Theory of Elasticity
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Lecture - 19
Constitutive Relation - II (Contd.)

Welcome this is the lecture number 19th for module 4, where we are discussing the constitutive relation. And in the last to last class actually we have discussed the an isotropic material, and in terms of mathematical constants that C ij is a mathematical constants we have discussed.

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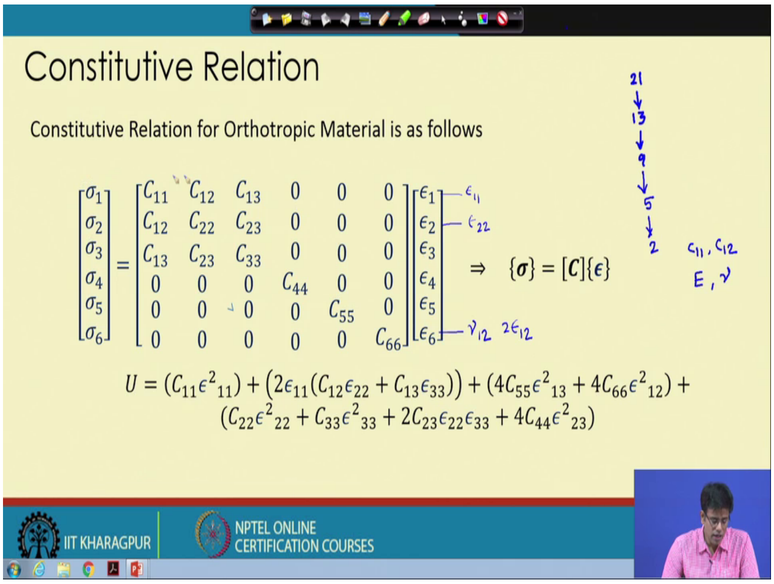

Now, essentially the constitutive relation here is a actually for an orthotropic material. We have already reduced the what does this mean this orthotropic means, and then this is in a how stress and strains are related. So, just to a quick recap these are the normal strains epsilon 11, epsilon 22 and these are the sheared strain.

So, for instance, this is gamma 12 and so on. So, essentially it is 2 of epsilon 12. So, we know these things. And you also know the strain energy term u how it looks like and for instance, the strain energy for the orthotropic material it will depend on the 9 constants.

So, the 9 such constants we have reduced from primary from 21 independent constants, and from the 21 independent constant we first reduce the 13 for a monoclinic material and then from that we have reduce the 9 constant for orthotropic material. And these 9 constants are these C 11, C 12, C 13, C 2 C 23, C 33 and C 44, 55, 66. So, that was the, our basics in the last class.

Now, again from that 9 constant we have also reduce the 5 constants which is for transverse isotropic and then finally, we have reduced to 2 constant for the isotropic material. Now, in this class what will learn is essentially how these constants are related with the engineering elastic constant. For instance, as we know for an isotropic material we know that there are 2 mathematical constant as well as 2 engineering constants. For instance, that if you remember the isotropic constitutive equation that C 11 and C 12 are the engineering constants mathematical constant, and corresponding to the mathematical constants are engineering constants are Young's modulus and Poisson's ratio.

So, our main objective here is to represent this constitutive matrix in terms of engineering constants that is what will be the Poisson's ratio Young's modulus and all those things.

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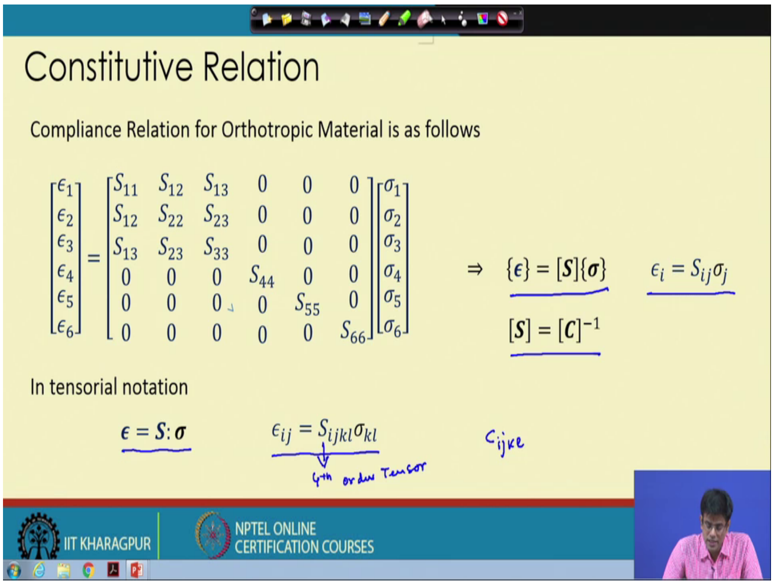

So, before going into that what I wanted to tell you is that the similar to the stiffness matrix or the constitutive matrix a compliance relation for the orthotropic material can also be written, that is the compliance is essentially the inverse of a stiffness. So, that instead of stress strain relation it is strain stress relation.

So, essentially this compliance matrix looks like this because it will have the same thing because the symmetry will be the same, and it will be related with the stresses. And then in a similar manner we can represent it in a vector matrix form. So, this is a in Voigt notation and S is a matrix here. So, essentially in a initial notation it is it looks like this. Now, from our understanding that is a S has to be C inverse.

Now, in a tensorial notation though it is not required for this class this is the tensorial notation similar to the stiffness relation and in a initial notation it looks like this. So, here S is a matrix, but here it is a actually S is a tensor. So, it is a essentially compliance matrix is a 4th order tensor. So, this has to be remember, like for the stiffness matrix, stiffness matrix also we know that is C ijkl, so similar to the i j k l, right.

Now so, S 11 and S 22 and S 12 S 13 these quantities can be written in terms of C 11 C 12 and C 33 because if we invert the C matrix we will get the S matrix. So, corresponding to S 11 the components of C ij will be there. So, this can be achievable very easily.

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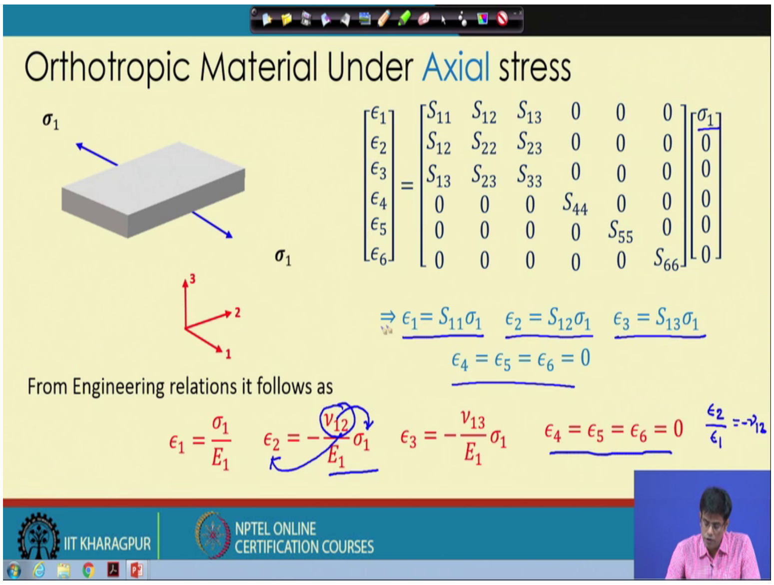

Now, to find out the engineering constants corresponding to this we will use this compliance relation. Suppose there is an orthotropic material now, there are 3 perpendicular axis are mutually orthogonal axis, so these axis 1 2 3.

So, along the 11 axis or 1 axis the first axis I apply a uniform normal stress. So, this is sigma 1, I apply in the direction 1. Now, due to this stress all and all other stresses are 0, I assume that all other stresses are 0. So, only sigma 1 is applied to the body. So, in that relation, so only sigma 1 will be there and all other stress components will be 0.

So, now if I multiply these with this compliance matrix then I will get the strains like this which is epsilon 1 is S 11 sigma 1, epsilon 2 is S 12, and epsilon 3 is this and all other strain components will be 0, because these components will have no effect because these as stress components will be 0. So, all sheared strains are essentially 0, and these components are the due to the Poisson's effects that we know.

So, now, from a engineering perspective the engineering relation we know if I stretch a body unidirectionly in one direction then the stress and stress relation is essentially the gives the Young's modulus in that direction. For instance, so, epsilon 1 can be written as sigma 1 by u. So, this is from our basic knowledge or basic strength of material knowledge. So, now, similar to that I know that epsilon 2 that is the transverse strain in the 2 direction will be minus nu 12 this term is nu actually. So, this is a Poisson's ratio and this is the sigma 1 by E 1.

And. So, this is a very consistent with the definition of the of the Poisson's ratio if you look carefully that is epsilon 2 by a epsilon 1 is essentially nu 12 or minus nu 12 because it is a contraction may is sigma S positive, sigma 1 is positive. So, that is the definition of Poisson's ratio we know from our knowledge of isotropic material and definition of Poisson's ratio the transverse strain by the longitudinal strain.

Now, here what I have done is essentially we have put 12 write the indices 12. So, 1 represents the stress direction and 2 represent the strain direction. So, 1, this 1 is represented is this stress direction so that means, the effect of first direction stress and the effect of second direction strain. So, the ratio between these 2 strains are defined as the Poisson's ratio.

So, similar to this there will be a contraction in the third direction. So, there will be a transverse strain in third direction which I represent nu 13. So, again this 1 indices represents the direction of the stress and 3 represents the direction of the strain. So, again E 3 by E 1, E 3 a epsilon 3 by epsilon 1 is essentially nu 13. So, this will know from our engineering knowledge that if we stretch one body and if we allow Poisson's effect to happen, then it will be like this the strain components will be like this.

Now, all other strains that is shear strains will be 0. So, now, if we compute this 2 these 2 relation we can easily say that is 11 is essentially 1 by E 1. So, these actually gives me the relation of S 11 with the engineering constant E 11. So, finally, this in this case S 11 we can derive it as a 1 by E 1.

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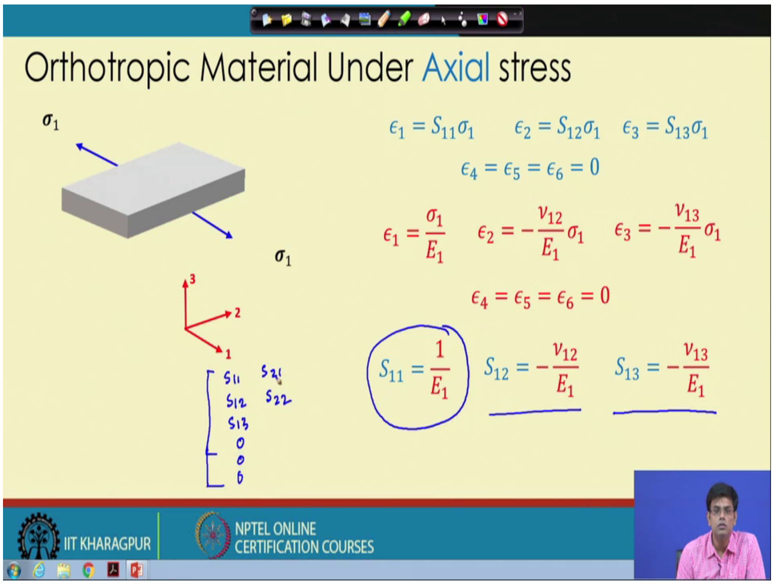

Now, similar to that is 12 is minus nu 12 by E 1 and S 13 is minus nu 13 by E 1. So, this is a case where actually I give a normal stress or the axial stress in the direction 1, and then I compute the S first column of the constitutive compliance matrix. For instance, if you remember the complex matrix the S 11, S 12 and S 13 we get and all other quantities are 0 for orthotropic material.

So, and then S 21 though the symmetric we will use this condition later. So, this similarly instead of giving a one direction if I giving the second direction I will get the second column of the complex matrix. So, that we will see now.

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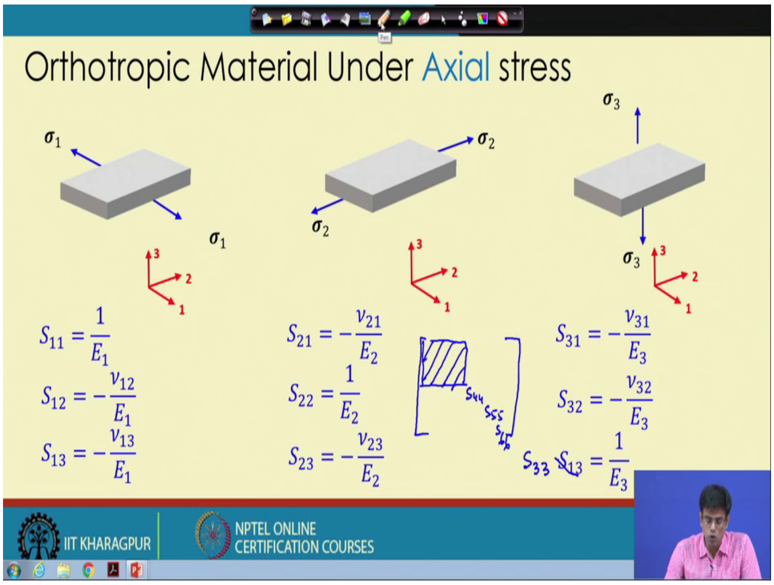

So, if now, instead of likewise if I give the first direction it will be S 11 is 1 by E 1 S 12 is this and S 13 is the this. So, nu 12 represents of Poisson's ratio nu 12 and nu 13 in the 13. So, similar to that if we give thus axial stress in sigma 2 direction following the same procedure I get the S 21, S 22 and S 33. Now, again third direction in the third direction, if you give then we have S 31, S 32 and S 13 sorry, this will be S 33 this will be S 33.

So, now what we got is essentially if you remember that. So, this portion of the compliance matrix we have got in terms of engineering constant. So, what is left is essentially S 44, S 55 and S 66 so, this is left. So, to find out these constant we will use pure sheared stress. So, I think all of you know what is pure shear so, we will see it.

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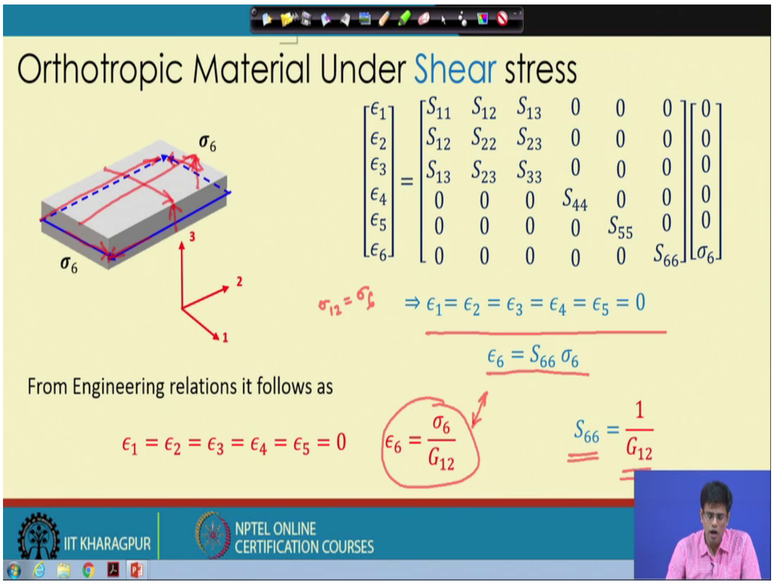

Now, in that material I just give a sheared stress or a pure shear in the direction of 12 direction. So, now, this pure shear is actually this is sigma 12. So, I represented in terms of sigma 6 as in the Voigt notation. So, it is essentially sigma 6.

So, similar to the normal cases or the axial cases I just put only sigma 6 and then all other stress are 0. So, the here it is essentially this and this direction, and similarly this to this direction this is the opposite side of the body.

Now, if I give this pure shear condition then I will have only sheared strain which is essentially gamma 12 which will be nonzero. Other strain components I can see it from this matrix multiplication with this matrix and multiply this vector then other strain components become 0. So, in a from engineering perspective the same we know that shear strain is shear stress by shear modulus. So, this essentially place the comparison and then if I compare these two relation then I can simply comment the S 66 G 12. So, G 12 is a shear modulus in 12 plane.

So, now, similar to this I can give sheared pure shear condition in these direction and these direction as well as in these direction and these direction. So, similar to this I can obtain the S 55, then S 44. So, to summarize here S 44 I can just obtain like this in the previous procedure, and which will be G 23 and then S 55 is G 13 and S 66 is G 12. So, by this actually we got the every component of the, every component of the matrix. So, every component of the compliance matrix.

Now, if I put it if I just use this engineering constant to represent the compliance matrix. So, this is the my strain vector and this is a stress vector. So, this is my compliance matrix. Now, if you look carefully that we know that orthotropic material is having 9 independent constants. So, engineering constant will also be 9.

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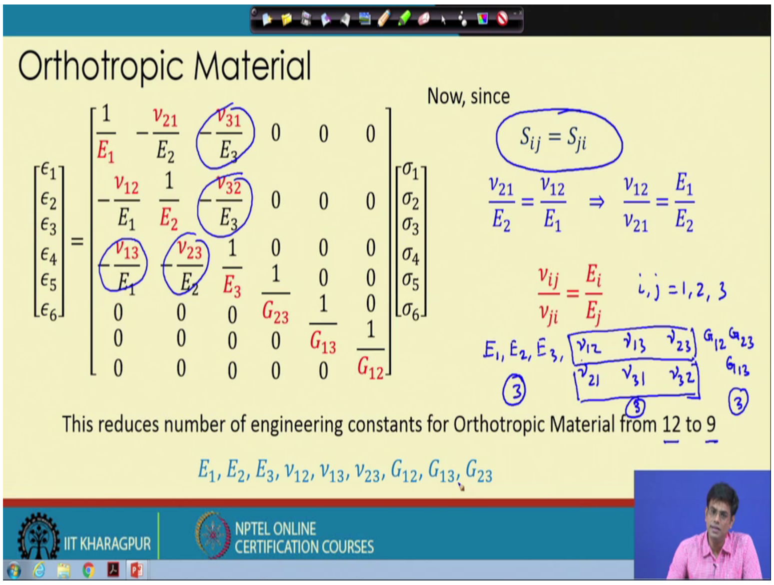

So, now, if you see the all rate quantities are independent I have written only one times here. So, if you look carefully 1 2 3 4 5 6 7 8 9 10 11 12. So, there are 12 components here, so which cannot happen because orthotropic material we have seen there are only 9 independent components. So, now, if we use this condition that the compliance matrix is symmetric matrix. So, now, these 2 quantities will be equal. So, for instance, nu 21 by E 2 has to be nu 12 by E 1 So, if you flip the side. So, nu 12 by nu 21 is essentially E 1 by E 2.

So, in general I can do it for this one these and these case also because this will be same and this case and this case also. So, this will actually reduce 3 mode component of this matrix 3 mode. So, finally, it will the a 9 independent component. So, in general nu ij by nu ji equals to E i by E j. So, this can also be proved from the (Refer Time: 17:47) reciprocal theory. So, this is reciprocal relation also.

So, for here ij can, ij is 1, 2 and 3. So, finally, it is what we have seen is a essentially there are 3 Young’s modulus E 1, E 2 and E 3, and there were 6 Poisson’s ratios which was nu 12 nu 21 nu 13 nu 31 and nu 23 and nu 32. So, among these with this reciprocal relation we can say that these quantities are only independent these quantities are independent or these quantities are only independent whatever it is and so finally, there are 3 independent here 3 independent here. And then there are another 3 independent is coming from the shear modulus G 12, G 23 and G 13. So, there are 3 here. So, finally, this reduces the engineering constants of orthotropic material from 12th to 9. So, these are my engineering constants.

So, what it means essentially is that there are 3 mutually perpendicular direction. So, along each direction there are Young's modulus which is E 1, E 2, E 3, and similarly for there are 3 planes nu 12, 13 and 23. So, there are Poisson's ratio is according to that strain and there were shear modulus in that (Refer Time: 19:36). So, in case of isotropic material what we have seen that Young's modulus and Poisson's ratios are only independent components. So, shear modulus can be represent in terms of E and nu so, will see that here.

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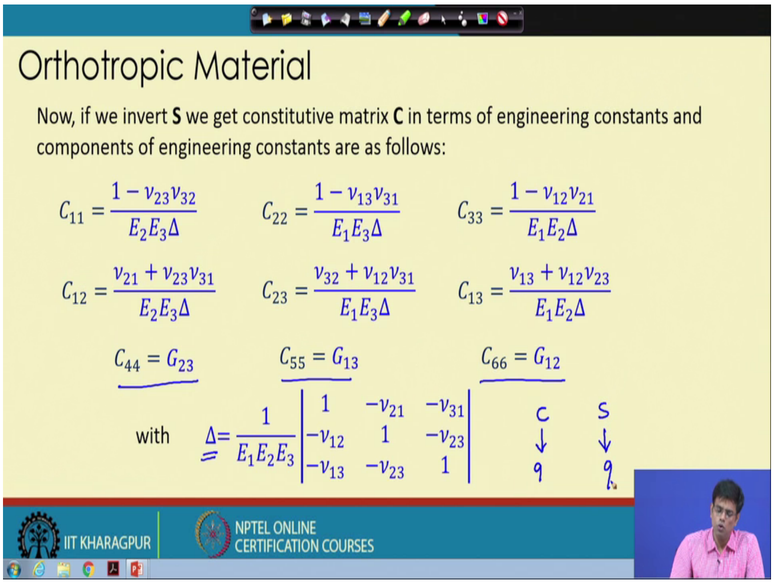

Now, if we look carefully, so once we get the compliance relation and if we write the independent constants only then we can invert that compliance matrix S to get the C matrix, and C matrix this will look like these. So, if we look carefully that C matrix contains the all engineering components which are E 1, E 2, E 3, nu 23, nu 32, nu 13 and nu 31 and etcetera. So, this is the and obviously, C 4; C 44, C 55 and C 66 are the shear modulus itself. So, because you know the diagonal terms if you invert it will be just 1 by

S 66, so which is essentially G 12 so, and this is the delta. So, delta is a also composed on the engineering constants.

So, what we learnt here? That is C matrix and S matrix that stiffness matrix and compliance matrix can also be represented in terms of 9 engineering constants. And those 9 engineering constants are 3 Young's modulus, 3 Poisson's ratio, and 3 shear modulus. So, now, with this knowledge we can proceed to derive for transverse isotropic material.

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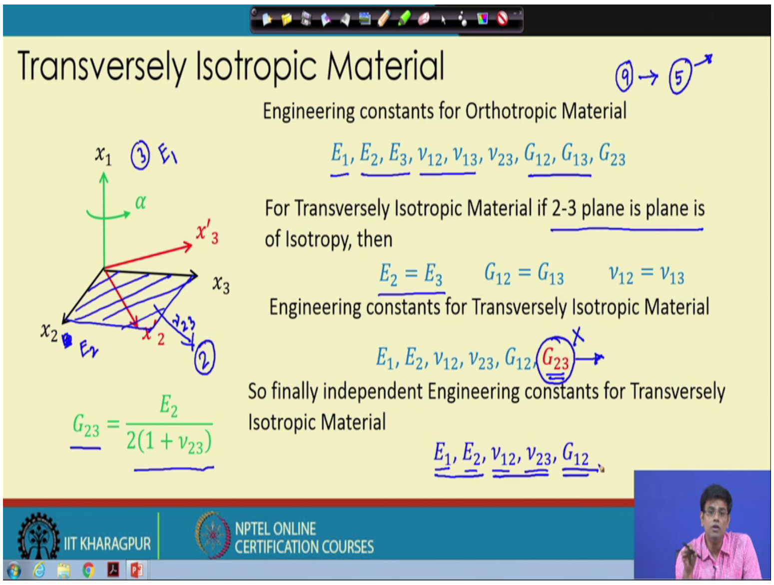

So, as you know the transverse isotropic material if we have discussed in the previous class that is if we take the 23 plane is a plane of isotropy. So, then probably you remember that if you rotate 23 plane about axis x 1 so we get the transverse isotropic constitutive relation and those transverse constitutive relation we will have, transverse isotropic constitutive relation will have 5 independent components. So, from their 9 to we have reduced it to 5, so 9 for orthotropic and 5 for transverse isotropic.

Now, if we assume that 23 plane is plane of isotropy then it will be E 2 and so Young's modulus along this direction and this direction in the same and shear modulus obviously, will be same G 12. So, this G 12 and G 13 will be same, so and nu 12 and nu 13 will be same.

Now, this reduces, so E 1 will be independent one of them will be independent and one of them will be independent and here G 12 and G 13, one of them will be independent. So, 1 2 3 4 5 and 6, so that is I will written here, so E 1, E 2, nu 12, nu 23 and G 12, so and G 23. So, you see we have 5 independent constants for transverse isotropic material, but with this reasoning we get the 6 independent constant. But this G 23 is not actually independent because this is a plane of isotropic. So, this plane 23 is isotropic plane.

So, now, as we know the isotropic plane in this isotropic plane I have 2 independent constant only. So, if I know the Poisson's ratio in this plane Young's modulus in this plane which is say E 2 or E 3, and then for in this Poisson's ratio is G 23 nu 23 if I know this then I can actually find out G 23 which is not then independent, so which is these actually G 23 is E 2 by 2 into 1 plus nu 23. So, essentially G 23 is not an independent constant.

So, these reduces finally, and it is consistent with our previous knowledge that is the transverse isotropic material is having only 5 independent constants, which are 2 in the plane of isotropic it is 2 constants. So, in the plane of isotropic in the 2 constants because isotropic material we know already from knowledge is there are 2 independent constants E and nu, and then along this axis there along the transverse axis of the normal to that axis, normal to that plane that is x one axis there are 3 constant.

So, one Young's modulus and Poisson's ratio and the shear modulus which as E 1, E 2 and nu 12, nu 23, and G 12; so, this actually very consistent with our previous module of isotropic, transverse isotropic constitutive relation of the transverse isotropic material.

So, now you see there in both the cases the relation between engineering constant and the mathematical constant the mathematical constant by which I mean that C ij or S ij, those both the cases the number of independent constant does not change. Even though the form or the how it relates the form is different but the number of independent constant in all cases in case of isotropic material or in case of a orthotropic material is same. So, with this knowledge we can also reduce the isotropic material.

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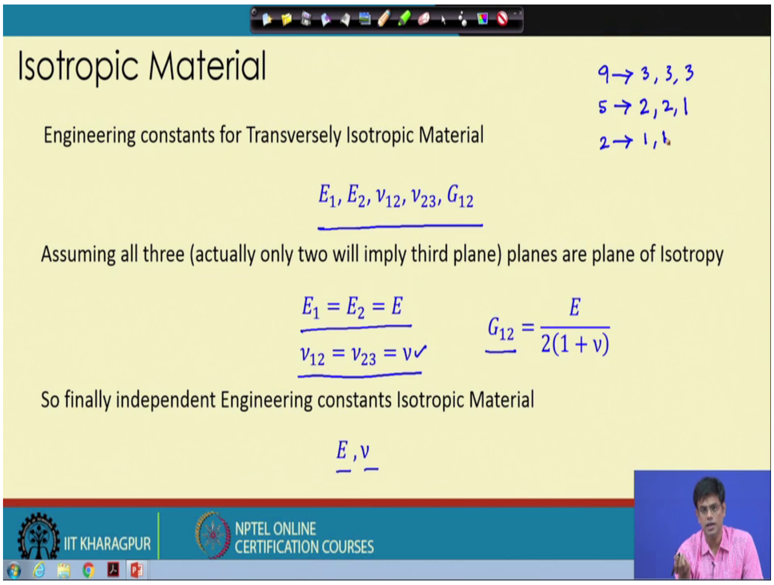

For instance, in the so finally, the engineering constant for transverse isotropic material is E 1 to E 12, nu 12, E 23 and G 12. And then assuming all planes are strain of isotropy so then E 1, E 2 will be let us assume E and nu 12 and nu 23 is nu which is Poisson's ratio, and then again we know that G 12 can be related with E and nu. So, finally, the independent engineering constants for isotropic material is E and nu. So, this actually completes the relation between the mathematical constants or the C ij or S ij with the E and nu.

So, to summarize for and orthotropic material what we what we have is essentially the 3 Young's modulus, 3 Poisson's ratio and 3 shear modulus. For the isotropic material we have, for transverse isotropic material we have 2 Young's modulus, 2 Poisson's ratio and 1 shear modulus. Similarly, for isotropic material there are 1 Young's modulus and 1 Poisson's ratio. There is no independent shear modulus, because shear modulus which can be related with the E and nu. So, this is for a orthotropic material. Now, the question is in case of a general an isotropy what is the case.

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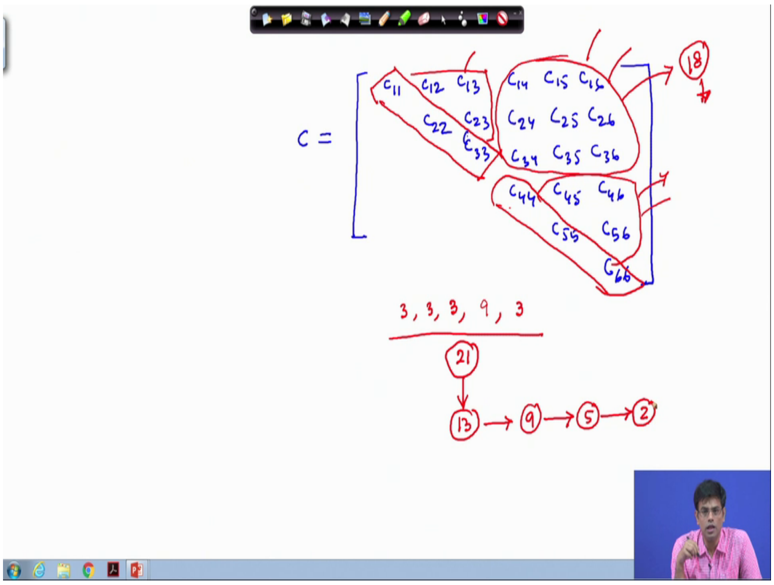

So, if you remember for a in case of a general an isotropic material, so the general an isotropic material in the constitutive matrix, if you remember for a general anisotropic material which is after in the introduction of major and minor symmetry. So, C 11, C 12, and C 13; C 22, C 23, C 23 C 33; C 14, C 15, C 16; and C 24, C 25, C 26 and then C 34, C 35, C 36 and then C 44, C 45, C 46, C 55, C 56, C 66, so original an isotropic material.

So, now if you remember in the class that we have previous class we have say that these are the normal stress components. So, these components are the normal stress component, and these components are responsible for the Poisson's ratio and these are the coupling coefficients, these are the coupling coefficients which are actually the coupling between normal stress and shear stress.

So, now and these are the sheared, pure shear components of the shear components, and these are the these and the components between in between sheared strains there are coupling and this is known as strains coupling strains from effect. So, these are the Poisson's effect, these are the stress of effect, these are the coupling coefficient which is the connected with the shear strain and the normal strain, and these are the pure normal strain, and these are the shear strain stress or shear strain here, so stress strain coefficient.

Now, corresponding to this coupling coefficient, there are coupling coefficient for the engineering constant. And similar to this instance of coefficient there are engineering constants corresponding to that. So, now, if you look for an isotropic material, so there will be similar to that orthotropic material, if you start from orthotropic material there will be say there are 3 Young's modulus, there will 3 Poisson's ratio, there will be 3 shear modulus and so there will be 18 such coefficients.

This case, this coupling coefficient there will be 18 such coefficient out of which we can again prove that by the symmetry of this we can have 9 engineering constants, and then from here actually there are 3 engineering constants. So, finally, if you sum up this is 21 independent engineering constants which is consistent with the our mathematical constant for general an isotropic body. In this way you can also derive for the monoclinic material.

Now, if we reduce this as per the symmetry condition that we have used in the previous classes we can also derive from this 21 engineering constant to 13 engineering constant for monoclinic material, then from 13 to we can reduce for 9, for a orthotropic material from 9 to 5 for a transverse isotropic material and then from 5 to 2 finally, we get the isotropic material. So, in the both cases in the mathematical constants or be engineering constants, the number of independent constants will not change at all.

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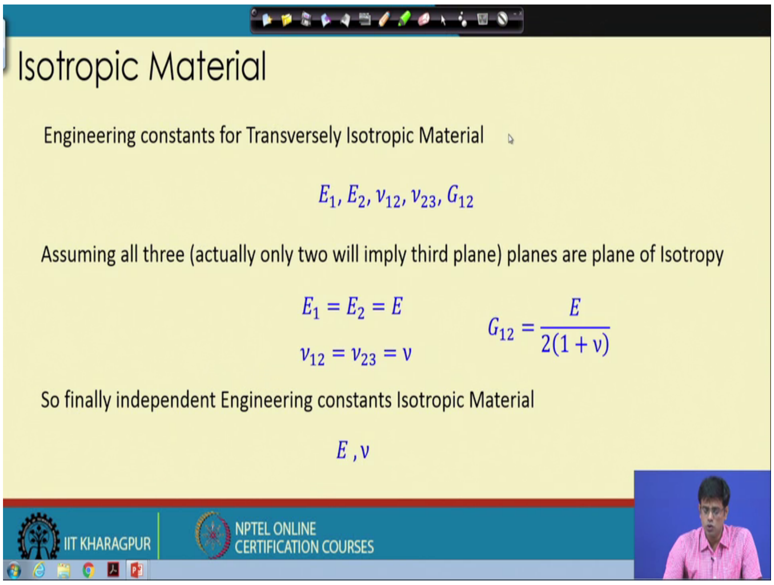

So, this actually completes the relation between engineering constant and mathematical constant. So, in the next class we will try to find out what should be the restriction on this constant or is there any restriction on these engineering constants.

Thank you.