this thing. So, it will be function of epsilon 11, epsilon 22 plus epsilon 33, epsilon 22, epsilon 33 minus epsilon 23 square and epsilon 12 square epsilon 13 square. So, now, following the orthotropic material, so we will start with the orthotropic material because in case of a transverse isotropic material the orthotropic assumption is valid. So, inspire a in addition to the orthotropic assumption there is a axial symmetry about an axis. So, the strain energy function is this.

Now, if you look carefully in this strain energy function. So, this I have manipulated a little bit to make the specific quantity in the specific bases. So, this quantity if you look carefully, so it has to be function of epsilon 22 plus epsilon 33. So, naturally C 11, C 12 and C 13 has to be equal unless this becomes equal. So, this cannot be epsilon 22 plus epsilon 33. So, first assumption is first thing is that that C 12 equals to C 13. So, this is one condition. So, first thing we have understood is that C 12 is C 13.

So, if you remember the constitutive equation of the orthotropic material that orthotropic incase an of orthotropic material that C 12 and C 13 are not same. So, it is different. So, if you remember that is constitutive matrix for an orthotropic material C 11, C 12, C 13, C 22, C 23 and C 33 and then C 44 C 55, C 55 and C 66. So, these are the 0s, and this is symmetry.

So, now see for an orthotropic material C 12 and C 13 was different. Now, if I assume this transverse isotropic condition then I have to have this C 12 equals to C 13. Now, similarly if the strain energy has to be function of this quantity epsilon 12 and epsilon 13 square then C 55 and C 66 has to be equal. So, this condition again gives me that C 55 equals to C 66. So, this to so we have reduced it now, from orthotropic we have a 9 constant. So, we have reduced it here 1. So, and we have reduced it here 1, so we are now, in the 7 constant. So, now, let us see how we can reduced it further. So, let us go to next slide.

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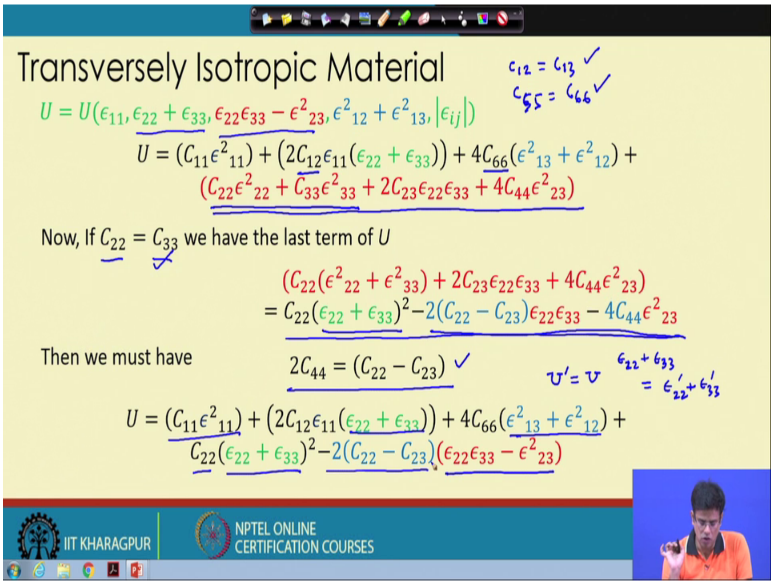

So, now this is this was my condition. So, now, here what I have done is I have incorporated the these to previous condition C 12 equals to C 13 and C 55 equals to C

66. So, it is in terms of C 66 and C 12.so, there is no C 13 and C 55. Now, if I in addition to, so this was the last term in the bases. So, we have finished with this term. So, these terms are as per the invariant quantities. So, now, but these term we have to make as per the invariant quantities.

So, now if we assume C 22 and C 33 again then what will happen? It will be a function of epsilon 22 and epsilon 33 square. So, here, so, if I assume this C 22 equals to C 33 then it will be function of epsilon 22 square and epsilon 33 square. So, which I can again, if I manipulate this with this assume assumption, so I can write this last quantity in terms of this, right. So, which is essentially a plus minus of some quantity which you can try simply you can pen and paper you can try it. So, once we have done with this then we can see that it is a function of epsilon 22 and epsilon 33 which is as per our assumption that is invariant quantities epsilon 22 and epsilon 33, and then again these quantity is left.

So, now you see it comes in the form of this, but we need to change the material constant. So, now, if this has to be a form of this then 2 C 22 these quantity C 22 minus C 23 has to be to C 44. So, now, this gives me another condition which is essentially C 44 equals to C 22 minus C 33. Now, here so now finally, what we got C 12 to equals to C 13 C 55 equals to 66 C 22 equals to C 33 and then again this condition.

Now, finally, with these thing if I plot, if I write the strain energy function it looks this. So, this is a epsilon 11 square epsilon 22, epsilon 33 and epsilon 13, epsilon 22, epsilon 22 plus epsilon 33 and this last invariant quantity. Now, see that means, if I write the strain energy in the transform coordinate system U prime so, U prime will be U why? Because these quantities does not change as we know that epsilon 11 epsilon 22 plus epsilon 33 is essentially epsilon 22 prime plus epsilon 33 prime. So, we know this, so similarly for these quantities.

So, essentially with this rotation plus orthotropic assumption orthotropic symmetry strain energy on change so that means, this is by these constants these C 11, C 12, C 66, C 22 and these are essentially my material constants. So, if I now, write it in a formal matrix form how it looks let us see. So, it looks like this.

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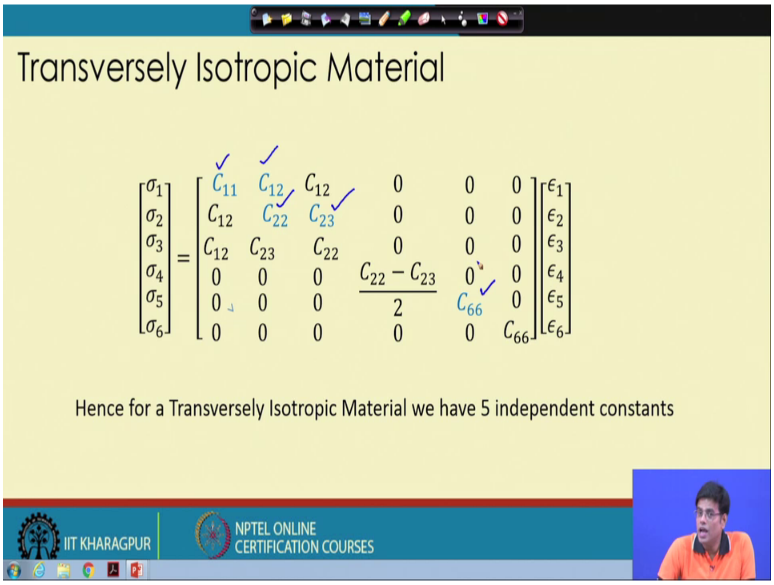

So, if you see carefully these C 11, C 11 is essentially this C 12, C 22, C 23 and C 66 is my independent constant. So, essentially isotropic material transverse isotropic material is 5 independent constant. So, this is how we represent any transverse isotropic linear obviously, linearly elastic transverse isotropic material which follows a Hooke's law.

So, now, if I you have derived for transverse isotropic then we can simply derive for isotropic very easily. So, it is the in case of a transverse isotropy we have seen that it is for the one plane of isotropy.

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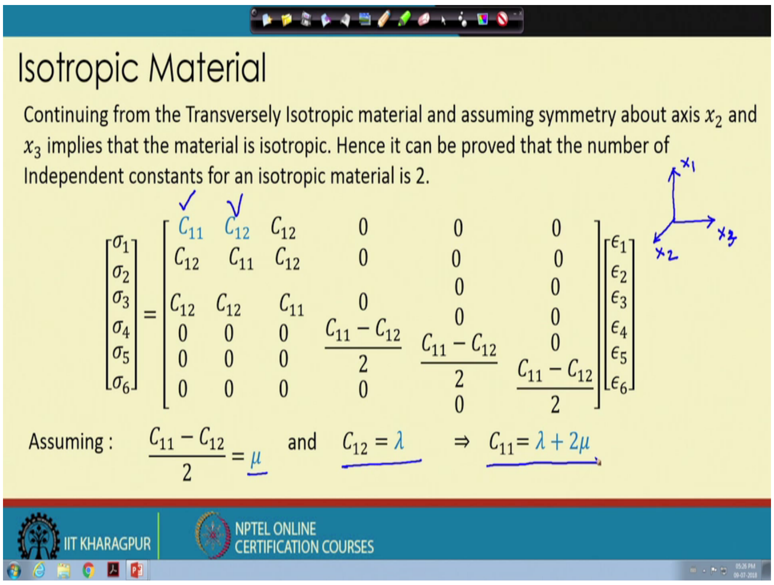

Now, if I assume instead of x 1 symmetry about the x 1, so in case of a transverse isotropy we have seen that we have assume that x 1 is my the about an axis for which that means, that x 2, x 3 and x 2 plane is my isotropic plane. So, now, similarly in addition to that if I assume about x 2, that is x 3 x 1 is also an isotropic plane and x 3 about an x 3 that means x 1 x 2 is also an isotropic plane. So, if I assume 3 mutually perpendicular isotropic plane, then it can be proved that isotropic material constant is having the only independent number is 2. So, because if you just do this then it is very simple exercise that all these quantities will be this and the this quantities will change it 2. So, finally, isotropic material will have C 11 and C 12.

Now, if we assume this C 11 and C 12 by 2 is mu and mu is what will define what is mu, and then C 12 is lambda, then C 11 is lambda plus 2 mu and this lambda lambda mu are called lame coefficient. So, mu is essentially some of you must have known this earlier that is mu is essentially the shear modulus which is E by 2 into 1 plus mu, 1 plus mu the Poisson's ratio. So, essentially we can write that isotropic constituted 3D isotopic constitutive relations and in the tensorial form also we can write it in this form that sigma ij, lambda ij, delta ij, epsilon kk and 2 mu epsilon ij.

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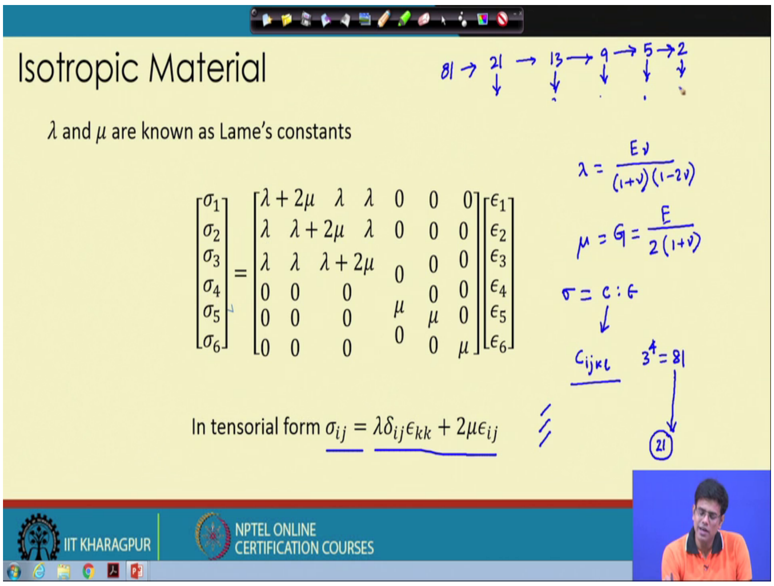

So, lambda is essentially known as the first lame constant. So, lambda is also some of you probably know which is E nu 1 plus nu into 1 minus 2 nu. So, and then mu is essentially the shear modulus G essentially E by 2 into 1 plus nu. So, these 2 constants we know.

So, now, what we have in a closing remark what we have done here. We have done here we have started with general isotropic material that means, the isotropic parameters which is essentially general isotropic material for which the constitutive matrix is a we know that C colon epsilon or the inner product with epsilon, where C is essentially 4th order tensor ijkl.

So, now to summarize it C ijkl will have 3 to the power 4 that is 81 constants. Now, from these constants what you have done we have we assume major symmetry and minor symmetry with these two and the existence of the strain energy function where we have if you remember where we have explored the partials that is del squared u by del epsilon ij del epsilon kl is essentially equals to del squared u by del epsilon kl and del epsilon ij. So, with these 3 conditions from 81 constants to we reduced it 21 constants 21 independent constants.

So, now, from these 21 constants is for a general an isotropy which is known as allotropic material or the triclinic material. So, from this triclinic material again we assume some symmetry condition. First what we have done? We have done for a symmetry reflection symmetry about the plane. So, first is known as the reflection symmetry of an of a plane, so which is known as the monoclinic material, so which is having the 13 independent constant. So, from 21 to 21 is for triclinic then you derived it from 13 to triclinic material, and then instead of a one symmetric plane if we assume 3 mutually perpendicular symmetric plane then we got the constitutive matrix form orthotropic material which is having the 9 independent constant.

Then if we assume from orthotropic material with that means, there is a 3 plane of symmetry are the 3 symmetry plane about 3 mutually perpendicular axis. And then from there additionally we assume that there is a axial symmetry, and the there is a plane of isotropy normal to the taxes.

So, if we assume that then we derive we got the transverse isotropic material for which there is a 5 independent constant and this is known as the transverse isotropic material. Then from the transverse isotropic material again if we assume that there is a axial symmetry about all 3 perpendicular planes and there is a reflection symmetry 2. So, then from there we achieve the isotropic material where the material constant is 2.

So, finally, if you look carefully that from 81 to 21, so 81 is not independent constant, 21 is independent constant. So, the process is 81 to 21, 21 to 13, and 9, and 5, and then 2, so, essentially the triclinic material, monoclinic material, orthotropic material, and then transverse isotropic material and then finally, the isotropic material. So, this was the most general way of; and also there are some other material for instance cubic material which is having 3 independent constant. So, this can be also done very easily. So, I left this thing on you. So, here today will stop here. So, we will follow in the next class.

Thank you.