Theory of Elasticity
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Lecture -18
Constitute Relation – 2 (Contd.)

Welcome this is a second lecture module 4. So, we are go, this is the lecture number 18. So, now, here what I planed is to find out a transformation matrix for elasticity matrix or the constitutive matrix. So, this is a very special case before I start this.

So, in general 3D rotation are not editing it is it is non commutative. So, essentially what it means that means, if you rotated a coordinate system first this and then this and then the again this. So, it is not commutative. So, 3D rotations we are not discussing here. So, in general what we are discussing here it is rotation about an axis or rotation of plane, so, rotation about an axis.

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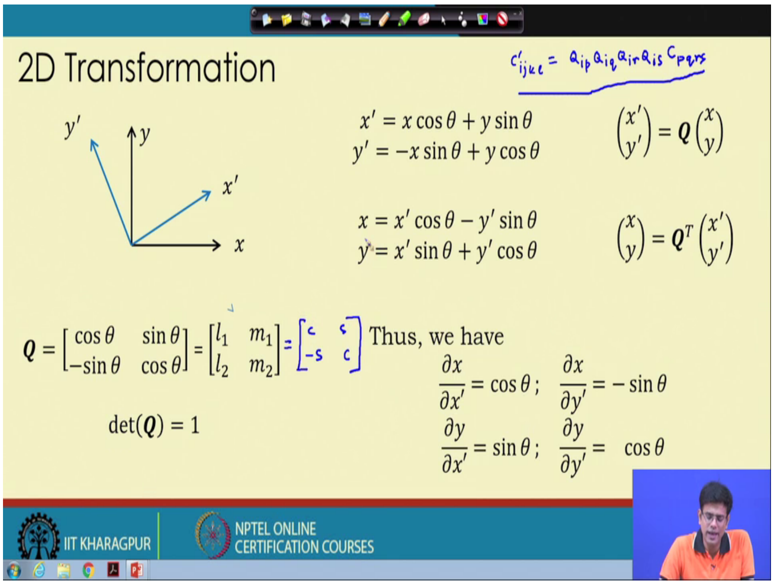

So, for instance this is a x and y. So, it may be important to point out here that rotation about an axis means that the rotation matrix or it is it that is 3 dimensional rotation can be rotation about the an axis or rotation of a plane. So, for instance we have seen in the last modules or last class itself that rotation matrix if I rotate a about an axis the transformation matrix is 3 cross 3 constitutive matrix.

So, here a first what is our objective today is to first to derive the stress strain transformation which all of you know from your basic theory of, basic strength of material or the solid mechanic course. But we will just briefly go through we can finally, our objective is since we transform stress which transform strain so what will be the transformation for the constitutive matrix.

Now, here we also know that the constitutive matrix is a 4th order tensor and this 4th order tensor how it transforms through the ijkl, if you remember this formula which is Q ip, Q iq and Q ir and Q is and C pqrs. But this is a, this formula it is impossible to write it in a matrix notation. So, sometimes it is very handy and we will be having several engineering application of that what we will study today. So, essentially if I rotate about an axis what will be the C ijkl or in the prime coordinate system. So, that is our objective.

So, now we can essentially transferred this quantities and equate with that the procedure will derive today. So, and then we have to find out the sequence of rotation and then we have to do this for all components of C ijkl instead of that if I have a simple rotation about an axis then what should be the my transform coordinate matrix that is my objective here.

So, let us start with 2D transformation. So, we know 2D transformation from class 2 level or class 10th level itself. So, 2D transformation is, if I rotate this is y axis to x prime y prime we can write x prime equals to x cos theta y sin theta and y prime is minus y x cos theta y sin theta. So, essentially Q is cos theta sin theta minus sin theta cos theta and this is this can be sometimes represented as l 1 m 1, l 2 m 2. So, there are various representations C s also a sometimes it is written in this form also in the book or any literature you will see this. So, C s minus s C in this form also it is possible to write.

So, here we have denoting l 1 m 1 l 2 m 2. So, now, here naturally this invert transformation is also possible which is x y equals to x prime cos theta. So, which is inverse transformation is Q transpose because Q is an orthogonal matrix. So, Q transpose is Q inverse and det of Q is plus 1. So, this is a rotation matrix. So, we know the properties of the rotation matrix.

So, now if we take the derivative, so partial with respect to the axis; so del x by del x prime. So, which is essentially cos theta minus sin theta sin theta and cos theta. So, this will know this will use it. So, now let us see if we; so essentially for what we have learned here we can rotate the coordinate system and we know the coordinate system.

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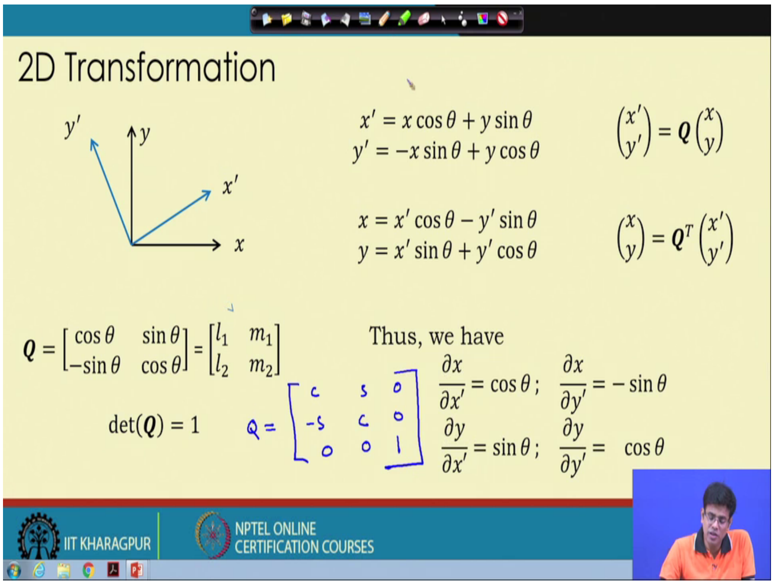

And one important thing maybe I should point to here that Q can be written in 3D format also so which is essentially C s minus s C and then 0 0 0 0 1. So, it represent a rotation about a z axis; so our rotation of x y plane. So, essentially it is an one parameter parameterization of the rotation matrix.

So, it is better to know here that 3D rotation cannot be parameterized by one parameter there are several ways to do it for instance quaternions is there, Euler angles is there. Those who study rigid body dynamics they may know it. So, we are not talking about those rotations, we are rotations our rotations at about an axis or rotation of the plane. So, that is our basic thing.

But this formula's with slight modification can be applied to for general notation also. So, that obviously, we will now discuss in this course and we do not have the applications for this course.

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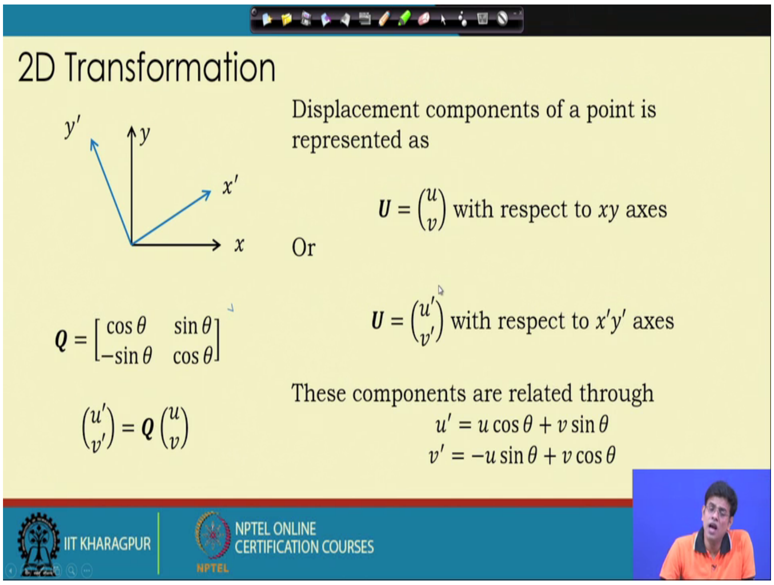

So, let us see how similar to the this thing to coordinate transformation we can transform a vector also which is a displacement vector.

So, u prime and v prime is related as u cos theta v sin theta and minus u sin theta v cos theta. So, in x y you may axis it will look like u v vector and u prime v prime. So, in a 3D case also it is same, in the 3D case it will be u v w. So, now, our main objective is to derived strains, so which we know from our strength of mutual concept and that is why I started with the 2D problem. So, let us see how we can transform the strains.

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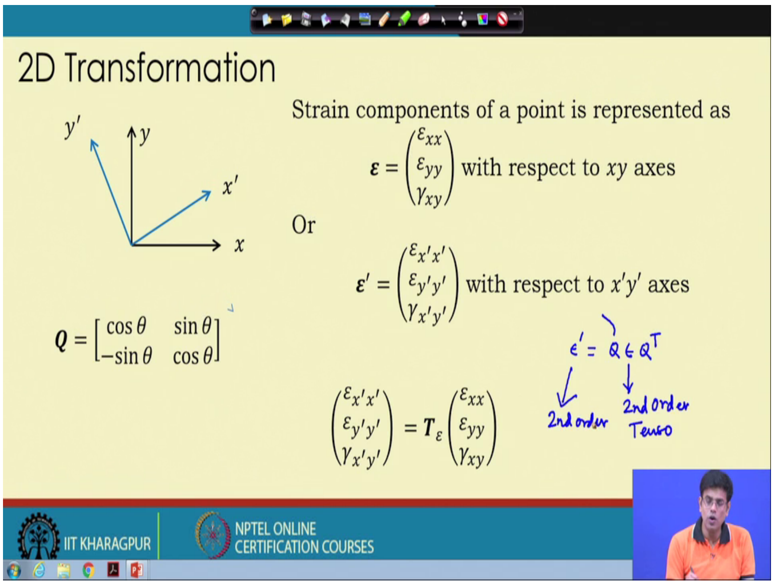

So, now essentially strains are here, it should be noticed here that I am using here the gamma x y in that shear strain. So, which is essentially 2 of epsilon x i, x y. So, we keep it in this format. So, here we will have the some benefit or doing that. Now Q is this so essentially strains will be transferred here. So, now, in a tonsorial notation we know this how to transfer strains.

So, if I have a rotation matrix then I know from our tonsorial this thing in the previous class itself we know that epsilon prime is essentially Q epsilon, Q transpose. So, here this epsilons are second order tensor, so second order tensor it should be remembered here. So, and this is orthogonal matrix. So, Q is similar to this or a also a second order tensor. So, it is a matrix format. So, this gives me a transform second order tensor which is a this things. But, we wanted to have this in a vector format or specifically this can be modified in Voigt notation also. So, the Voigt notation also we know this is a epsilon xx is essentially your epsilon 1 epsilon, yy is a epsilon 2 and so on. So, we know this so from what earlier lectures.

So, now how to essentially, how to find out this T epsilon or the transformation matrix for the strains so that is my main objective. So, T epsilon you will see that this is a matrix we are not talking about tensors here anything. So, you are talking about a matrix. So, essentially this will be 3 cross 3 matrix because this is a 3 cross 1 vector and this is the 3 cross 1 vector. So, this T will dimension of T epsilon will be a 3 cross symmetrics, and which we can arrive from our strength of material knowledge itself. So, let us see how we can arrive this.

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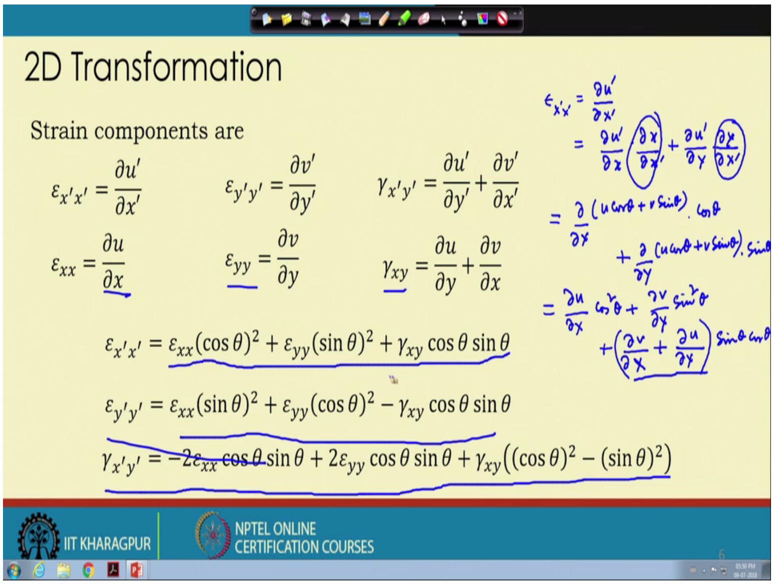

So, essentially strain components are del in prime coordinate system it is del u prime del x prime and so on, and non prime coordinate system del u by del x and so on. So, if write what is the relation between these two; so, epsilon x prime x prime is essentially this. This probably you have this is very familiar expression those who have studied more circum arctic sent of material according transformation or strain transformation.

So, you know this formula, and how to derive this formula you derived essentially epsilon x prime x prime is essentially del u prime del x prime. So, this item write it simply del u prime by del x into del x by del x prime and then del u prime by del x del y into del y del del y by del x prime. So, you know this from the coordinate transformation that we have seen.

So, and then del u prime also you know because if we substitute del del x of u prime or (Refer Time: 12:06) what we know is u cos theta plus u sin theta v sin theta and then del x by del x prime is essentially your cos theta. And then similarly for del u prime again you substitute this del del y of u prime u cos theta plus v sin theta into del y y del x prime which is sin theta. So, like this we can arise this cos square u cos square theta. So, del u by del x will be first term cos squared theta. And then del from here it will come to the del y del v del y by sin squared theta, where it will come here this quantity. And then, cos theta sin theta that is del v by del x plus del u by del y cos theta sin theta. So, this will be two times of this. So, this will be if I write it in a engineering strain component it looks like this.

So, essentially we can simply write here del u del x cos squared theta plus first I write this quantity this quantity. So, del v by del y sin squared theta and then plus del v by del y plus del u, sorry del v by del x, del u by del y. So, this is into sin theta cos theta.

Now, you see carefully this is del u del y and del v by del x is essentially gamma x y which is a shear strain, and then del u del x is epsilon x 6 and then del v by del y is epsilon yy. And similarly for epsilon y prime y prime you will also can we write and this equation also you know it from of strength of material knowledge. So, this also we can derive for the shear strain. So, if you remember this thing, so we can express it from a from the in a matrix format. Probably we have studied this earlier also. So, let us see how it looks like.

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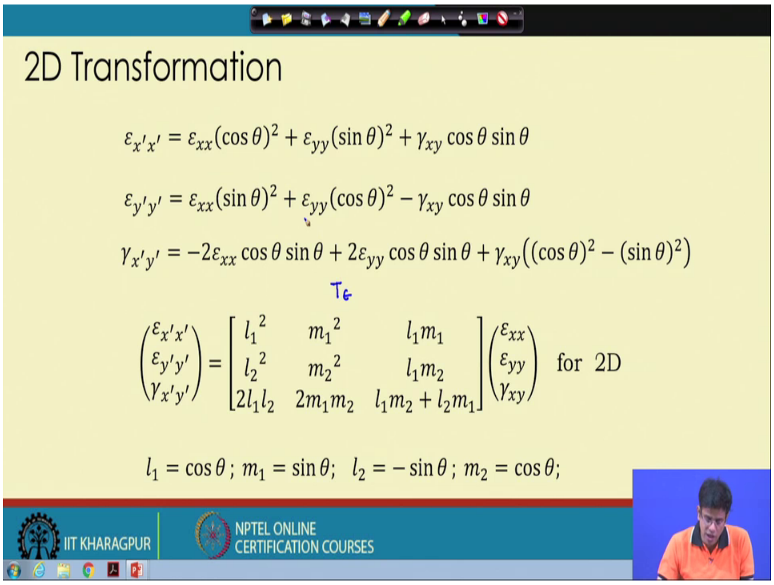

So, this is a formula and then if we substitute l 1 equals to cos theta m 1 equals to sin theta l 2 equals to minus sin theta m 2 equals to cos theta then we get this T sigma T epsilon. So, T epsilon we will get it from the this formulas. So, this is applicable for 3D also.

So, with the in a similar way if you can express the if you can express the strain quantities in a rotation about an axis or rotation of a plane, then we can also derive T epsilon or the transformation matrix for strain for 3D case. So, this is the we know it from the our basic knowledge. Now, from our strength of material knowledge.

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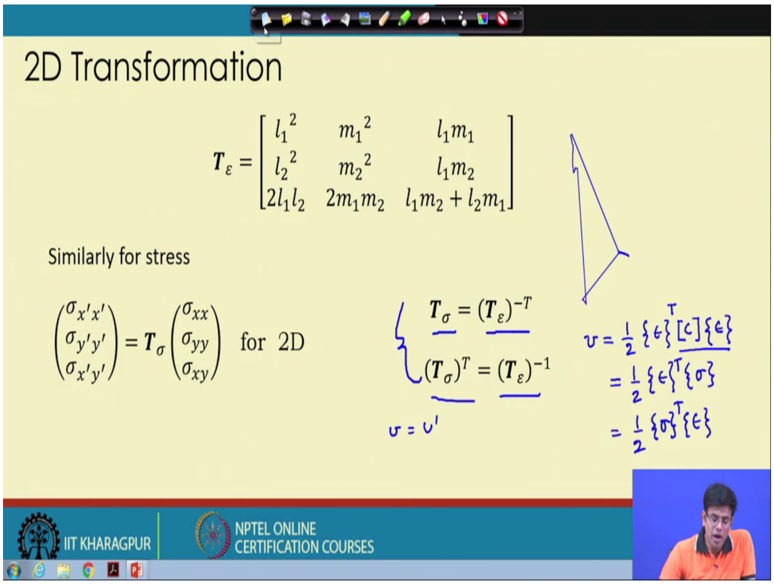

Now, with this T epsilon we get the strain transformation matrix. So, similarly stress transformation the stress transformation also we can go with the similar procedure and if there is a rotation about a axis it can be proved by T sigma is essentially T epsilon inverse transpose. And T sigma transpose essentially if you take the transpose then T epsilon inverse.

So, this is this can be a cubed very easy way also there are two way if you know the virtual displacement or virtual work principle that will also give you the same result. Otherwise you can use the in variance of the strain energy function which will you know the strain energy function is essentially, strain energy function essentially is half of epsilon into C into epsilon transpose C into epsilon.

So, now if I write half of sigma or the epsilon transpose in place of this I write sigma. So, I can write it is half of sigma transpose E, epsilon. So, if you do this for a transform coordinate since, we know u and u prime are same so because it is an in variance relation a strain energy and is invariant. So, now this we can achieve for that and we know the epsilon prime we know. So, we can achieve the these relations for the rotation about an axis.

So, now with this in mind we can approach for a constitutive what will be the constitutive matrix. If I now, what you have seen we have seen how to transform a coordinate how to transform vector general vector for instance displacement strain displacements, and then we transform a second order quantity essentially written in Voigt format that is strain and stress. So, which will have the specific transformation matrix essentially that is not a rotation matrix. So, basic purpose is T sigma T epsilon are not rotation mattresses. So, but it contains the component of the rotation matrix.

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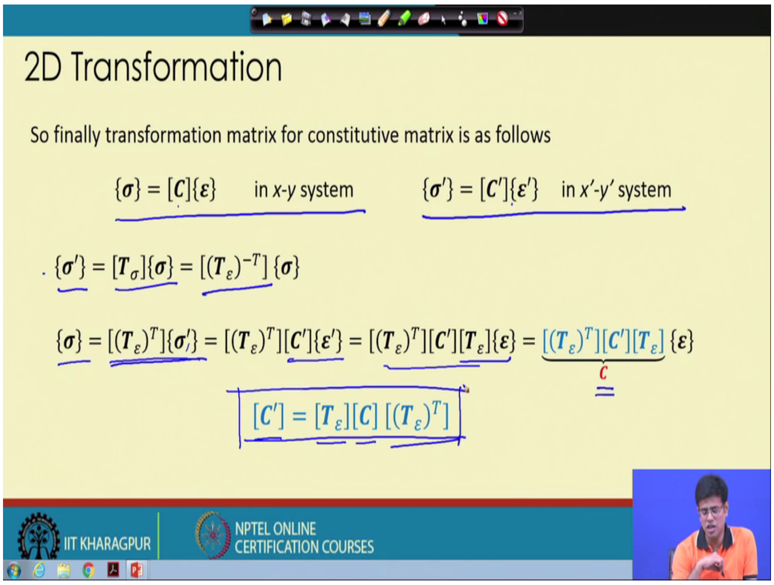

So, now in a so finally, the transformation matrix for the for constitutive matrix is as follows. So, in a x y system we know what is the Hooke's law; in a x y x prime y prime system also we know what is the Hooke's law.

Now, if we write sigma prime is essentially T sigma into sigma. So, stress transformation then we can substitute T sigma the previous relation that we have discussed T epsilon my inverse transpose we can substitute this and then with this is T epsilon inverse transpose I can again write it in T epsilon transpose sigma prime. So, because I just shifted I just multiplied with the in T epsilon transpose, here in here I multiplied with the T epsilon transpose and here I multiplied T epsilon transpose. So, which is a in a inverse or T inverse kind of thing, so which is identity matrix. So, I just flipped the site so, sigma is essentially T epsilon transpose sigma prime.

So, now this sigma prime again I can write C prime epsilon prime from this relation. So, this C prime epsilon prime again epsilon prime I can write it T epsilon into epsilon. So, I transfer it to the non prime coordinate system.

Now, if you look carefully, this quantity this quantity which is in the second basis this is actually the C because sigma is essentially C into epsilon. So, this is essentially C into epsilon. So, I can write C is equals to this and then again if I do inverse I multiply inverse and then transpose. So, I get C prime is essentially T epsilons C T prime; T epsilon transpose. So, this is a general formula for transformation of constitutive matrix, where again I am pointing out that these formulas are actually valid for the rotation about an axis. So, in general this is not true for a 3D rotation.

So, now what exactly it will be different is this. So, we may not able to use this formula for in general 3D rotation. So, there we have to have a different formula which may not be seen as this. So, now we have achieved the rotation about an axis or rotation of a plane where the constitutive matrix will transform, and then we can extend it for the 3D case the 3D case is very simple.

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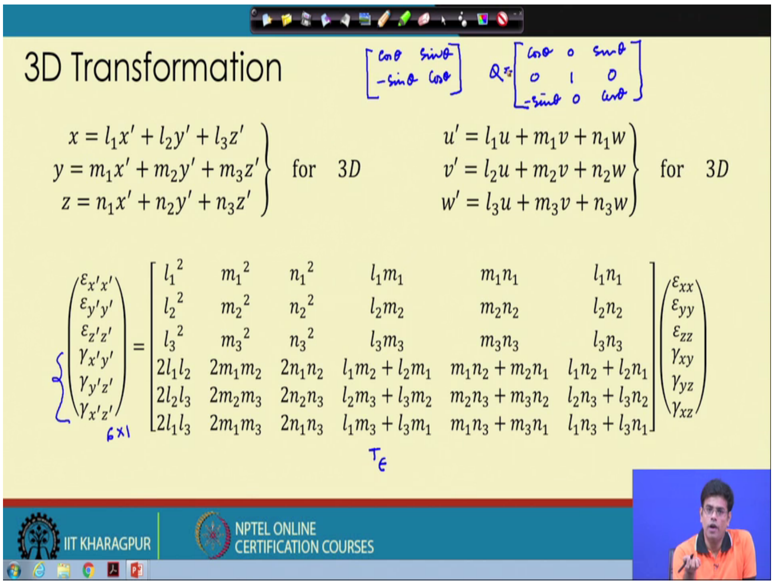

You just have 3 coordinate system, where l in or the any vectors and then or the direction components of the direction is these cosines. So, you write x y z in this form and then you can construct the rotation matrix which is l 1, l 2, l 3, m 1, m 2, m 3 and n 1, n 2, n 3. So, similarly we can transform the vectors, vector is l 1, l 2, l 3, m 1, m 2, m 3, n 1, n 2, n

3. So, the vectors also we transform in a similar manner. So, here now in a 3D case there will be 6 cross 1 strain vector in a Voigt notation, so again I am using it here I am using the shear strain components.

So, now again this quantity is in a prime coordinate system. So, this can be in the 3D rotation, 3D matrix for the strains transformations. So, it is essentially T epsilon, and this is the non prime quantity.

Remember here that in of probably in a 2D case we have seen the rotation matrix in cos theta sin theta minus sin theta cos theta, so in this form. But in a 3D case it may be like this cos theta 0 sin theta and then 0 1 0 sin theta, so minus sin theta and 0 cos theta. So, in this format it is a. So, this matrix is a in (Refer Time: 23:59).

Essentially the this Q matrix is one parameter family. So, that is a rotation about an axis rotation about a or rotation of plane. So, in case of a finite rotation or 3D rotation means where it is non community commutative that one parameter is not possible. So, essentially it is a 2D rotation it is not a 2D 3D rotation. So, but rotation matrix or the Q is written in terms of 3 cross matrix. So, it is an about an axis which is the which does not rotate and which so, it is an about an plane.

So, why I am doing this because we will be using mostly composite materials. So, where will see the use of these mattresses very often and it has a you very large amount of engineering application.

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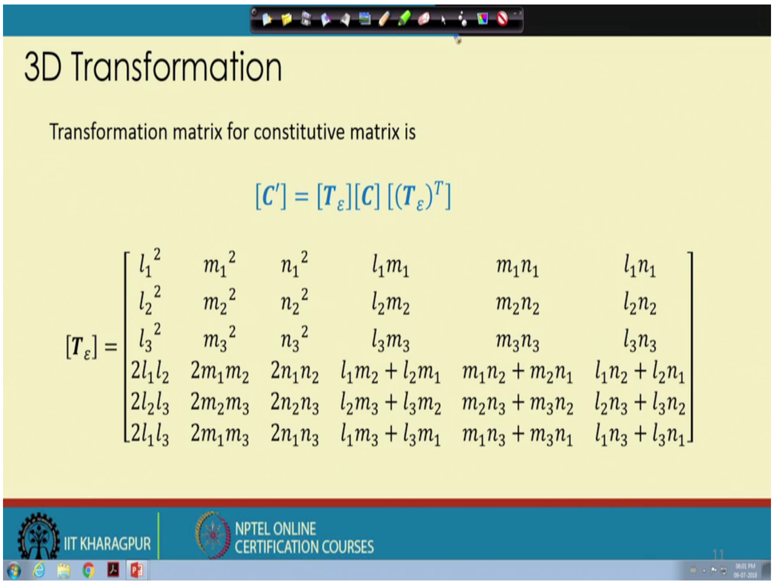

So, now the constitutive matrix can be written in the same from the constitutive matrix here will be in the same form. So, this T epsilon is essentially this. So, now, for a 3D case this matrices has to be multiplied with the proper non prime C and you have to get the prime C.

So, now this is why we have learnt this essentially is to because we will be essentially sometimes we will be using probably I discussed it earlier why we have I seen.

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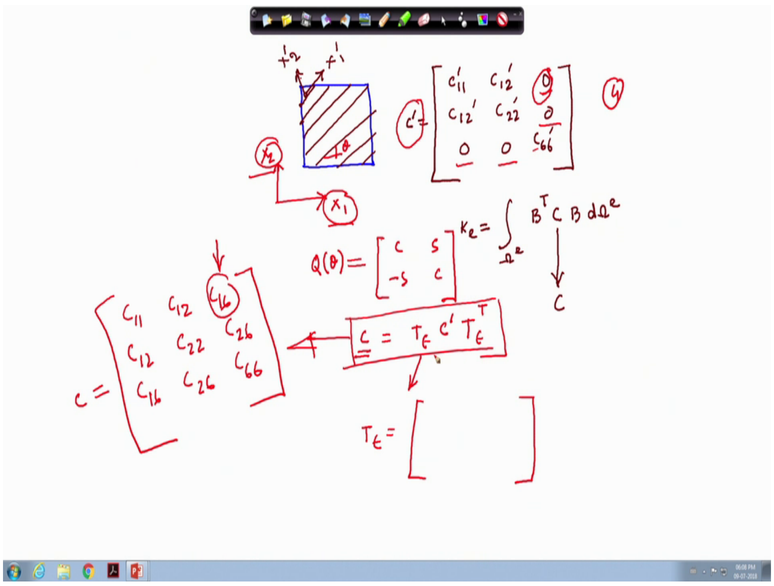

So, let us give to a 2D example, where the coordinate axis is like this. So, essentially see it is x 1 and x 2, so x 2. So, it can be 3D material also. So, I am giving you the example of 3D, 2D material.

Now, this is my coordinate axis and now here it is made of an isotropic material, but what happens the anisotropic material an isotropy or the for instance it is carbon fiber or glass fiber material where the fiber directions are in this directions. So, these are my fiber directions.

Now, what is available essentially is the material parameter or the material parameter along this; material parameter along this what is available essentially is material parameter along these axis, so which is see x 1 prime, x 2 prime; so, material parameter along these axis. So, essentially what will have we will have the C 11 so, since it is a 2D materials.

So, C 11 prime, C 12 prime, C 12 prime and C 22 prime and C 66 prime. So, if I take an orthotropic material. So, this is my 2D constitutive equation. So, it can be represent in terms of u 1 E 2 and E 3 prime; E 1, E 2 and new 12 this quantities that is engineering constants Young's modulus, Young's direction 1 prime, Young's modulus, Young's direction 2 prime these are also possible.

So, now this is my constitutive matrix along the fiber direction this is available, but when we solve through finite element or any this structure when we solve we will be evaluating the elementals stiffness matrix say K e. So, which is elemental stiffness matrix those who have studied finite element look element stiffness matrix need to be evaluated over the element domain which is essentially B transpose C B, so here over the integration over the element. So, this is C prime.

So, now, you see this, this matrix is essentially need to be evaluated we need C here. So, here to solve for C you would these integral is along these x 1 x 2 to coordinate system x 1, x 2 coordinate system. So, we need this C to be transformed in these coordinate system that means, this C prime need to be transformed in non prime coordinate system.

And to do that actually if we know the rotation that is the how this fibers are oriented along x 1, x 2 plane so which is essentially say angle theta. So, I can define Q theta that is the rotation matrix along this and these can be written in terms of cos theta sin theta minus sin theta cos theta, so for 2D case. So, now, this can be a essentially we need to transform this C. So, C will be essentially again T epsilon and then C prime, here it is C prime and then T epsilon transpose.

So, now T epsilon again will find out from the formula that we have derived earlier for 2D case. So, we know this vectors very easily. So, l 1 square, m 1 square and so on which we have derived here. So, this is a very useful to solve this and we need to transform the material property to the structural axis, so this is this is why it is important. So, this C is along the x 1, x 2 and you will see if you do it on your own with MATLAB or any other software if you do this multiplications. So, this is an orthotropic material where the 4 independent constants are there.

But if you this C may not be look like this form that means, these elements of this matrix may not be 0, that is C a here this C maybe in general C 11, C 12, C 16, C 12, C 22, C 26, C 16, but C 26 and C 66. This will be symmetric because this is a symmetric transformation. So, now, this C will look like this so that means, that you see that the coupling coefficients which where along x 1, x 2 plane will be it was a 0 1 6 coupling coefficient. So, this is essentially it is a shear strain with the a normal strain coupling coefficient.

So, this quantity which was 0 earlier in the transformed coordinate system at that is x 1, x or the structural coordinate system or the body coordinate system which is not 0, so which is natural if you have a inclined sheet. And you are pulling in a different direction. So, it will offer some coupling between shear and the normal stress. So, this is evidence from this transformation here.

So, essentially this is important because we may not have always the material orientation and the structural orientation is same. So, if it is not same we need to transform the elasticity matrix to the structural coordinate system, and to do that essentially we need to follow this formula. So, this is the basic objective for this lecture. So, we will, I stopped today here, and then we will meet you in the next class.

Thank you.