Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 02
Introduction to Tensor

Welcome this is the second lecture of module 1. This in this lecture, we will start introduction to tensor as discussed in the last week, last class ah. So, here basically the objective is familiarity with the initial notation and the basic tensor algebra and tensor calculus which will be helpful for learning this course. So, before we directly goes to tensor operations as we will, let us first understand what is tensor.

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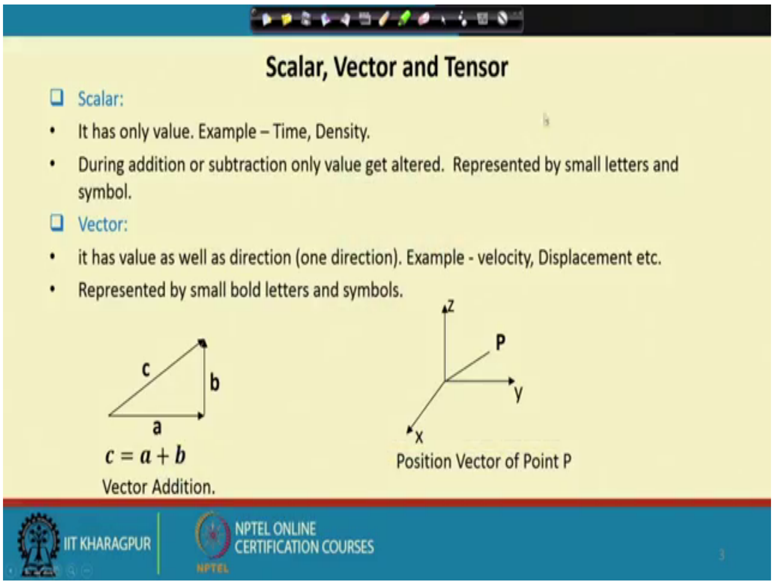

So, for instance all of us know that scalar vector and the new quantity here is actually the tensor. Now, scalar as we know, it has only one value ah means it has magnitude, it does not have any direction or anything ah. So for instance time, density these are the scalars. Now this is the very important that we should or we will represent scalars with the small letters or symbols in this course; in a different book you will find a different convention, but in this course we will represent scalar with a small letters or symbols.

So, in case of a scalar addition and scalar ah subtraction or multiplication, its values gets only altered, but in case of a vector, it is always a always have a direction as well as magnitude; for instance velocity, displacement, these are the vectors.

So, we know what is vector addition, we know what is vector subtraction or multiplication. So vectors will be represented by the small bold letters and symbols. So, this is important because we can distinguish from the quantity itself or type of quantity it is. So, now the tensor, so tensor is actually the new quantity we need to understand what is tensor.

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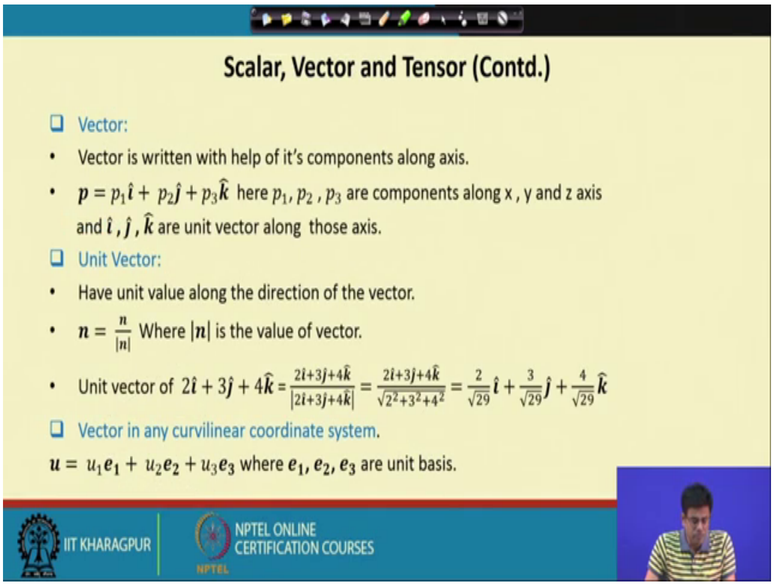

So, let us first ah understand the vector little bit more. So, vector is the always written in terms of its component along the axis ah; for instance if it is a Cartesian coordinate system, our unit vectors in the Cartesian coordinate system are i j k and then a vector p can be represented in terms of i p 1 i p 2 j and p 3 k, so p 1 p 2 and p 3 are components along x y z directions.

So, similarly we can define unit vector. Unit vector is the vector for which the magnitude is 1. So unit value ah but the, it has the direction same as the vector. So, any vector n can be transformed into an unit vector by dividing its magnitude. So also magnitude know what is magnitude of a vector, it is the ah; for instance in this example 2 y plus 3 j and 4 k are the vector; we can transform it to unit vector of this form.

So, now any vector in any general coordinate system may be spherical may be cylindrical or general curvilinear coordinate system. We can express is as component wise; so e 1, e 2 and e 3 are the basis vector on that coordinate system and we can represent any vector ah according to its component along u 1 direction u 2 direction and e 3 direction. So, in general vectors can be ah, in general vectors are represented in terms of component. But we will learn how to ah ease this process through the indicial notation.

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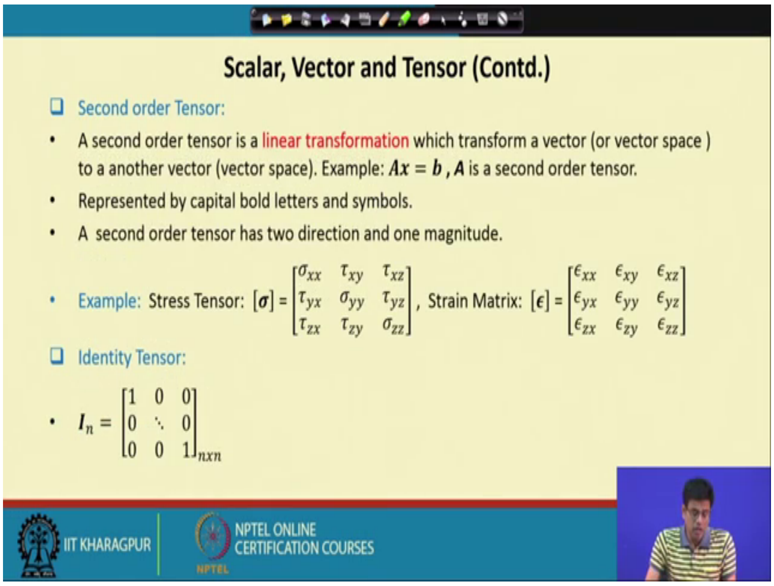

Now, first let us introduce what is tensor? So, tensor among tensor we will first introduce what is second order tensor. So, let me first ah read out the definition, the a second order tensor is a linear transformation which transform a vector to another vector. So, this example, this definition is very helpful to understand the higher order tensors actually. For instance in case of a second order tensor or any matrix, any square matrix can be considered as a second order tensor ah.

As we know that matrix vector equation Ax equals to b ah, the a matrix A transforms the vector x to another vector b, so A is a second order tensor. So, here so if this matrix A is known as the coefficient matrix of the ah solution of the system of equation, this matrix A is known as the second order tensor.

So, second order tensor or any tensor we will represent by the capital bold letters or symbols. So, for instance we have learnt the scalars have no directions, vectors have only one direction. So a second order tensor have two direction. So, in a similar argument if we say fourth order tensor, so it will have a four direction components.

So, even though it looks a very unphysical from a 3-D 3 dimensional space, but we will understand it for our this course. So, in for example, the stress tensor which all of you probably know from the solid mechanics course or stress matrix is a, it has a this form and this tensor, this matrix is known as a second order; this matrix is a second order tensor.

So, similarly the strain matrix is also a second order tensor. Now why it is called second order tensor, because it has a two direction component. So, first component as we know it represents the direction of normal and second one is the direction at which it is the acting. For instance sigma 1 2 or sigma x y if I try to understand, what is sigma x y ah, then the normal direction is along the x and it is acting on the y direction. So, similarly sigma x and other ah components can be understood.

So, in general ah, so if I trend the if I want to define higher order tensor, so for instance if I want to define a higher order tensor or a fourth order tensor, so fourth order tensor how can I define? I can define in this way that a fourth order tensor is a transformation which transform a second order tensor to another second order tensor likewise the definition of second order tensor which is a vector to another vector.

So, essentially ah in this way we can define matrix itself. The matrix is essentially a linear transformation, which transform a vector space to another vector space. So, ah several examples of second order tensor we observe; for instance a identity tensor. Identity tensor for of which a identity matrix of n'th order we represent it by I n; so this is also a second order tensor. Now ah, right now we will only learn about second order tensor and then in the later part of this module, we will also learn what is fourth order tensor and it is used for us.

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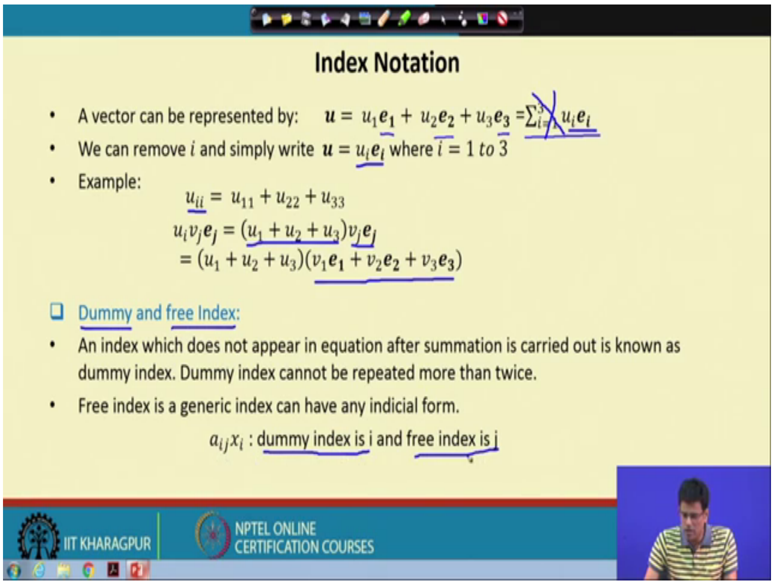

So, so a vector ah as we were discussing in the last slide, a vector can be represented in the component form. Now, if you look carefully that if I want to, if I fix the coordinate system for which the basis is e 1, e 2 and e 3, then we can simply write it as e summation of i 2 3 u i e i. Now this e i is the unit vectors. So now for our ease of writing we can simply remove this summation symbol, and if I write u equals to u i and e i and if we assume that i repeats from 1 to 3, then this type of representation is very helpful for us ah, because we will ah we will not to write this summation and so this type of summation in the removal of this type of implied summation is known as Einstein convention or the initial notation.

For instance if I want to write u i i and if I assume that i repeats from 1 to 3, then it is simply u 1 1 u 2 2 plus u 3 3. So, similarly if I want to write u i v j e j e j is the e is the ah unit vector, then u i is first sound and then v j e j. So v j e j is again summed ah as per the ah as per our understanding of i j repeats from 1 to 3.

So, in this way we can reduce our computation or we need not to write every vector or every tensor in a component form. So this is essentially for ease of our notational notation or also it is known as tensorial representation. Sometimes we will also use only tensorial notation and notation. So in a component form, we will understand it through the indicial notation.

Now while doing this we introduce two indices; one is dummy index, another is free index. So, what is dummy index and index which does not appear in equation, after summation is carried out is known as dummy index. Dummy index cannot be repeated more than two times. So, we will see all these during this course and we will also mention there. So for instance, if I want to write this summation then ah it is implied that u i e I; it is ah repeated ah till i equals to 1 to 3. So the free index, free index is a generic index which can have any initial form. For instance if I want to write a if I write this expression what this mean; a ij x i is the i is the dummy index here and j is the free index.

So, ah here are we, we can assume the value of j and then do all summations over i. So this is the two types of index you have to keep in mind and we will always use this type of ah notation to represents our quantities; for instance the kinematic quantities or the derived quantities.

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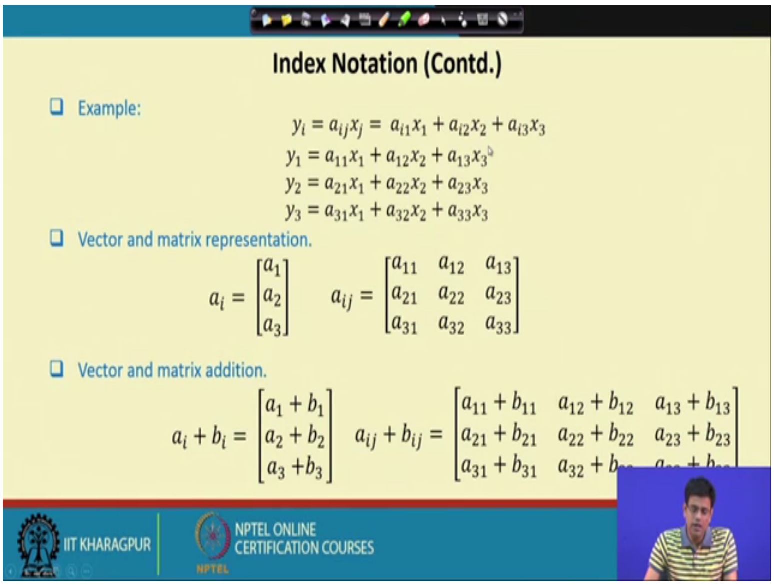

So, now let us see some example, if I write y i a i j x j, then I can simply assumed i or I can simply write in terms of i a i 1 x 1 ai 2 x 2 and i 3 x 3 so on. So, similarly for a different i which is y 1 equals to a 1 1 x 1 and so on. So, in a vector representation if I write a i, it means that it has three components, because we will mostly work with the three component form in 3 dimensional space; so, i i varies from 1 to 3.

Similarly if we want to write a component of a matrix a i j, so ai j means it is all component of a matrix and if the matrix is of dimension or order three ah, then it runs from i equals to 1 to 3 to j equals to 1 to 3. So, this type of representation will shorten our writing and this is the only reason we are doing this. So, similarly vector addition if I write ah ai bi that means, each component of ai is added with the corresponding component to of the bi, so a 1 b 1 a 2 b 2 and a 3 b 3. Similarly matrix addition is also component wise addition and which can be understood from this ah symbolism.

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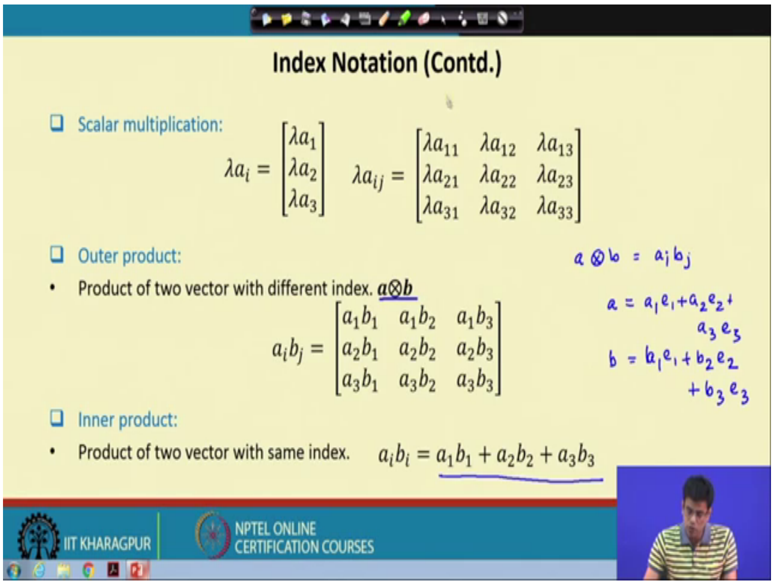

So, now scalar multiplication; the scalar multiplication is also similar to that, so if I multiply ai with a scalar lambda, then it is multiplied with the all component. So, ah another important thing which probably we have not learned in solid mechanics is the dyadic product or the outer product or sometimes it is also known as tensor product, so ah it is ai bj.

So, if I write a tensor product b a tensor product b, so it is simply a i bj. Now what does this mean ai bj, so where I runs from 1 to 3 and j runs from 1 to 3. So, similarly a 1 b 1 a 2 b 2 a 3 b 3 and so on, so this type of symbolism also will be using here and another ah which probably all of you know, the inner product or dot product of two vector which is a dot b.

So, a dot b is we write it a 1 e 1 is the two vectors if we write in this form; that is a 1 e 1 plus a 2 e 2 plus a 3 e 3 a and then b is also similarly we write a 2 ah, sorry b 1 e 1 and plus b 2 e 2 and b 3 e 3.

So, if we take dot product between two vectors, we know that e i ej equals to 1 when i equals to j, and this dot product will ah ultimately produce a scalar which is ai bi; so which is essentially if you write a with the summation convention a 1 b 1 a 2 b 2 and a 3 b 3. So, this is the way we will represent here. So, it is very important to understand these notations, because ah in this process we can shorten our ah representation.

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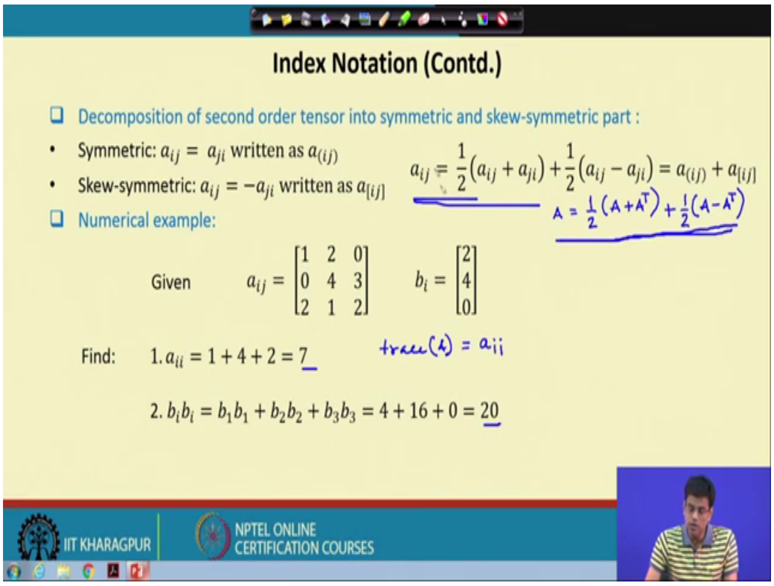

Now so for instance, decomposition of a second order tensor into the symmetric and skew symmetric part. I think ah those of you who have learn the matrix algebra, then probably you must have known this that any vector or a any matrix can be represented in terms of a skew symmetric matrix and a symmetric matrix. So, any matrix A can be represented represented like this A plus A transpose plus half of A minus A transpose right.

So, this if we want to write it in terms of initial notation, it looks like this. So, a ij is the component of a and a ji is the component of a transpose, and similarly the this part also we can write. So, here generally we represent the symmetric part is with a first bracket and anti symmetric part or skew symmetric part with the third bracket. So, this expression is actually equivalent with the matrix expression this. So, this is essentially the tensorial notation and this is essentially the indicial notation. So we have to ah we, we may use both of them to write the our quantities. So, when we write in terms of this so it is implied that each components are added properly. So, for instance if I give some example, so if this is a matrix a and this is a vector b, then its component wise it is right like this. So, if I write aii which is the all summation of all diagonal elements 1 4 2 which is 7.

So, for your information this is also known as the trace, trace of matrix a which is a ii right and then if I want to take the dot product with the vector v or b dot b, then as we have seen earlier which is bi or the square of each components of b,; so which turns out to be the 20. So, similarly we can use ah, we can give other examples also.

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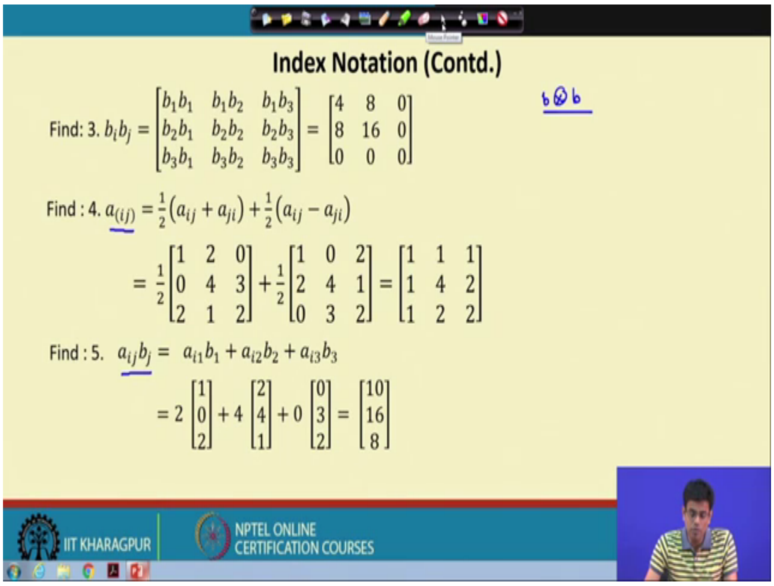

For instance b i bj, so if I want to do tensor product between the same vectors or the for instance b, tensor product b if I want to do; so this we will turn out to be the b i bj. So this will ah be this, so component wise it looks in this form. Now one should understand here this dyadic product or the tensor product increase the dimension. So, essentially b was a ah single direction or a b was a vector and then when we do its tensor product it is a, it becomes a tensor and it is a second order tensor.

So, similarly aij or the symmetric part of the matrix we can write it in this form. So, similarly if I want, if I want to multiply vector b with the matrix a which is essentially a ij bj, so we can write the component form in this way. So, the first row this is if I take the

I and then, similarly we can multiply with the corresponding vectors of b. So, this is helpful in understanding the tensorial notation.

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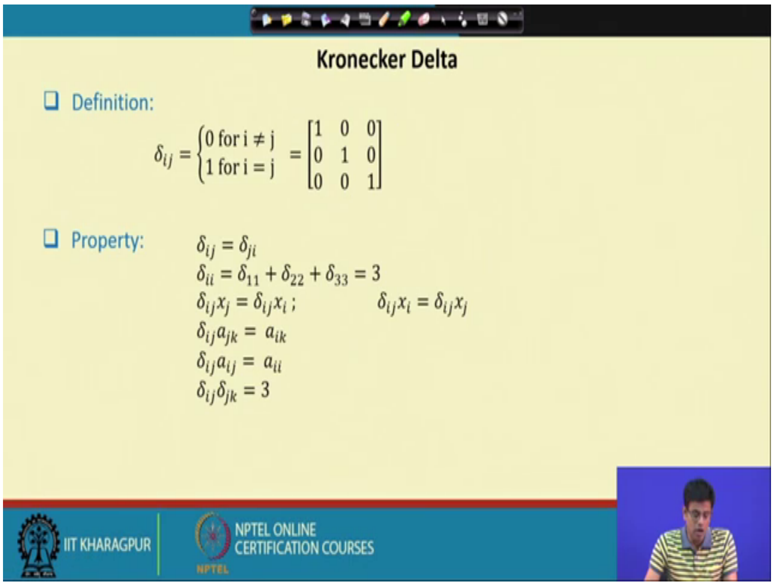

Now, we will define what is Kronecker Delta. I think ah you have learned this in your solid mechanics course, but just to review, this Kronecker Delta is a actually and operators or in a similar to ah what we have used in solid mechanics. So delta ij or it has very interesting property, the property is when i not equals to j then it is zero; when i equals to 1 it is i equals to j it is 1. So, generally we will use 3 dimensional Kronecker Delta which represents the identity matrix because all i j's are different here so this will becomes zero, all of diagonal elements.

So, the properties of Kronecker Delta, since its a diagonal matrix, its transpose will be the same diagonal matrix. So, ah; that means, delta ij is delta g i and its stress will be always 3, and if I want to multiply with a vector with the Kronecker Delta the vector will remain same. So this ah, and if I want to a multiply a matrix the matrix will not change. So, similarly delta ij delta jk I can write 3, so this is the property of the Kronecker Delta. We need to remember this properties, because sometimes we will use these properties.

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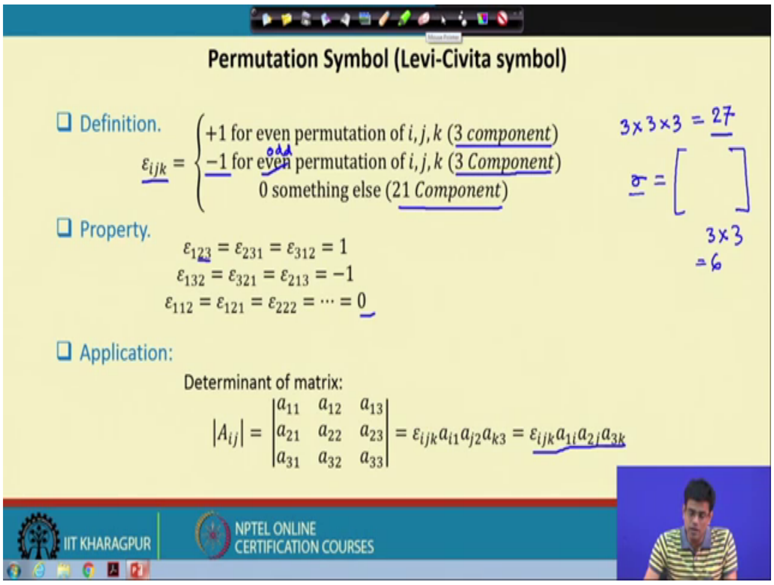

Now, another symbol or another operator we will use this permutation symbol or Levi Civita symbol. So, for instance the permutation symbol is known as the epsilon ijk. So, it has you see the i runs from 1 to 3, j runs from 1 to 3 and k runs from 1 to 3, so total 3 cross 3 cross 3; that means, there are 27 components of this ah operator.

So, if you look carefully that a second order tensor has total 3 cross 3 component. So stress matrix for instance or stress tensor for instance, is have 3 cross 3 component right; so it has 6 component, but this permutation symbol is essentially 27 components. So, this is actually if sigma is the second order tensor, then permutation symbol is actually harder a tensor, because it is the 27 components.

So, we understand what is third order tensor here. Now this permutation symbol like Kronecker Delta has a property that if i j k ah, if we writing even permutation, then this permutation symbol will be 1 and if i jk if we write in odd permutation ah this will be on, odd permutation i, then it will have also three component which will be ah minus 1.

For instance if I write 1 to 3 epsilon 1 to 3 this will be 1, then 2 3 1 this will be also 1 and epsilon 3 1 2 this is 1. So, if I change this flip these two that is if I write 1 3 2, then this will be minus 1 and if it is any one is repeated here, then it will be always 0. So, if you look carefully then you will see that 21 such components will be zero and there will be 6 such components which will be nonzero which is either 3 of which will be plus 1 and 3 of which will be minus 1.

So, for instance we can use this permutation symbol two evaluate the determinant of A matrix. So if the matrix A is ah we know the determinant of a matrix already. So if we write it, so it will be like this. So, you can use pen and paper to expand this expression and check whether this will ah, this goes to the determinant of a matrix or not.

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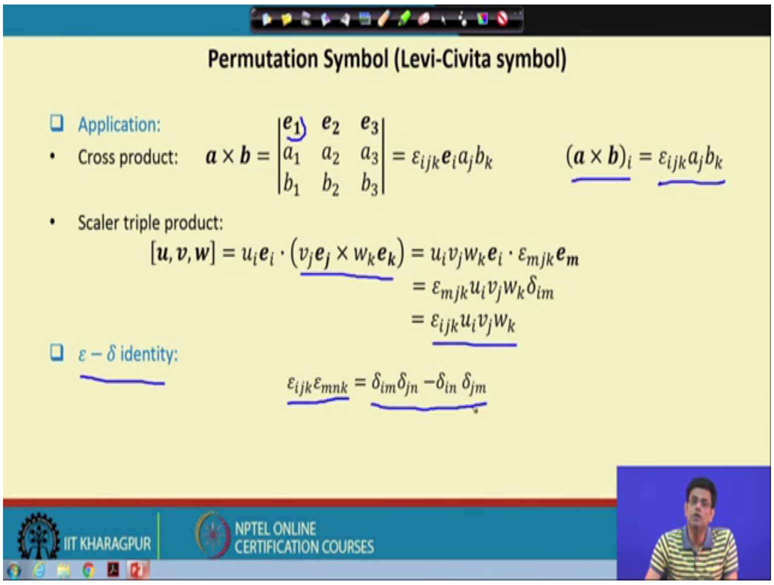

So, now for instance another application is cross product. So, all of us know what is cross product; so which is e 1 e 2 and e 3 a 1 a 2 and it will determinant of that. So, if I write it in terms of permutation symbols which is essentially epsilon ijk ei and then aj bk. So, the component wise form is just you remove the ei, the vector unit vector term.

Now similarly scalar triple product; so, scalar triple product ah we know which is a dot b cross c something like that here you dot v cross w, so v and w if we write it initial form it will just we can write v j ej and w ke k. So you if I write v j e j and w ke k in terms of this form or this form then finally, we can write this scalar triple product is epsilon ijk uj vj w k. So, this is another important relation or identity is that relating this permutation symbol with the Kronecker Delta. So, if I if I write or if I want to relate this so epsilon ijk is epsilon ah into epsilon m and k should be of this.

So, this is an important relation and this requires to prove some of our relations or some of the theorems, so ah. Then we will see the coordinate transformation.

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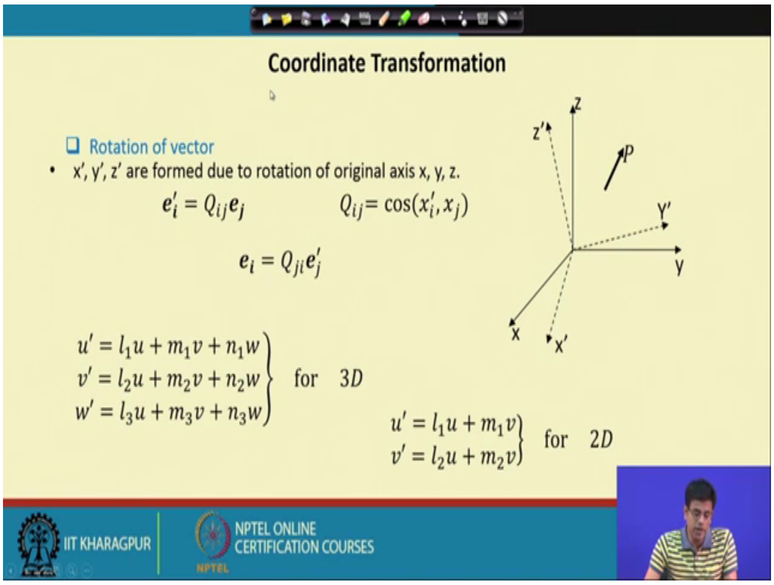

So, coordinate transformation already you have understood in the ah solid mechanics course, so we know if we rotate a coordinate axis about something or about a vector, so we can write we arrive a new coordinate system x prime y prime z prime and our initial coordinate system was xyz. So, relation between these two coordinate system are already known to us which is e i prime is Qij ej, so e is the basis vectors in x y z coordinate system and prime basis vectors are e prime j.

So, now, we know that Qi j's are the direction cosines right, so this Q becomes a rotation matrix. Now for 3 D this looks, if I want to instead of a basis vector, if I want to transform a vector then which is simply a we know from the calculations that l 1 u m 1 v n 1 w, so l 1 m 1 n 1 are direction cosines. So, similarly in 2 D we can transform a vector which is l 1 and l 1 u and m 1 v. So this is well known to us, but if we want to transform a tensor how should I do?.

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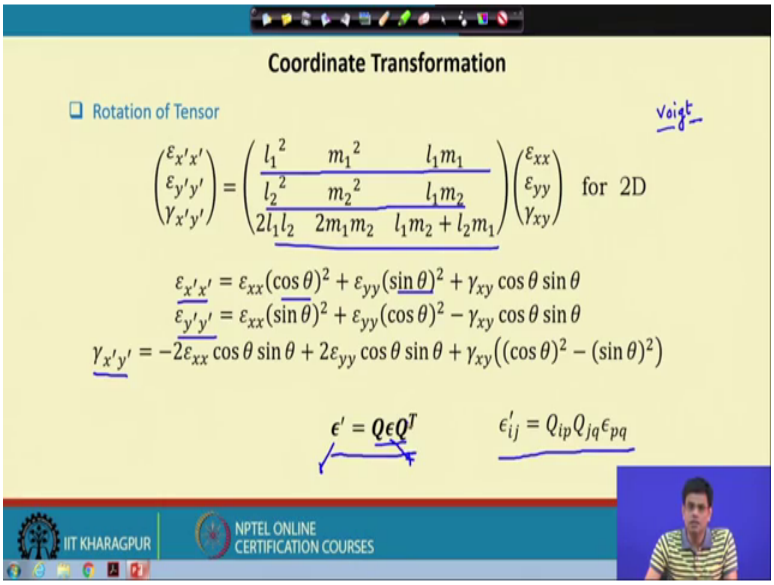

So, before we do it, do that let us see that how we transform our strain or stresses in our solid mechanics course. If you remember carefully that that prime quantity strains and prime quantities x 6 strains and prime quantities yy strains and x prime y prime the engineering stress, the strains,. These are related with the non prime quantity or non prime coordinate axis strains with this. So, if you look carefully these are the cos theta square sin theta square cos theta sin theta and so on.

So, if we write now the strains in vector format which is also known as the void notation ah. So this tensors we can write in a vector form which is of this form; so this is a void notation. So, if you write prime quantities in a vector format and the non prime quantities and vector format, then these two are related with the matrix of this form. This is not a rotation matrix because the rotation matrix is l m ah l 1 m 1 and n 1 in 3 D, and for 2 D which is l 1 m 1 and m 1 l 2 m 2, so ah, but these matrix is the transformation matrix for the strains.

Now you see a so we can understand from this configuration or the transformation of strains that each of these direction cosines are multiplied twice and the reason is that, because strain is not a vector strain is a tensor. So, it has two direction. So if I write it in a more formal rotation so it comes out to be strain is this. So prime quantities strains which is here it is a tensor and the non prime quantity strains are related with the rotation matrix Q is in this way. So Q is epsilon prime, it is Q epsilon Q transpose and if I write it in initial notation it is of this form. So, ah the reason why this transformation is like this, because strain has two directions, not one direction.

So, for instance in the previous slide we have seen how to rotate a vector. So a vector is essentially a hm one direction ah, it has one direction. So, if I want to rotate a vector, so it is essentially multiplying the pre multiplying the rotation matrix. Since strain is a tensor our transformation of trends looks like this.

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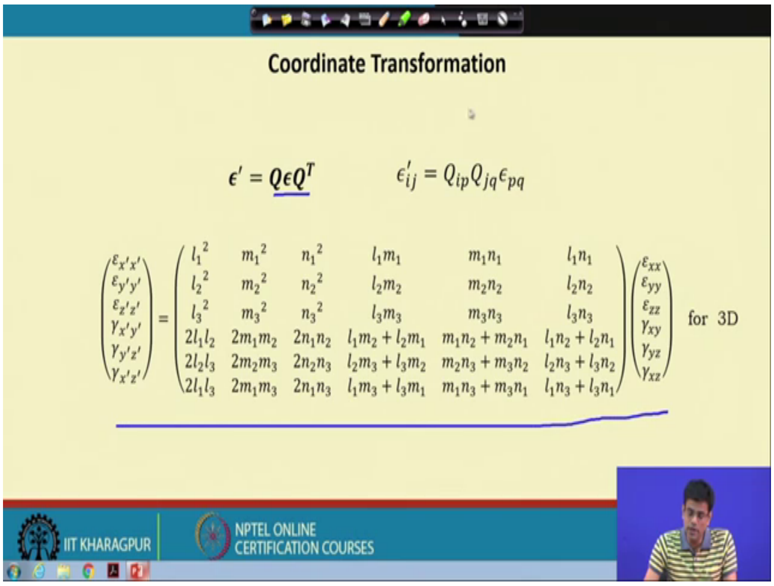

So, similarly for 3 D we can formulate it. So, if I write it in this, I think this ah expression you are familiar in solid mechanics. So this is essentially this; so these strains, this is in tensor format and this is a void notation. So, these actually ah formalize how we learnt ah, how we represent or how we rotate tensors in 3 D.

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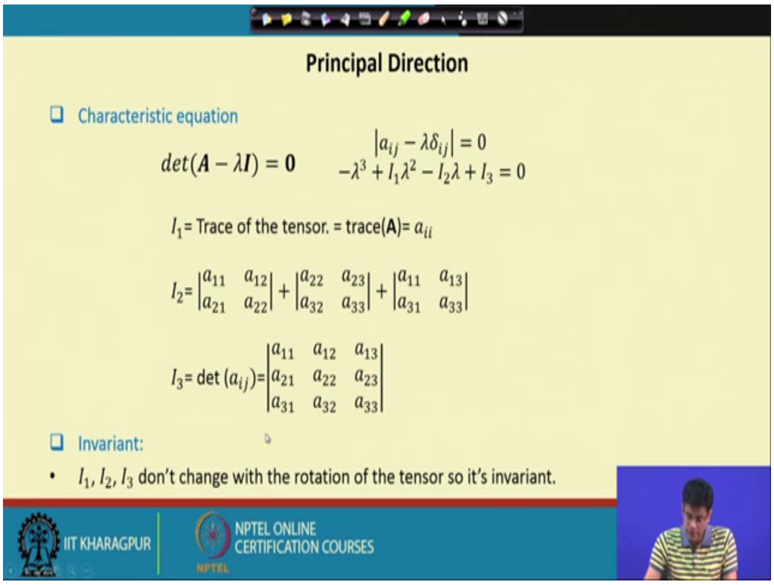

So, now we will go for what is called principal direction. So, principal direction is all of us know what is principal stress on which that there is no shear stress right. So how it comes? So as we know that ah that it is an Eigen value problem. So, if I write the corresponding Eigen value problem which is if a is a matrix its Eigen value is det of A minus lambda I equals to 0 that we know and this is essentially the characteristics equation which represents the three roots of a Eigen value lambda and then, this I 1 I 2 and I 3 are also we know from our solid mechanics knowledge that trace of a is I 1 and trace of and I 2 is all summation of all (Refer Time: 33:58) and then, I 2, I 3 is essentially the determinant.

So, these I 1, I 2, I 3 are known as the invariant of matrix a, which is a second order tensor and this ah, what is this invariant mean? Invariant means if you rotate this matrix a, these invariants I 1 I 2 and I 3 will not change. The component of this matrix may change, but these quantities will not change.

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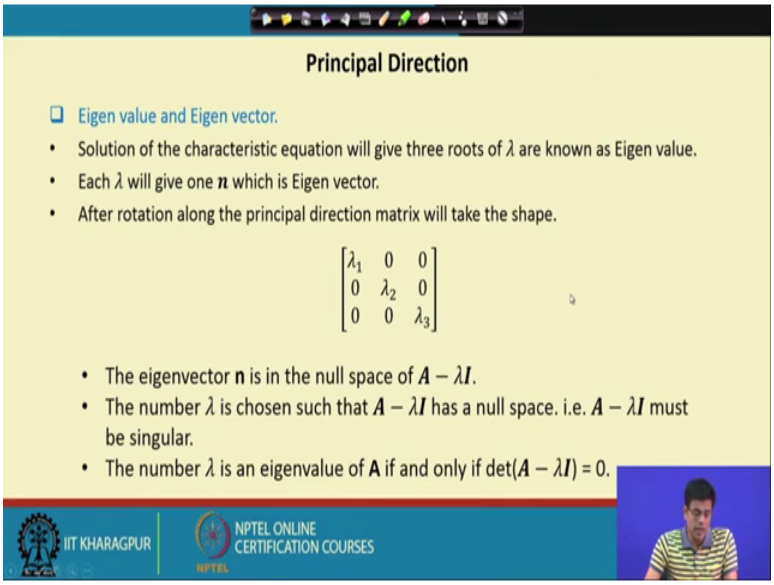

Now so as we were discussing Eigen value and eigenvector, so solution of characteristics equation will give three roots which are called Eigen value and these Eigen values are ah, each, for each Eigen value;there is a eigenvector. So, after finding out the Eigen value the Eigen value matrix will look like this. And so those of you who have little knowledge about null space or the vector space knowledge.

So, we can say these eigenvectors are in the null space of A minus lambda I. So, the number lambda is chosens are such that A minus lambda I has its null space and A minus lambda I must be singular. So, this lambda is a Eigen value o if and only if this (Refer Time: 35:26) of A minus lambda I equals to zero, so these things we know.

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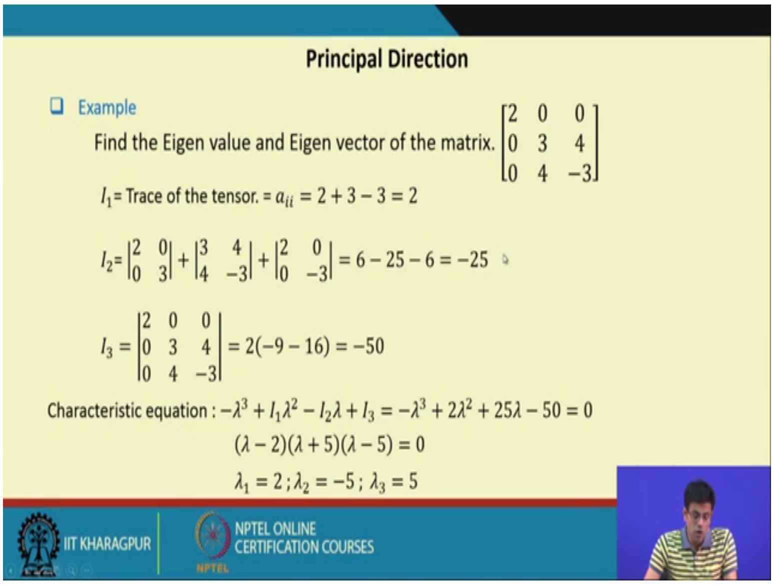

So, now with an example we can see that how to calculate this Eigen values. So, this is a matrix, so the all invariants we can compute and this then characteristics equation we can find out and we can find out the roots of the characteristics equation which is 2 minus 5 and 5, and then ah we can compute its ah eigenvector.

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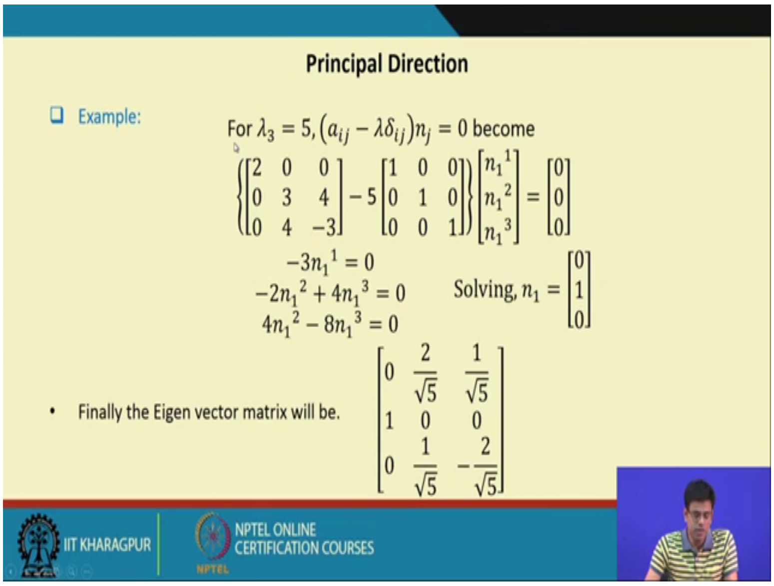

For instance the computation one, computation of one of the eigenvector is this and then finally, we can compute the eigenvector matrix right. So, I stop here today, so in the next class we will also ah review some of the matrix algebra and tensor analysis, and then we will mostly discuss about the tensor algebra.

Thank you