Theory of Elasticity

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Lecture -20
Constitute Relation – II (Contd.)

Welcome. This is the lecture number 24 module 4, where actually we are discussing constitutive relation II. So, in the previous class we have learn, we learn that how to relate the mathematical engineering constant mathematical constants of the constitutive or complains matrix, which is C i j or S i j, in terms of the engineering elastic constants or engineering constants, which are Poisson's ratio and the Young's modulus or the coupling coefficients.

So, now, in this lecture what we will learn is that is at all is this engineering constants or engineering material constant on engineering elastic constants have any restriction on the on their values. For instance some of you probably know, that the Poisson's ratios cannot be greater than half and cannot be less than minus 1, this is probably you have learn in your strength of material or solid mechanics course.

But, what would be the case in case of a large number of Poisson's ratio for instance the orthotropic material. There are 3 Poisson's ratios, what is such restriction? So, is that a Poisson's ratio for such material will have the same, kind of restriction or it is having something different. So, and similar to that we you have also learn probably that the Shear modulus and the Young's modulus will be always greater than 0. So, and how this relation comes so, will see this in from this lecture.

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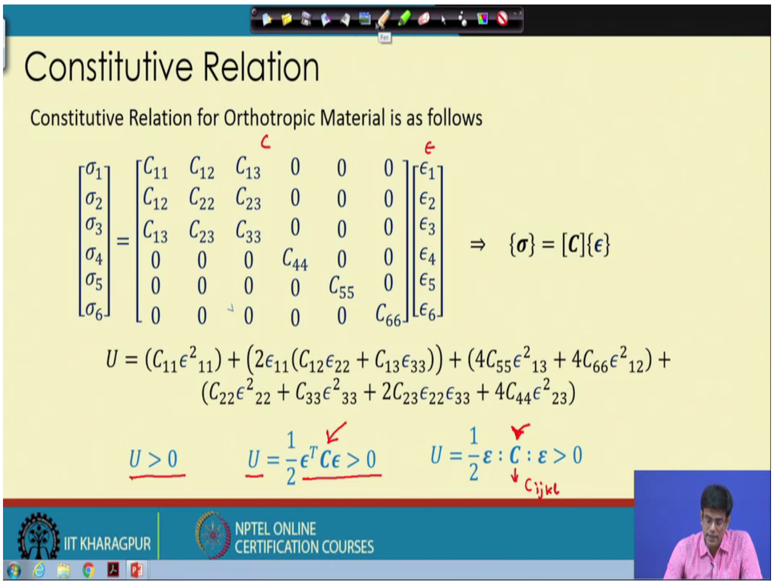

So, before doing that let us quickly recap, that will start with the orthotropic material, for instance the constitutive relation for the orthotropic material can be written in this and this is the strain energy function. Now, probably in our previous class we have discussed that strain energy function, that U should be greater than 0. If U is not greater than 0, then what will happen that; it will not produce a physically meaningful deformation. So, meaningful stress or deformation whatever you say.

So, now so, we know this the strain energy density should be greater than 0. Now, this strain energy density if this is greater than 0 so, we know the form of the strain energy which is half of epsilon transpose C epsilon, which is essentially the vector format or the void notation, where epsilon is essentially this is here the epsilon. Hence, C is this matrix. So, now, here in a tensorial notation though this C and this C looks same, but this is a fourth order tensor which is C i j k l.

So, now half of epsilon contraction or epsilon inner product C inner product epsilon so, which gives me the scalar function. So, this we know from our tensor algebra knowledge. So, this quantity has to be greater than 0 for a realistic deformation of the realistic stress whatever you call. Now, to have this condition so, to have this condition satisfied this can be for any strain. So, this finally, leads to this component or the C matrix, which has to be the positive definite matrix

So, unless this quantity is or this matrix is a positive definite matrix, this cannot be greater than 0 for any epsilon or here, that C should be positive definite matrix, unless C is positive definite matrix the, for any epsilon this quantity cannot be greater than 0.

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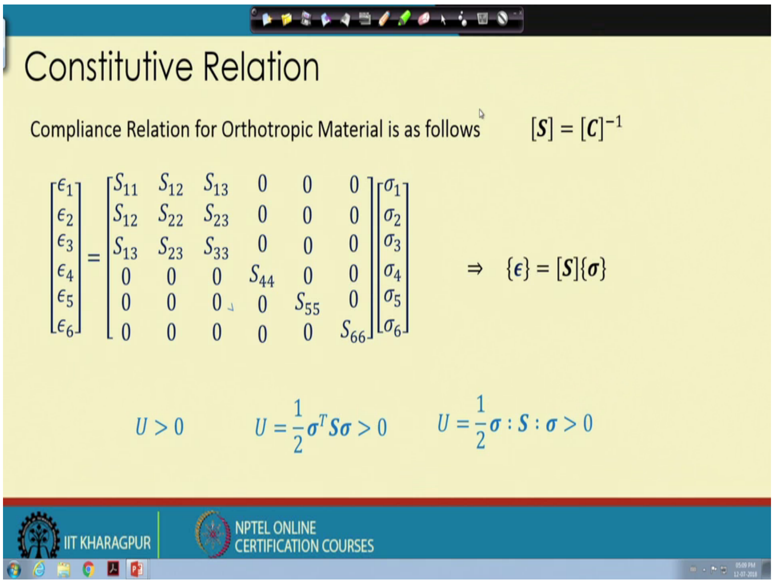

So, similar to this actually, we can have from the compliance relation, which is epsilon and stress how it is related. So, we can also just substituting C inverse and then, we can also say that half of sigma transpose is sigma or in a tensorial notation here remembered that S is S i j k l, which is a fourth order tensor. Here it is a S i j right. So, now with this quantity has to be 0 greater than 0 so, this if had this quantity has to be greater than 0, then this matrix S, this matrix S has to be the positive definite matrix.

So, these actually imposes some restriction on these case S i j and C i a finally, C i j. Now, if there are some restriction on the C i j and S i j then finally, we can put this restriction on the engineering constant Poisson's ratio and the Young's modulus and the Shear modulus. For instance, before discussing those issues let us see, what does this mean that positive definite matrix?

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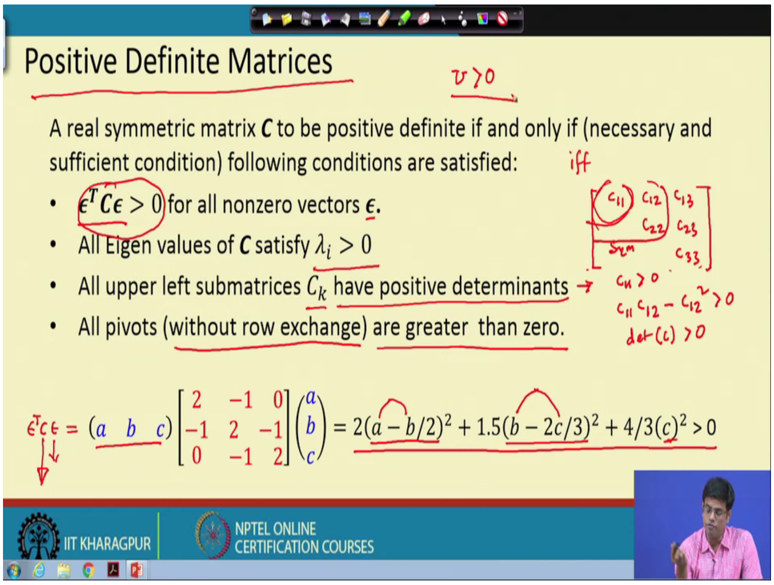

So, for instance a positive definite matrix a real symmetric matrix C to be positive definite, if and only if so, necessary and sufficient condition if so, sometimes it is written in terms of i f f so, i f f. So, it means if and only if and then necessary and so, it means necessary and sufficient condition. So, following conditions are satisfied; so, for any non-zero vector epsilon so, this epsilon transpose C epsilon should be greater than 0. So, for any non-zero vector, now, if epsilon is 0 vector naturally this is equals to 0 so, which is not the case we are looking for.

So, epsilon transpose C epsilon is greater than 0. And, then all Eigen values of C, which is lambda I should be greater than 0. So, probably all of you know the Eigen value analysis. So, the all Eigen values is greater than 0 and all upper left sub-matrices of C k have positive determinants. So, what does this mean? This means that suppose let us do it for 3 by 3 matrix. So, which is C 1 1 C 1 2 and C 1 3 C 2 2 C 2 3 and C 3 3 suppose and this is symmetry.

So, this means that C this means that C 1 1 should be greater than 0 C 1 1 into C 1 2 minus C 1 2 square should be greater than 0 and determinant of this C matrix should be greater than 0. That means, essentially the this part is greater than 0 determinant of this part, determinant of this part is greater than 0 and finally, the 3 cross 3, determinant of 3 cross ratio bigger at than 0. So, now, in addition to that all pivots without any row exchange, without any exchanges are greater than 0. So, those who have who are no question elimination methods, then what is pivots?

So, if you do a row reduced echelon form of a or reduce to echelon form of a matrix then you get the pivots. So, these pivots cannot be negative if the pivots are negative, then matrix cannot be see as the positive definite matrix. So, in a summary so, a positive definite matrix is a matrix which is a real symmetric matrix; obviously, it is symmetric matrix and it is a real matrix So, we are talking about real case only, we are not talking about the Hermitian case here So, here that epsilon transpose or for any vector in a non zero vector epsilon, it is greater than 0 that is first condition. All Eigen value of that matrix should be greater than 0. And, all the upper left sub-matrices that is C 1 1 C 1 2 C 2 2 and this first sub-matrices is a second sub-matrix and this is a full matrix since it is a 3 cross a matrix.

So, this will have the positive determinant and then all pivots without any row exchange this should be keep in mind that it is without row exchange. If all pivots are greater than 0, then the matrix is said to be a positive definite matrix. So, finally why, we are discussing this? Because, if this matrix C is a positive definite matrix; we get this condition automatically, and in the previous slide we learn that our strain energy U has to be greater than 0. So, this is essentially implies this condition. So, let us with an example suppose there is a vector a b c and this is a matrix.

And, then if I take the transpose of this vector a b c a components are a b c, and then multiply and then again multiply the vector then we get this quantity. Now, you see carefully this is the square term. So, irrespective of the value of a and b this quantity will always be greater than 0. Now, similarly b and c irrespective of value of b and c this quantity will be greater than 0. Similarly c is a square. So, this is greater than 0. So, whatever be the vector a b c these quantity finally, this quantity is essentially say this vector is epsilon. So, epsilon transpose c epsilon this quantity is essentially greater than 0; that means, what it means mathematically is that for any vector here this quantity will be greater than 0.

Now, what it means physically for us, physically for us is whatever; be the form of the strains. These strain energy epsilon transpose c epsilon half multiplied by the half. What whatever be the case of epsilon or the strains or the strain steps. The strain energy has to be greater than 0. And, this will be greater than 0 only if c the quantity c is positive definite matrix only then this can be satisfied. Otherwise, if C is not a positive definite matrix, we cannot ensure that the strain energy density is greater than 0. So, based on this criteria will actually drive the restriction of the material Constance. So, let us see how we can use these concepts?

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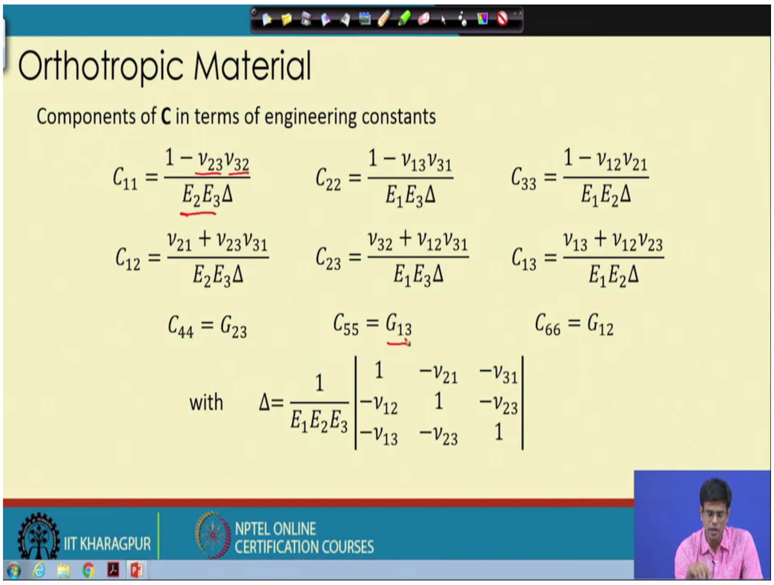

Suppose so, this is from our previous class that constitutive matrix. Finally, our objective is to pose restriction on these engineering constants E and nu and shear modulus G. So, finally, we know how these mathematical constants are related with the engineering constants? So, similar to that we can write the compliance relation also in terms of engineering constants which of this.

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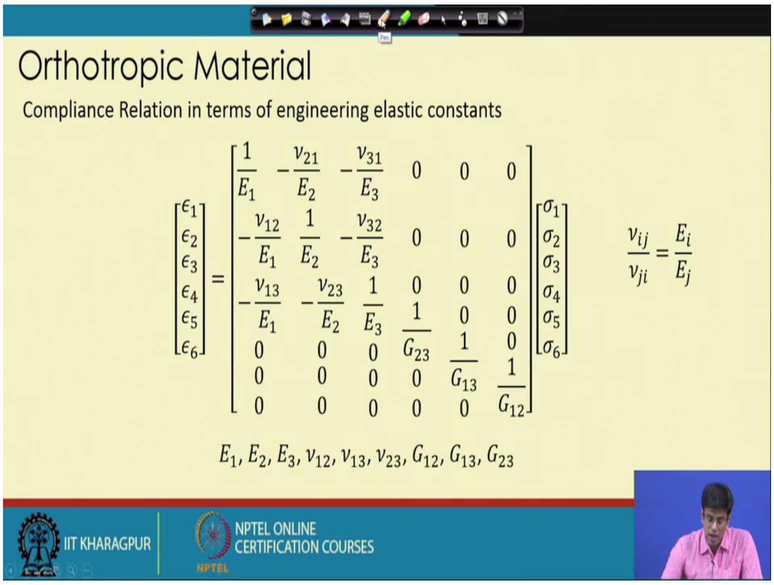

So, know we know there are 9 independent constants. So, there will be this can be the this is the orthotropic material.

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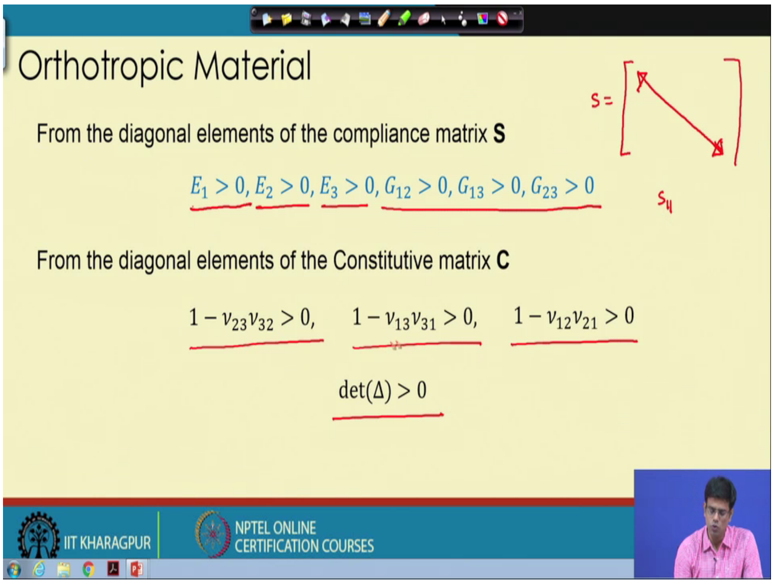

So, now with this in mind so, let us ash derived the restriction on the engineering constant. Suppose we test in the previous lecture we have seen, we give a normal stress in one direction. So, which is sigma 1; so, now this sigma 1 will results in epsilon 1 is a 1 my sigma 1 by E 1. Now, based on the sign on the sigma 1 actually then epsilon one has to be the positive. This is very understandable thing, because if epsilon 1 is negative, then if you stretch the material, then it will not it will compress which is on physical here. So, similar to that other components if I give x at sigma 1 I gives the other directions separately. And, then we can say that these the diagonal of the compliance matrix or the S matrix the diagonal of the compliance matrix can be all diagonal entries are greater than 0.

So, similar for the shear cases also S 1 1 2 S 6 6 so, which is essentially S 1 1 if S 1 1 is greater than 0; that means, 1 by E 1 and so, E 1 has to be greater than 0. So, E 1 cannot be negative. So, E 1 is greater than 0, E 2 is greater than 0, E 3 is greater than 0 and then Shear modulus are greater than 0. So, you see from the 9 independent constant we can all, we have already a found out the restriction of the 6 independent constant which is E 1 E 2 E 3 G 1 2 G 1 3 and G 2 3. Now, from the now if you remember the diagonal entries of the constitutive matrix of the stiffness matrix see. Then, we can also see that these quantity will be greater than 0, this quantity will be greater than 0, and this quantity will be 0, additionally this determinant of delta will be has to be greater than 0.

So, if you remember that which is essentially 1 my C 1 1 is essentially 1 minus nu 2 3 into nu 3 to by E 2 E 3 into delta. So, it will 3 we have already proved greater than greater than 0, this quantity should be greater than 0 and this quantity should be greater than 0. So, similar that other diagonal entries if we compute for C 1 1 C 2 2 and C 2 3. So, these quantities will be greater than 0. Now, what it means actually for us, let us see?

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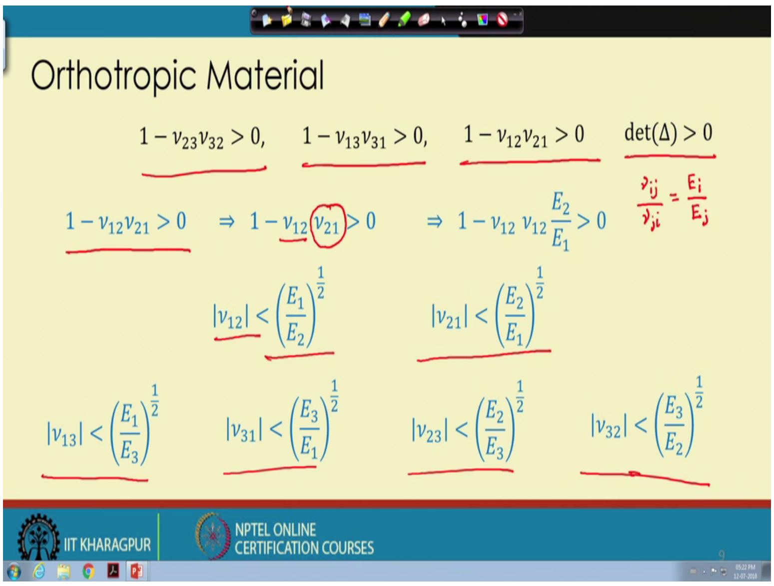

So, if this greater than 0 means, what restriction it process? So, suppose I take this and analyze it. So, suppose this is greater than 0, it means that if I substitute in place of nu 2 1, if I substitute the reciprocal relation we know, the reciprocal relation which is nu i j by nu j i equals to E i by E j. We have learned in the previous class this, the reciprocal relation of the constitutive matrix in terms of engineering constants, for an orthotropic material.

So, if you do that than if you substitute it then this becomes greater than 0; so, now if you just do the sum manipulation that is nu 1 2 square is essentially E 1 by E 2 and then you take the square root essentially. So, mod of nu 1 2 should be less than E 1 and E 2 the ratio square root of the ratio, E 1 by E 2. Similarly, we can instead of nu 1 2 substitution, we can nu 2 1 substitution in terms of nu 1 2 we can substitute nu 1 2 in terms of nu 2 1, which will give me this relation. So, essentially you see that Poisson's ratio cannot be arbitrary actually.

So, Poisson's ratio can be from here that Poisson's ratio can be negative also, but mod of these Poisson's ratio, you have should be E 1 by E 2 square root of E 1 by E 2 it should be greater than this So, Poisson's ratio should be less than this. So, similar to that I if I use this relation and this relation I will get this, this, and this, this. So, you see this relation these poses a restriction on the Poisson's ratio given the E 1 and E 2; that means, young's modulus and if I give the young's modulus in the 3 axis. So, I can have a proper restriction on the Poisson's ratio of the orthotropic material. Now, will see what this condition means?

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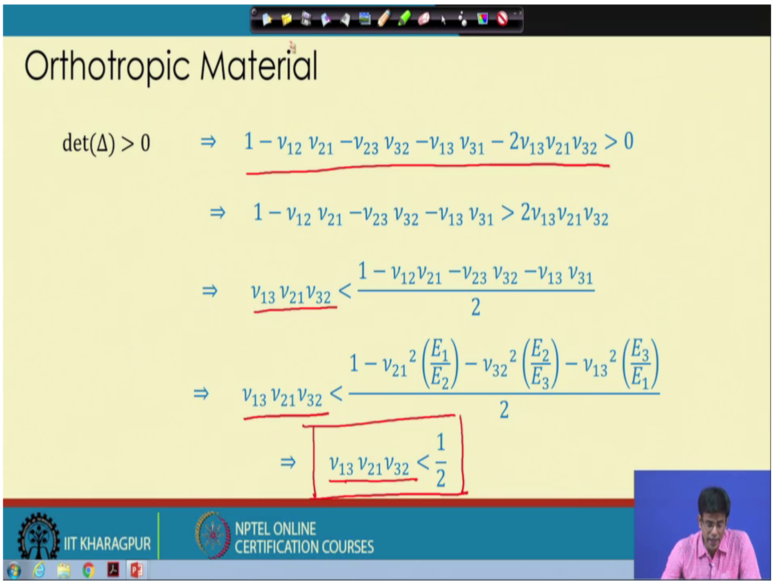

So, let us go for this. So, the determinant of a delta should be greater than 0. Now determinant of delta should be greater than 0, means if I express the determinant. So, it will be this the this should be greater than 0. Now, if I do some manipulation I think this term this side and then I take to down here. So, these nu 1 3 1 2 1 and 3 2 should be greater than this. Now, if I again use the reciprocity relation, then nu 1 3 nu 2 1 and nu 3 2 will be nu 2 1 square E 1 by E 2 nu 2 3 square E 2 by E 3 and minus nu 1 3 square E 3 by E 2 E E 3 by E 1; so, divided by 2.

Now, see these quantities since E 1 E 2 E 3 is are greater than 0 and these are the square. So, this finally, gives me the restriction that product of all 3 Poisson's ratios should be less than half. So, this actually says that Poisson's ratio is cannot be arbitrary large. For instance one of the Poisson's ratio is very large, other one has to be less, otherwise this condition cannot be satisfied.

So, the Poisson's ratios cannot be arbitrary large. This poses the restriction on the Poisson's ratios, but there is a think if one of them is negative, then actually on the other 2 there is no restriction. So, this we can observe from this relation. Now, again what this means for the transverse isotropic material?

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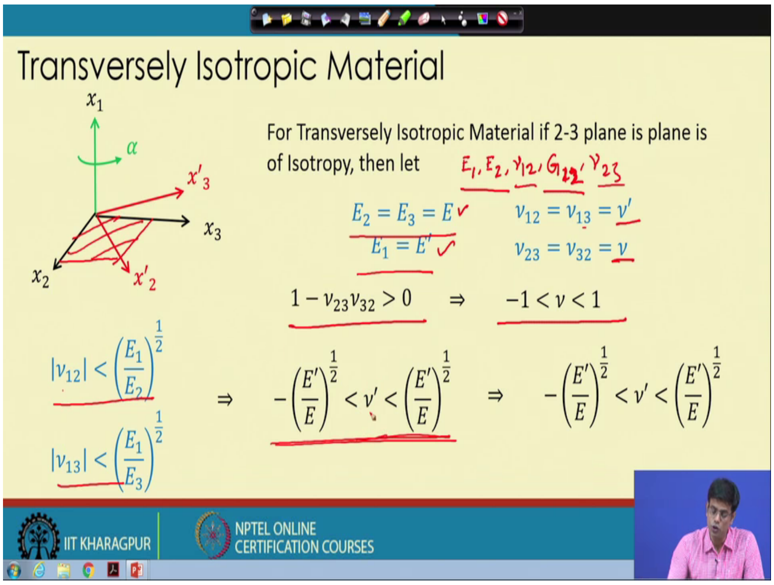

For instance for the transverse isotropic material, if we assume the 2 3 plane is symmetric plane again with this plane of isotropy is this is the plane of isotropy 2 3 plane. So, if we assume that we know what are the material constants for this, transverse isotropic case which is E 1 E 2 E 2 nu 1 2 g 2 3 I and nu 2 3. So, if you assume this transverse isotropic case then if you so, we know from our previous discussion that, E 1 E 2 and G 2 3 will be sorry G 1 2 it will be not G 2 3. So, these quantities will be greater than 0.

Now, then what is the restriction on this nu 1 2 and nu 2 3? So, suppose I assume that this is the condition E 2 and E 2 E 3 are same. So, I assume it is E and E 1 is E dash. And, then nu 1 2 and nu 1 3, which is equal for the transverse isotropic material, which is a nu dash and then for the in plane that x 2 x 3 plane the isotropic plane Poisson's ratio is nu.

Now, if you use this relation which is we have seen from the orthotropic case, we can prove that in plane case for the plane of isotropic the Poisson's ratio should be minus 1 2 in between minus 1 to plus 1. So, now similar to that if we use these 2 relation and then substitute these E and E dash. So, if you substitute these you will get these and then you will get first this and then if you substitute this you will get these. So, this relation actually again poses some restriction on the nu dash that is the other Poisson's ratio.

Now, if we again use the previous case that is these quantity, that nu 1 3 nu 1 3 nu 1 3 nu 2 1 should be less than half.

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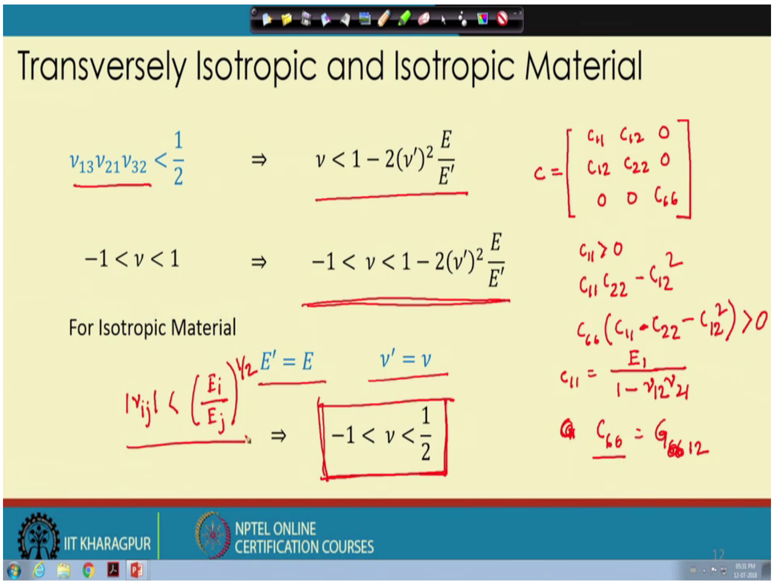

Then, we can just substitute these 2 Poisson's ratio, which will look like in this form and then again if you include this here then again we will get this. So, this is a very string entry restriction compared to this. Now, similarly so, that what it means that the material constants is essentially a engineering constants cannot be arbitrary. So, this arbitrary in else non arbitrary in else is coming from the positive definiteness of the stiffness matrix. And, this positive definiteness of the stiffness or the compliance matrix, implies some restriction actually on the engineering constants.

And so, if you again choose the isotropic material where this is this E E dash equals to E and nu dash equals to nu, then this will actually lead to the result to we already know from our strength of material knowledge. So, essentially what we have learned? We have learned that we can represent engineering constants, with the mathematical constants, and then we a can pose some restriction on the engineering constant as well as mathematical constant. For instance, if I if you have seen the, if I a consider 2 d case now so, for instance for a 2 d unidirectional material.

So, orthotropic material, which is essentially say C 1 1 C 1 2 0 C 1 2 C 2 2 and 0 and then C 6 6 and then 0 0. So, this is the in case of a 2 d material. So, these are represented in terms of engineering. This can be also represented in terms of engineering constants.

So, C 1 1 essentially from our positive recommend this is my C matrix. So, this C matrix has to be the greater than means the C matrix has to be the positive definite matrix. So, to become a positive definite matrix it is all Eigen values has to be greater than 0. And, then for any vector these all those 4 condition positive definite matrix is always there, beyond that I can also say the third condition which is C 1 1 should be greater than 0.

And, then C 1 1 C 2 2 minus C 1 2 square should be greater than 0. And, then C 6 6 into C 1 1 minus C 2 2 sorry into C 2 2 minus C 1 2 square this has to be greater than 0. Now, using this also we can pose this restriction. For instance we can find out that C 1 1 is actually E 1 by 1 minus nu 1 2 2 into nu 2 1. So, similarly this G 2 2 G 1 2 is essentially C 6 6. So, with this also we can find out sorry G 1 2 here. So, C 6 6 is G 1 2. So, with this also we can pose the restriction on the engineering constants.

But, essentially in a summary that all young's modulus and all shear modulus will be greater than 0, because of the positive definiteness of the matrices and the Poisson's ratios, there are for the isotropic material we know this for an transverse isotropic material also we know this. And for the orthotropic material, that is nu product of 3 Poisson's ratio should be less than half. And, then the Poisson's ratios mod of the Poisson's ratio should be the square root, that is mod of i j should be less than E i by E j to the power half.

So, this condition is also imply. So, if you remember these things we can actually you see that a material it given a engineering constant, a material actually represent the physical material on a in a real or the not. So, that is the basically idea. The idea is essentially that we need to model the material or we need to model to model the material mathematically we need to choose the intrinsic properties of the material. And, how these values will choose? So, this has to be some restriction on those intrinsic values.

So, this says some views of representing those conditions. So, in the I stop here today and the next class we will discuss in next class is the last class for this module. So, in the next class we will discuss the properties of laminar or some aspects of the laminar.

Thank you.