Theory of Elasticity
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Lecture – 27
Solution of Boundary Value Problems

Hello everyone. We are going to start the 6th week of this course. This week we will be discussing the Solution of Boundary Value Problems in elasticity. Particularly in this week we will discuss plane stress and plane strain problems ok.

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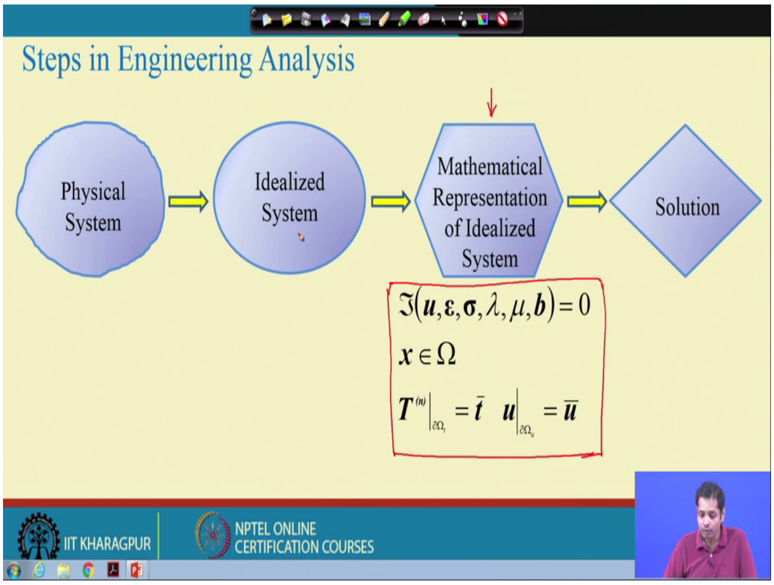

Now, let us start with this slide. This is the slide that I have shown several times during this course. The reason being is a very important slide and it is the core of the analysis that we perform in engineering or in science as well.

Now you see what we have discussed last week is if this is the representation, this is the difference these are the different steps, we discuss this how to write mathematical model for a given problem. So, this step we did last week ok. When we derive this step, we did not mention anything about the idealization of the system.

What we did mention about the idealization, but in terms of some assumption like the plane your it is a linear problem, the material is linear isotropic the stress strain relation is linear the strain displacement relation is linear, say linearity is one of the main assumption that we used while deriving this equation, but these equation is derived for 3 dimensional continuum ok.

Now, what we do, before we start solving this equation, or rather before we start using this equation to solve several problems, in flexural torsion and so on. Let us just go back to idealization of the system and let us see that these 3 dimensional equation can be depending on what information we need from the system, depending on the what kind of structure we are dealing with what kind of continuum, we are dealing with we can simplify this equations ok. Because now we can simplify this we can further simplify this idealized system and that is we are going to discuss this week.

Now all these examples, all these equations are for 3 dimensional continuum, but there are some problems there is nothing like 2 dimensional problem in real life like, it is all problems are all continuum are 3 dimensional, that is there is absolutely no doubt in it, but there are some problems there are some object which can be idealized as 2 dimensional problem. The reason is the one dimension either is very small or the one dimension is either very large; so, that the distribution of stresses or the distribution of the strain in that particular direction can be neglected.

If we can neglect that, if we compare to other component of stresses or other component of stresses, other component of strain then this problem can be idealized as 2 dimensional problems. So, before coming onto the how to simplify this equation, for those class of problem let us see some of the examples of those problems ok.

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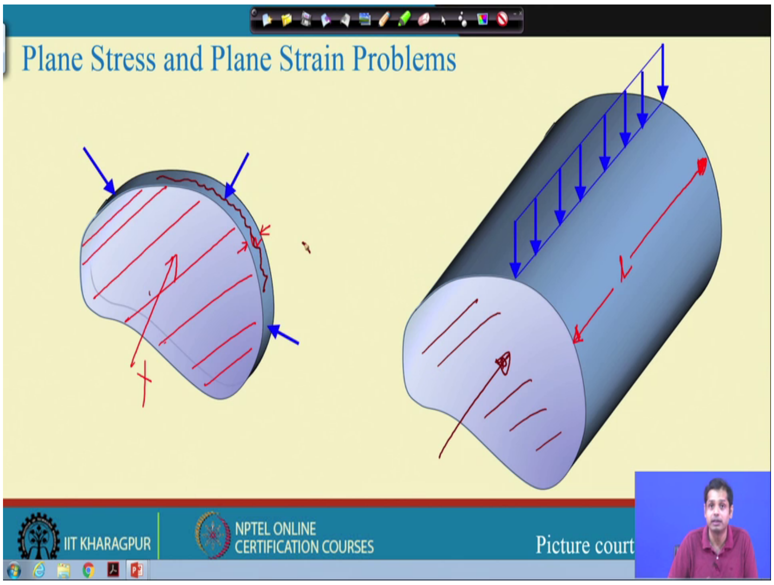

So, for instance take one example, 2 example one is let us not formally give any name to this example this kind of problem we will subsequently do that shortly.

Now you see the 2 example, there is in these example these dimension this thickness of this plate is very small. If this is h then h is very small and where if this dimension if this is l this dimension is very large ok. Now one dimension is very small and one dimension is very large it does it does not qualify any system to be to be system to be idealized for a particular idealize as plane stress or plane strain problem.

But there are it depends on the loading condition as well. Now you see in addition to that this dimension is small this thickness is small, or in addition to that these length is large as compared to the other dimension look at the loading on the structure ok.

Loading on the structure is on this plane it is if we take a plane. So, all these forces are acting on a plane ok. Means what I want to say is there is no load, if we take this is the this is the phase of this of this object. So, there is no load normal to this phase, there is no load normal to this phase. All the loads are acting on this surface and these loads are essentially on a plane and that plane is this plane.

Similarly, here you see all these loads are acting on this. So, there is no load acting in this direction. So, if I take. So, there is no load in this case acting in the longitudinal direction. Again there is no load here acting on the thickness direction. Means normal to this plane here also normal to there is no load on normal to this plane ok.

So, load is acting always on this in this ok. So, 2 things we can see, one is in this problem the first one it is the thickness is very small and the second one is the one dimension is very large. In addition to that the loading is acting such that there is no loading acting on this plane on the normal direction. If it is a length, then the normal to this length in this case it is a normal to the thickness there is no load ok.

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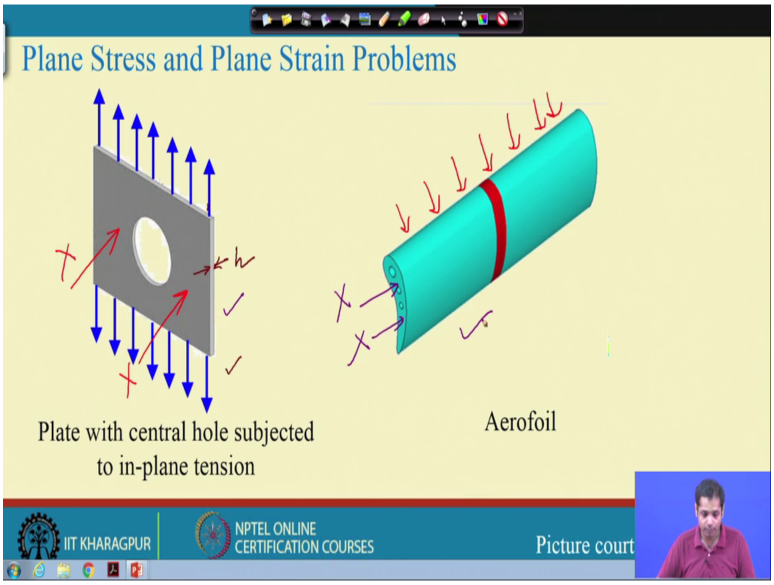

Now, some example of this for instance if you take a plate. If you take a plate and that plate is subjected to the first one. The plate is subjected to in plane loading. This is in plane tension or in plane compression whatever.

So, all these loads this thickness is very small. This thickness is very small this h is very small compared to the other dimension. And in addition to that these loads are acting on this plane. There is no load acting on this on normal to this there is no load acting normal this ok. So, loads are also acting on a plane. Now in this case what is in this is an example of an aerofoil. Here also you will see when we see the vibration of all the dynamics of this aerofoil.

Now, there is no load acting all these loads are acting on this, in this direction one dimension is very large, but there is no load acting in this direction. Normal there no load ok because if you if you apply load in this direction then the things are different you cannot idealized as plane strain and plane strain problems ok.

Now, what happens if you have a system, if you have an object or you are dealing with a problem, which has this kind of characteristic, ok. Now let us see let us start with first this and then we will come to this.

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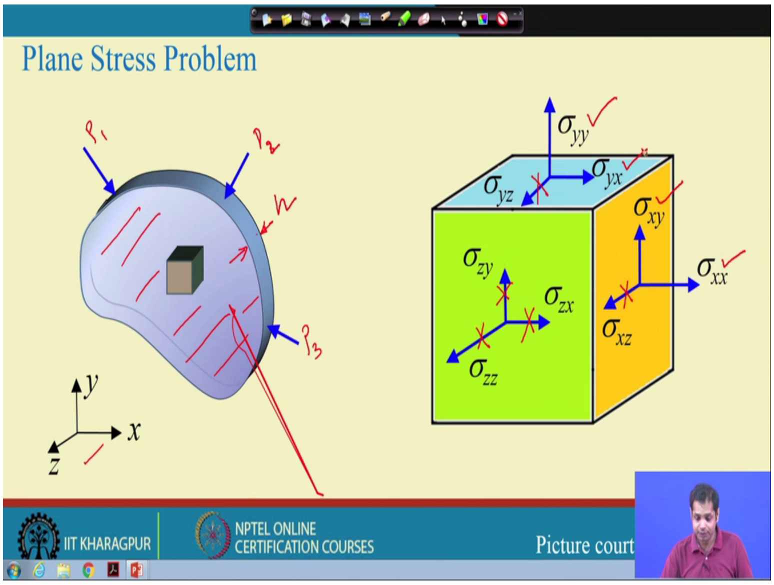

Take any arbitrary object. Suppose this is any arbitrary object whose thickness is small. This is the thickness and this thickness is h is very small ok. Now this is the x y z direction and the loads are acting all these loads say P1, P1 P 2 P 3 all these loads are acting on a plane.

Now, you we take a small object like a small infinitesimal volume element, and then you take this let us draw the let us show the stresses on this volume element.

So, you are familiar with this is how the stresses can be can be represented on this volume element ok. This is coordinate axis. This is the coordinate axis. Now in this case you see these dimension is very small. Now when this thickness is very small now if you recall that we see the point the stresses strain they are point wise description, if you move from one point to other this the their value changes isn't it.

Now, what happens if these thickness is very small, then what happens the change in the stress in that particular direction in this case this the z direction, the change in this stress is very, very small is. So, small that you can assume it is constant ok.

Now So, if we say that the change in stress in this direction is very small. Therefore, we can assume this is constant now if we do not, if this is constant then if you do not have if you do not have a stress on the surface, this if surface is stress free you recall we said that only stresses are acting on this on this, plane there is no stress normal to this plane ok.

So, this normal plane this plane is stress free. Similarly, the opposite side if you see that plane also there is no normal stress, that plane is also stress free. So, we say 2 things one is the thickness is very small and therefore, the change in stress along that direction whatever it could be shear stress, it could be normal stress, but the change in that particular direction the components of stresses we assume we can assume that no change in that stress

And then second thing is all these 2 phases are stress free then what does it mean. Stress free mean there is no stress on that surface what does it mean the stress is constant and on the surface stress is 0. It means that along that direction all the components of stresses are 0.

But if you apply a force on this on this surface for instance, if you take if you take this example of this mobile, now you we apply a load like this and then apply a load like this, this is a plane there is no load in normal direction. We do not have any load in this direction. We have only have load in this direction and this direction and then. So, this surface is stress free and this surface is also stress free. Normal stress in directions stress free means there is no stress in this direction ok.

So, and then along this thickness is very small. So, along this thickness there is no change in stress. The there is no normal stress in this direction or there is no stress in z direction here. There is no stress in z direction here and therefore, all the stresses any point you take within this in this in this object at that point if you see the state of stress, then the all the components of stress in this direction along the thickness direction is 0 or very small that you can neglect ok.

Now, if you if it is then what happens then this stress will be 0 there will be no stress in this direction. So, this stress will be 0. So, this stress will also be 0 and this stress will also be 0. So, all these stresses are normal in the z direction.

Now you see if sigma x z is 0, then sigma z x also has to be 0. Isn't it? That is what we come directly come from the moment balance equation. Now if sigma y z is 0 then this is also 0 ok. So, only stress component that we have if you take any infinitesimal element at any point only stress component we have is sigma x and then sigma y y sigma x y and sigma y x.

So, the stress component we have is all these stresses are gone ok. So, these are the stress components, we have similarly if we take the surface other surface also we have sigma x x sigma x y and this surface if we take there also sigma y y and sigma y x. Now if it is then the stress can be represented here as this.

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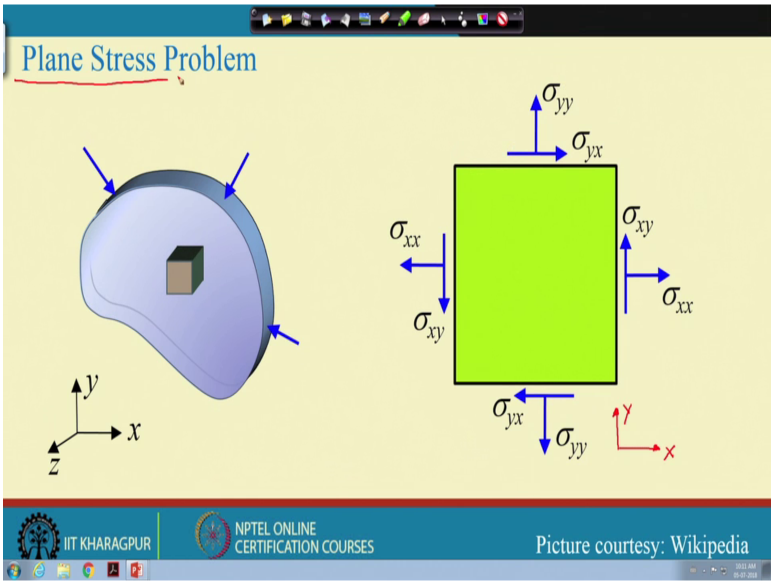

Now, all these stresses this is the z direction normal to this plane is now z direction and this is x, this is x direction and this is y direction ok. So, all these stresses we have only on that plane. On the on this plane, there is no stress normal to this plane. Therefore, the all these stress are confined in that plane. That is why these kind of problem is called plane stress problem. The components of stresses are only acting on a plane. Now this is called plane stress problem.

Now, remember this is also a 3 dimensional problem, but you can idealized as 2 dimensional problem. Now when you idealize as 2 dimensional problems the number of stresses that you have to deal with is now less number of equations are less. So, your solution becomes relatively easier. And when you idealize that you are not if really that assumption is applicable for that particular problem, then the solution you get which is reasonably good reasonably represent the solution of the actual system.

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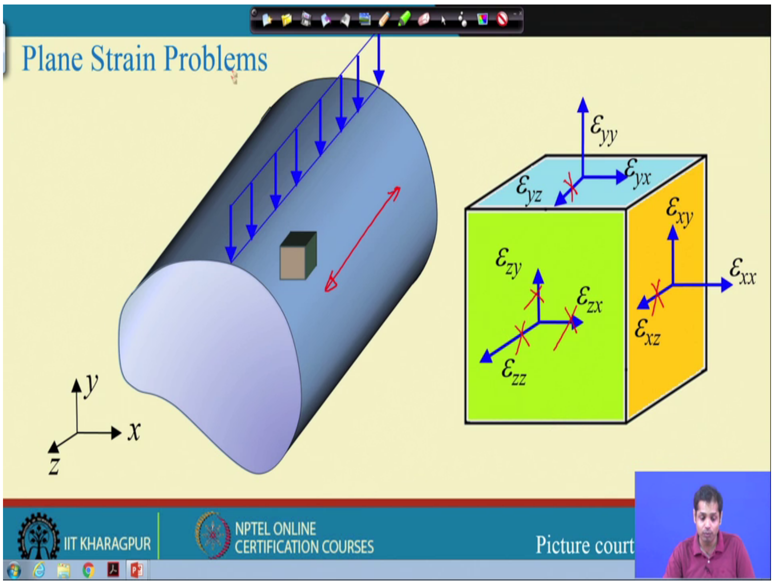

Now similarly take another case this one. Now what happens in this case again the condition is one dimension is small and one dimension large, it is not the condition it is yes. In such cases you may find that, but that you have to decide whether stress and strain in that particular direction can be neglected or not irrespective of the fact whether these dimension is large or these dimension is small. It also depends on what are the loading system what kind of deformation that is taking place.

Now, take an example. Now this another example where these this dimension is very large. Then what happened if you take again a small volume element here, and then on this volume element the strain components will be something like this ok. So, we have strains on different surfaces in different directions. Now what happens since this dimension is very large then the deformation in this direction will be very small isn't it.

Strain in this direction will be very small, as compared to the strain in other directions. So, if you apply a load like this then the material mainly will deform in other 2 direction.

This material may not deform or the deformation in this direction deformation in this direction will be very small deformation in this direction will be very small ok.

And the strain will be in this direction will be very small. And therefore, the strain in this direction is very small. Then what are the stain components we have in this direction we have sigma z z. So, this there will be no strain in this direction. There will be no strain in this direction; there will be no strain in this direction. So, similarly there will be this will also be 0 and then this will also be 0.

So, what are the components eventually you have for this? All these component gone. So, eventually you have these are the strain components you have. And then what are the then the you can represent this as these are the strain components you have ok.

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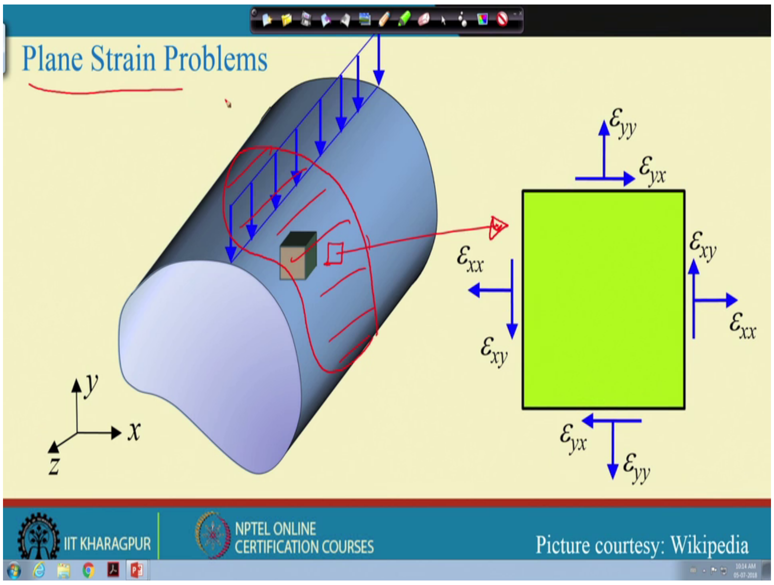

So, this is called plane strain problem because strains are confined on a plane. Now when we analyze it this kind of problem, suppose we have to analyze it then instead of taking it entire problem taking as 3 dimensional problem, we just take a section, any section we take we just take a section. And then this section is analyzed. We take a section and this section is analyzed ok. As 2 dimensional problems so, we have all these stresses at on this on this section, if you take a small if you take a small element this is the representation of the stresses or strain on that element. So, this is called plane strain problem.

Now, for instance, let me give you one example of plane strain problem take a scale and suppose is a cantilever beam ok. And you apply a load like this and this bending it is bending the bending takes place like this ok. Now you see this is the length direction which is the largest in this case and this is the width of the beam and thickness of the beam is this.

Now, but if you apply load like this and if bending takes place like this, you see what happens all the deformation all the deformation here takes place either if you look at across the width the deformation which is there is hardly any deformation takes place. What deformation is takes place only on this plane the beam is bending that you can see that you can see that is on this plane on this plane.

If this is x direction this is y direction this bending is taking place in on xy plane. Whereas, in the z direction, which is along the width direction this there is no deformation taking place ok. So, if you take any either you take this line or you take this line in both cases the bending is same. So, along this direction there is no deformation there is no strain. So, this can be idealize as plane strain problems.

But remember one thing, while idealizing this problem as plane strain problem why did not blindly rely on the fact that one dimension is large or one dimension in small. When you idealize a problem as plane stress or plane strain you see that how the deformation is taking place, what are the loading conditions and then you idealized the problem as plane stress or plane strain problem. Just dimension one is small and other one is large cannot qualify some problem to be plane stress and plane strain. Please keep that in mind ok.

Now, once we have defined what is plane stress and plane strain problems, let us discuss something on we say for plane stress problem we say that stresses in one direction one particular direction is 0. What about strains? Similarly, for a plane strain problem we discuss the strains in a particular direction is 0, then what about stresses let us find out that.

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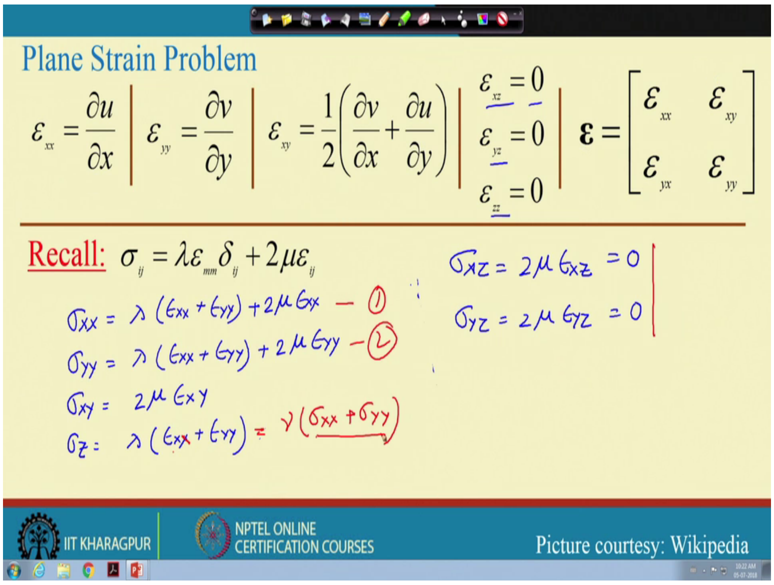

So, let us start with plane strain problem.

So, these are the 3 components of components of strain in plane strain problem. Epsilon x and then epsilon x epsilon y, epsilon x y; all these z components all these strains are 0. So, essentially the strain tensor you can have like this you can write as 2D tensor, but all other components will be then 0.

The third row and third column will be will be 0. So, that is why I did not write it here. So, these are the only non-0 components of strain we have. Then what about stresses? Strains are this that is directly coming from the coming from the idealization. Then what about stresses is that? So, that the stress in if strains are 0, and in a particular direction whether the stresses are 0 or not, that let us find out that. That is very simple we just write, you recall this is the constitutive relation the.

What is the stress strain relation on this stress strain relation you if we write the what are the stress strain stress component, let us write that the stress components are if I have to write sigma x x write sigma x x, sigma x x will be lambda then epsilon m m is epsilon x x plus epsilon y y repetition of y. There will be no epsilon z z because it is 0 plus 2 mu epsilon x x.

So, this is sigma x x right. Then sigma y y sigma y y will be same lambda epsilon x x plus epsilon y y, and then plus 2 mu epsilon y y, and what about sigma x y? Sigma x y will be lambda there will be no lambda this will be 2 mu epsilon x y right; 2 mu epsilon x y. So, these are the 3 stress components. What about sigma z? We know that epsilon z is 0, epsilon z is 0 epsilon y z is equal to epsilon x z is equal to 0.

Let us find out what is epsilon z z. Epsilon z z will be lambda into epsilon x x plus epsilon y y, plus epsilon 2 mu 2 mu epsilon z z, but since epsilon z z is epsilon z z is 0. So, essentially we have these term will vanish. So, your epsilon z will be this.

You see here one important point though in plane strain problem your strain in particular direction is 0, but stress sigma z z is not 0. What about the shear stresses let us find out that epsilon epsilon you delete this and then epsilon sigma you write sigma x z, sigma x z will be 2 mu into epsilon x z. Now epsilon x z is equal to 0 which will be equal to 0. And sigma y z which is equal to 2 mu epsilon y z which is again 0; so, only these 2 components of stresses are 0.

But all other component of stresses are non 0, but remember one thing. Epsilon z z again these if you substitute epsilon x y and epsilon y y x, epsilon x x and epsilon y y from this equation and this equation, these equation take this form nu into epsilon sigma x x plus sigma y y you can do this exercise.

Substitute solve equation number one and equation number 2 for sigma for epsilon x x and epsilon y y, and substitute that in this expression you get this. Now what it shows that though you have sigma z sigma z z, but that sigma z can be obtained from sigma x x and sigma y y. So, therefore, you can idealize this problem as 2 dimensional problem where your unknowns are sigma x x sigma y y and sigma x y.

Now, if we when you have to find out the stresses in other direction in z direction by knowing sigma x x and sigma y y or by knowing epsilon x x and epsilon y y, we can calculate we can compute sigma z z. Therefore, we do not need any explicit equation for sigma z z. Only equation for sigma x x and sigma y y will be enough. So, this is for plane strain problem ok

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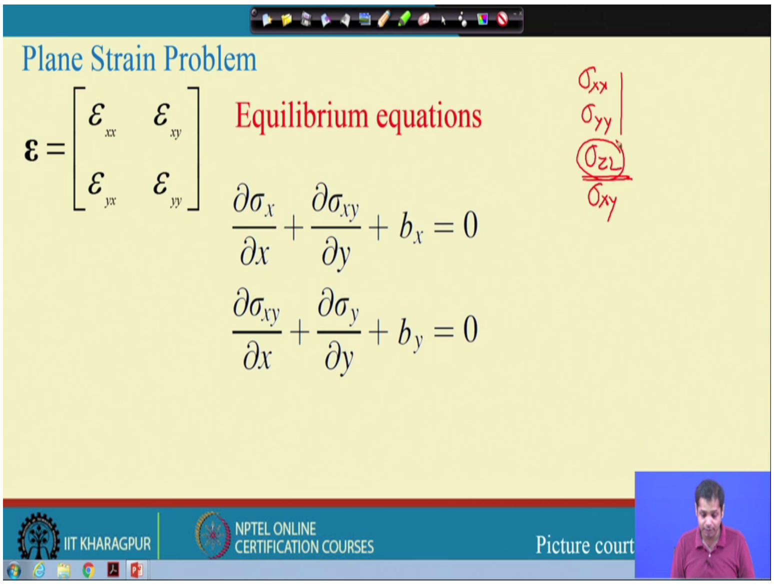

Now, let us the same thing, now once we have that for plane strain problem once we understand that only non 0 these 2 equations are enough, just now we have seen that we have sigma x x sigma y y sigma sigma z z and sigma x y. All these are non 0, but what happen is if we know this sigma z z you can compute this ok. And therefore, we do not need any explicit equation in z direction.

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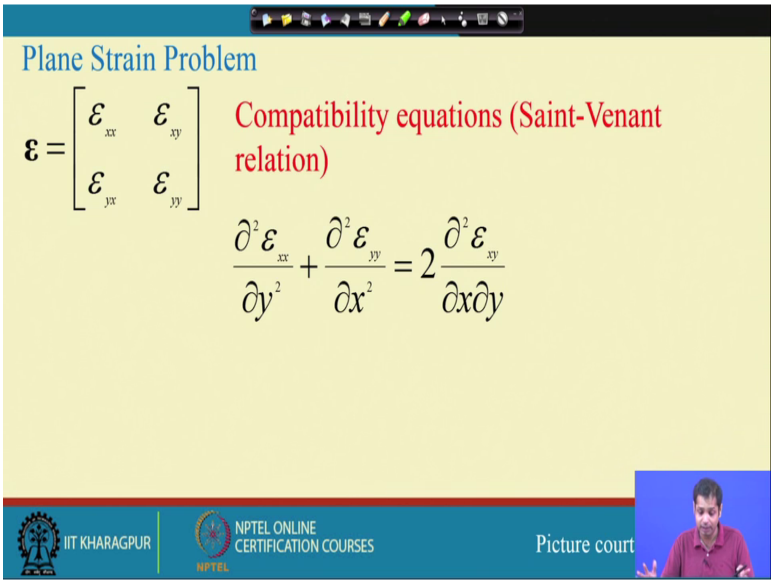

So, this equilibrium equation becomes this for plane strain problem. And then you can try this, then once you have the equilibrium equation then the compatibility equation the

Saint Venant's relation become this, from the 3 dimensional compatibility equation. If you reduce it and the with the only non-0 components of strain epsilon x x y y and z z you get this expression.

So, this is the plane strain problem compatibility equation.

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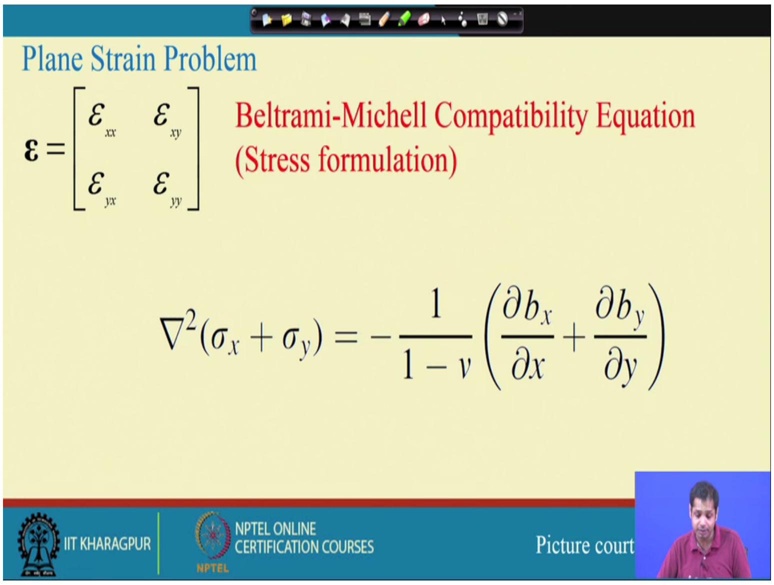

Now, once we combine compatibility equation equilibrium equation and and constitutive relation, we get, we get this the stress formulation this is the relation we get and for if you take here also if it is absence of body force or constant body force, this right hand side become 0. So, eventually you have this. We do not need equation for a sigma sigma z z.

Here it is actually when we write it it is sigma x x. So, this is sigma x x, this is sigma y y ok. So, this become this. And similarly if you write the Navier's equation which is the displacement formulation, then the Navier's equation become this.

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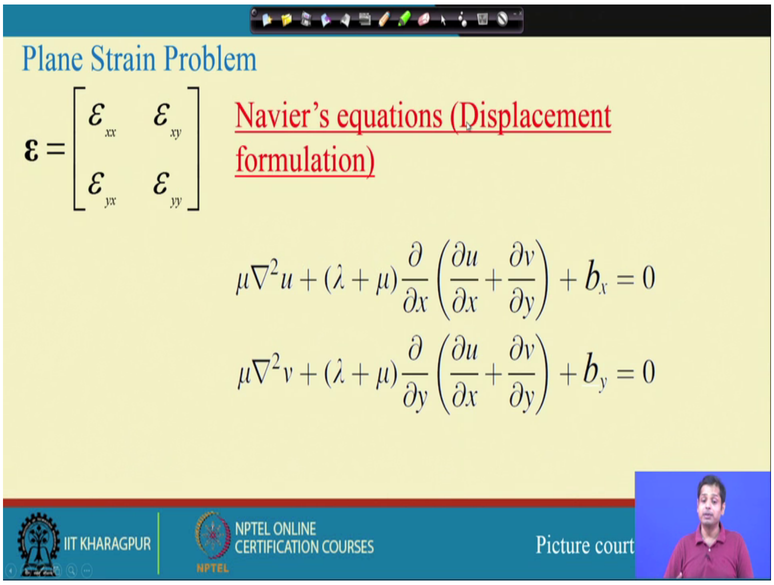

So, what you can do is, once we know what are the strain components and then stress components equilibrium equation and compatibility equation. Then we have a 3 dimensional equation general equation for stress formulation and displacement formulation. In that 3 dimensional formulation substitute the relevant only take the non 0 term and removing the term associated with 0 stresses and strain you get this reduced form.

Now, similarly. So, this is for plane strain.

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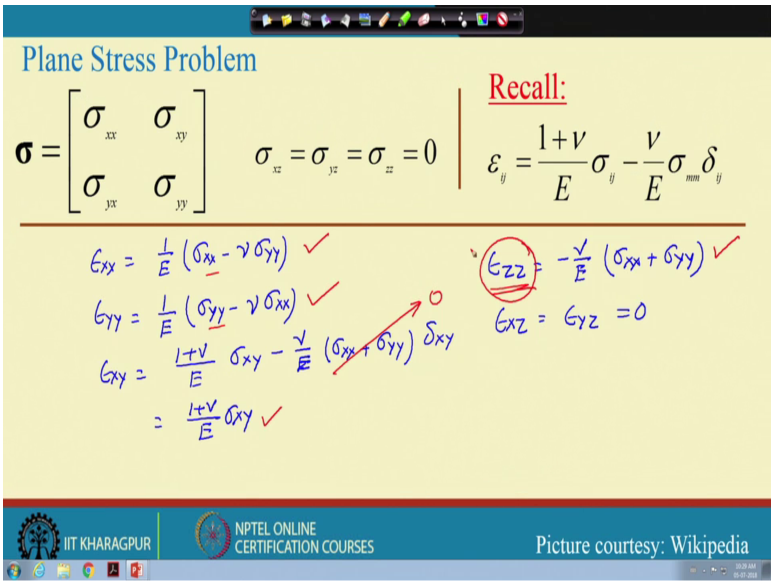

Now, for plane stress problem, we know that the stress all the components of stresses in a particular direction is 0. So, essentially your stress tensor become this. It is not a second or it is not a tensor second order tensor if you have to write it then you have to write the third column and third row as well, but they are 0 that is why it is not shown explicitly.

Now, then the same question come what about strain in this case ok. Now we can compute the strain, the same way we know the relation between strain and stress is this. Recall in the plane stress case we use the stress as a function of strain, but now we have to use strain as a function of stress. So, if you see if you substitute that, if you if you substitute the sigma x x and all these thing what expression we get is epsilon x x we will get 1 by e sigma x x minus mu nu into Poisson's ration sigma y y.

Similarly, epsilon y y, we will get sigma y y minus Poisson's effect sigma x x ok. And similarly epsilon x y it will be 1 plus it will be 2 into if you write epsilon x y; it will be 1 plus nu by e sigma x y sigma x y minus nu by e, into nu by e into your sigma x x plus sigma y y plus sigma z z. This is sigma x x plus sigma y y sigma z z will be 0.

And then what you have essentially is, this term will be this term will be delta i j now this is sigma x y. So, delta i j this term will be 0. So, this will be delta x y. So, delta i j is the Kronecker delta. So, it is true when i is equal to j. So, this term become 0 this term become 0. So, eventually you have this 1 plus nu by e into sigma x y. So, this is for 3 stress components on that xy plane.

What happen to epsilon epsilon epsilon z z. Epsilon z z if you if you substitute that here then what we get is minus nu by e epsilon x x plus epsilon sigma y y. You can try this and then epsilon x z is equal to epsilon y z, epsilon y z all these equal to 0 ok. So, then only non 0 term will be, this is the non 0 term this is the non 0 term and then, this is the non 0 term this is non 0 term, but again the epsilon z z once you know these thing sigma x x sigma y y and so on, you do not have to write do not you do not need an explicit equation for epsilon z z. Epsilon z z can be obtained from this ok.

Now so, you can follow the same approach once you once you know what are the non 0 components of strains and non 0 component of stresses you can simplify these 3 dimensional equation by substituting those non 0 components, and removing the term associated with 0 components and similar to plane stress problem you can derive the simplified form or the plane stress version of the compatibility equation equilibrium equation and then the saint Venant's relation.

Ah you can write the same with for stress formulation displacement formulation Navier's equation which is. So, you can do that exercise that I leave it you please do that exercise. Even for plane strain problem, I just showed you the final final expression. Please verify whether you are getting that expression or not any if you take any book all the derivations are given in the book ok.

So, what we discussed today is we have discussed that though some of the problem though all the problems are 3 dimensional problems some problems can be idealized as 2 dimensions. And there are 2 such idealization one is plane stress idealization plane strain idealization. We defined what is plane stress and plane strain idealization, and we have seen what happened to the stresses and strain in this see idealization and subsequently we have seen what happens to the governing equations for this idealization. Now next what we do is we use that simplified governing equation for plane stress and plane strain problem. We try to solve some problems from some problems which can be idealize as a plane stress and plane strain.

Now, before that one important point you see here particularly for plane stress case, in plane stress epsilon z z is not 0 right. What it means that if you idealize as a problem as plane stress, you have load suppose this a plane stress problem and you apply load normal to this, but since the stresses in these direction is 0, but strain is not 0 in this direction. You have a normal strain in this direction shear strain in this all the shear strain in this direction is 0, but you have normal strain on this direction. When you have a normal strain in direction, means you have some deformation in this direction which deformation is called out of plane deformation. Like it is a plane, but normal to this plane you have some deformation because the strain is not 0 there ok. So, this is out of plane deformation.

So, the deform shape longer remain just a plane isn't it. So, purely this problem cannot be said at least the deform configuration cannot be said, cannot be coined as a plane problem there is still some out of plane deformation. Next class what we do is we will discuss more on this epsilon z z, and then see how these epsilon z z can be treated. What is the further refinement of plane stress problems, some variants of some different formulation of plane stress problem; because of this out of plane deformation epsilon z z that we will discuss in the next class. And then in from the third class of this week we will start solving some examples on plane stress and plane strain problems. Ok I stop here today, see you in the next class.

Thank you.