Theory of Elasticity
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Lecture – 29
Solution of Boundary Value Problems (Contd.)

Hello everyone, this is the third lecture of this week; today we will discuss we will continue discussing Solution of Boundary Value Problems, but today's topic is airy stress function.

You see what we have done so far is we have derived the equations for plane strain and plane stress problems, governing equations and those equations are written in terms of stresses and also the governing equations are written in terms of displacements ok. We are almost when the verge of solving those equations, applying those equations for solution of different boundary value problems.

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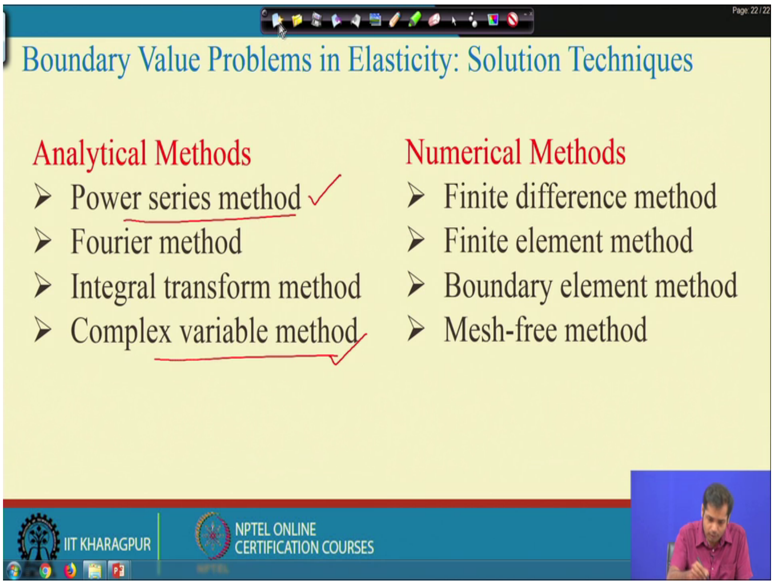

Now if you recall, there are there are many solution techniques available they can be every methods have their own advantages and limitations; we will not discuss that in detail in this course. Instead what we do is we will try to understand two methods one is power series method and then towards the end of this course we will spend some time on complex variable method.

We will we will be having very disk very brief discussion on numerical methods; what are the available numerical techniques towards the end of this of this course. In most of the time, we will be spending on power series method and then complex variable method ok.

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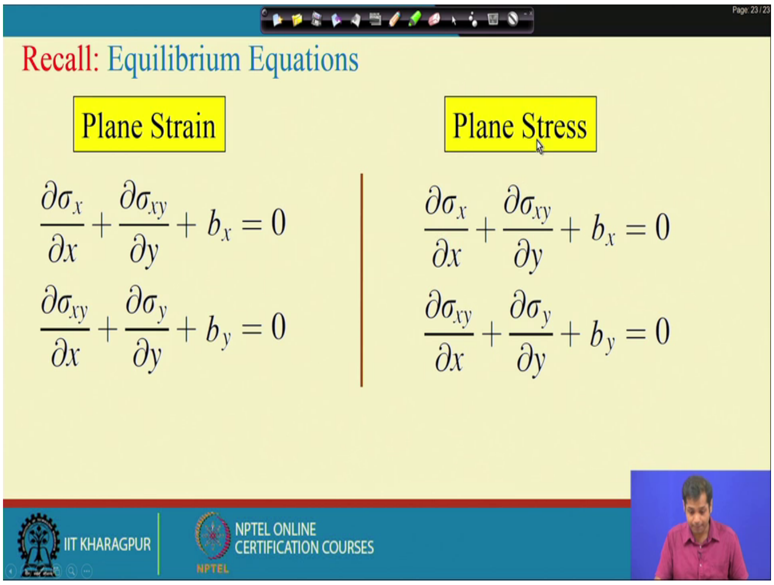

Now let us first let us let us have a quick review of where do we stand right now. Now you see these are the equilibrium equations for plane strain and plane stress problem; equilibrium equations remain same.

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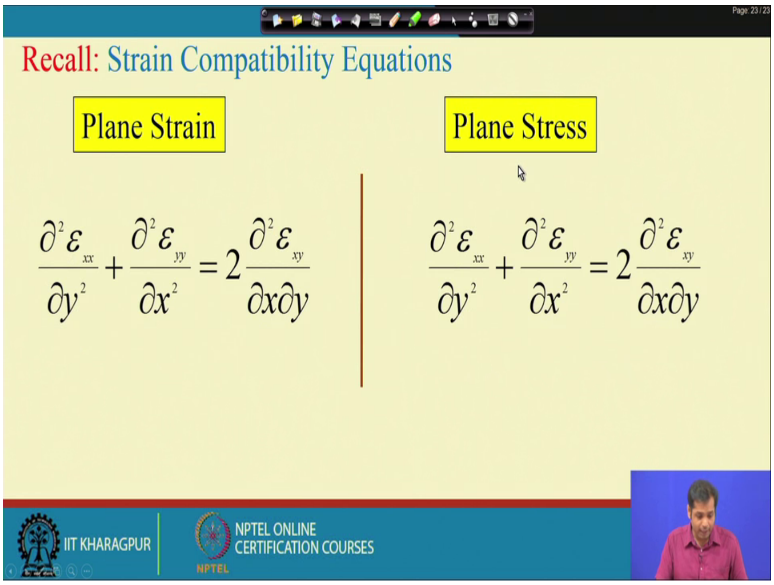

And then we have the compatibility equations, strain compatibility equation and again for plane strain and plane stress problem; this strain compatibility equation remain same.

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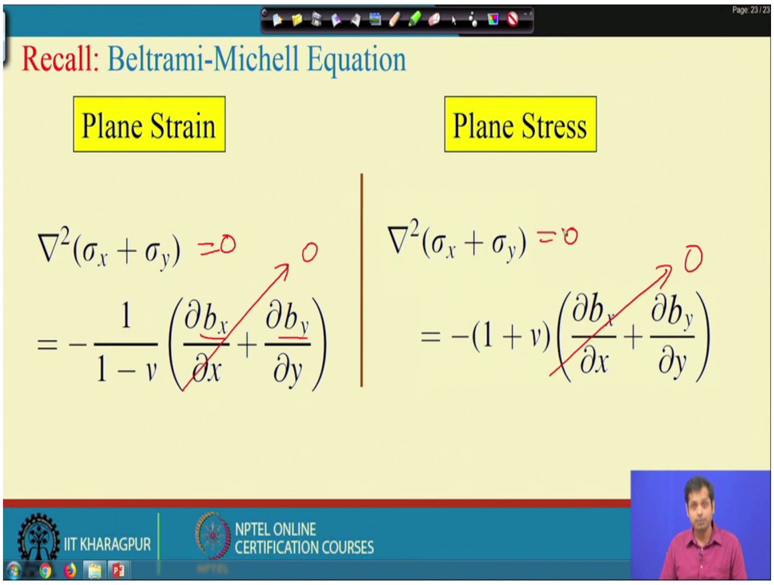

Now when we write the compatibility equation in terms of stresses; then the plane strain and plane stress becomes like this, where b is the b b x and b y are the body forces. Now, if it is the body force is 0 or body force is constant; then what happen? This part become 0 and this part become the entire part become 0 and the equation become this is equal to 0 and this equation is equal to 0. So, for plane stress and plane strain problem if there is no body force and or the constant body force; then the again this equation is also same.

Now, next so these are the equations that we have already discussed in previous classes; derivation of these equations and the assumption which constitutes the basis of the derivations.

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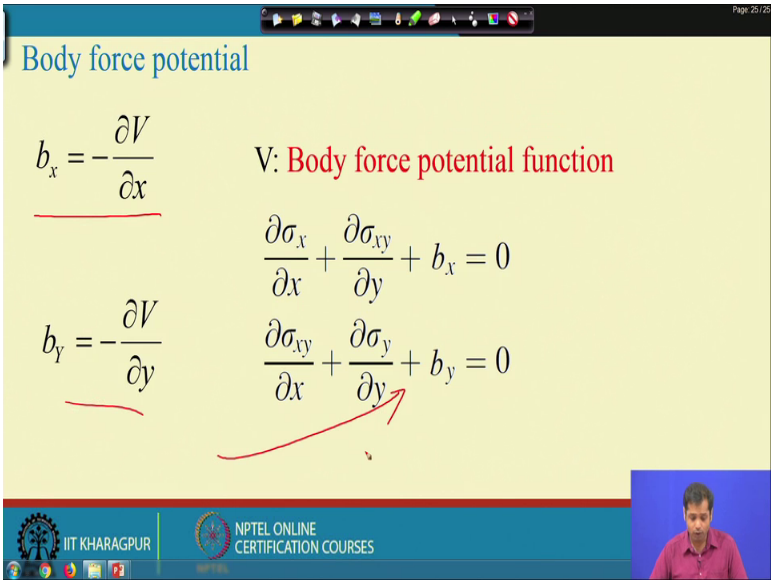

Let us now define one potential called body force potential and which is defined as like this. Suppose if we if V is the body force potential then if we differentiate V with respect to any coordinate axis gives the body force in that direction.

So, if we differentiate with y it requires; it gives me b y. What we do here now is let us not bother about what is that potential and how to get that potential for different kinds of body forces. Whenever we come across that, whenever we apply these all the techniques that we are going to learn to solve some problems and in that problem whenever we encounter any body force, then we will come to this point what is that potential and depending on the body force and the direction how to construct that potential.

Suppose we have the potential V which is body force potential and derivative of that potential gives us the body forces in respective direction. Now if it is then if we substitute now this is the equilibrium equation, if you recall, now if we substitute this body force potential into this equation; this potential into this equation then what we have is finally, we have an equation like this ok.

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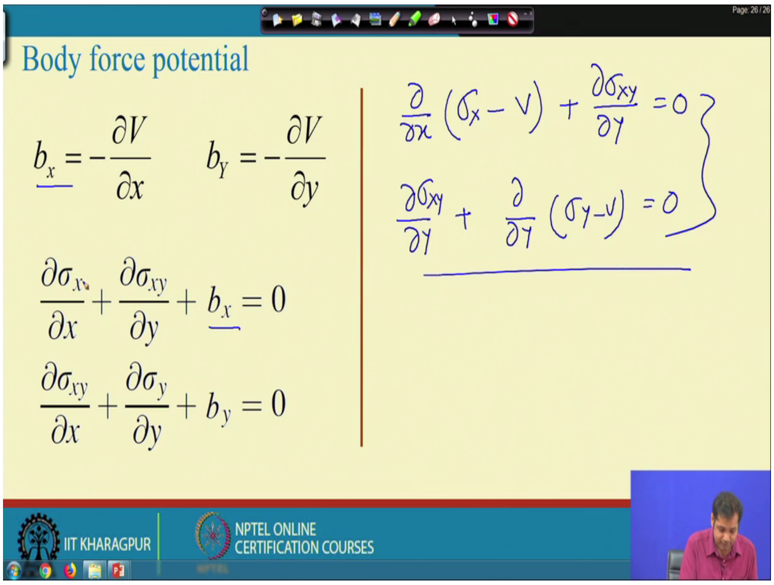

Now, what we have if we do it then the first term becomes this thing; if we used different colour. So, if we substitute b x in this equation then we have del del x, the first equation that becomes sigma x minus V plus del sigma x y by del y that is equal to 0; 0 means we assume ok.

And then del del sigma x y x y del y plus del del y sigma y minus V that is is equal to 0. So, if we apply the body force potential into the equilibrium equation; we have this please correct these are all sigma x x, this is sigma y y and so on that is how we represented stress components ok.

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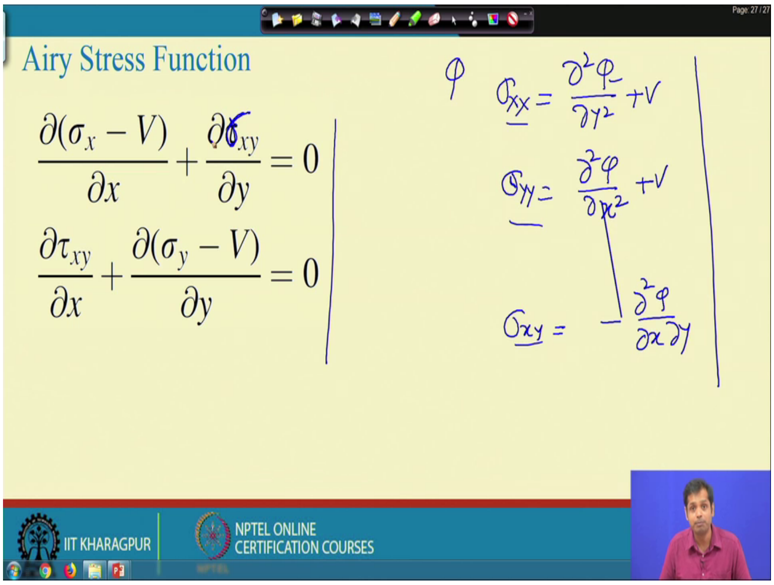

Now, once we have that then this is the your equations that we have. Now let us introduce another important term which is airy stress function.

You see here this equation the governing equations are written in terms of stresses ok. This is the equilibrium equation then we have the compatibility equation as well. Now if we have to directly solve it then what are the how many unknowns we have? The unknowns are 3 unknown in plane stress and plane strain plane problems. We have sigma x x and then sigma x y and then sigma y; this should be sigma this should be sigma x sig sigma x y ok.

Now let us define a potential where is potential phi such that that the how to whether that potential exist or not for a given problem; how to a construct that potential? We will see that suppose say phi is a potential which is defined as sigma x x is equal to del 2 phi del y 2 plus V. And then sigma y y is equal to del 2 phi del x 2 del x 2 plus V and then sigma x y is equal to minus del 2 phi del x del y.

Now, you see if I define a potential phi and its relation with stress component sigma x x, sigma y y and sigma x y; you check that this phi if that phi exist if that phi if we substitute in these equation phi automatically satisfy this governing equation. You can try this exercise; now what we have done is say phi is the potential which is related to stresses different component of stresses this. If we have some phi then that phi automatically satisfy the equilibrium equation, if we substitute this substitute sigma x x, sigma y y and sigma x y from this into this equation you will see that equilibrium equations are satisfied.

Now, if we have some phi equilibrium equations are satisfied now in addition to that if we make that phi satisfy the compatibility equation; then we can say that phi carries the information about the solution of this 2 equation. You see governing equations are 2; one is equilibrium equation and the second one is compatibility equation. By choosing phi like this governing equations are automatically governing equations are already satisfied.

Now, next job if we can show if we can make this phi, if we can find out a condition such that the phi can satisfy if that condition is met that phi also satisfy the compatibility equation; then we can easily say that phi satisfy both the conditions compatibility and equilibrium equation. And therefore, phi carries the information about the solution of this governing equations ok.

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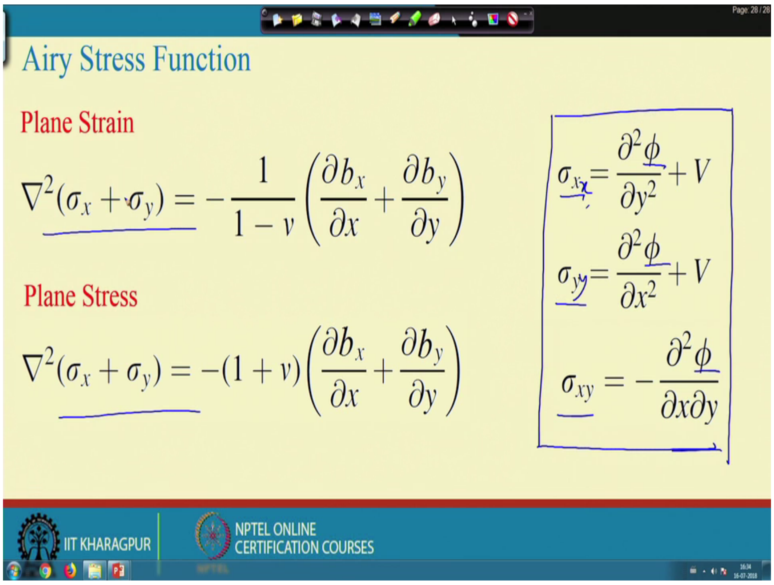

Now to make it clear suppose if you recall these are the governing equations, these are this is the governing equation for plane strain and this is the governing equation for plane stress problem ok.

Now we have the phi is the stress potential, it is called stress potential because it is related to stresses. And the sigma x x sigma y y and sigma x y are related to phi as this and we have already checked; we have not done that exercise here, but you can do that.

We have already checked that or discussed that these definition of phi this relation of phi satisfies the equilibrium equation; now here we have the compatibility equation written in terms of stresses.

What next we do is we substitute these sigma; sigma x x and sigma y y and sigma x x y in these equation and find out that what condition phi needs to be satisfy such that the equilibrium conditions are satisfied. Let us do that exercise.

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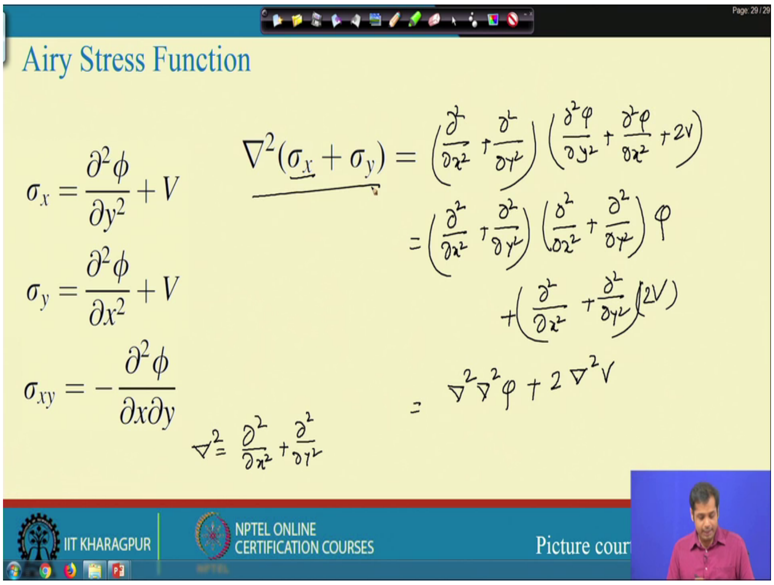

So, now if we take the first term in this equation; if we take the first term in this equation which is same for plane strain and plane stress problem, then what we have is this. Now, if you recall del 2 is what? Del 2 is the; now del 2 is if you recall del 2 is del 2 del x 2 in 2 dimension plus del 2 del y 2.

So, what does it what it becomes is del 2; del 2 del x 2 plus del 2 del y 2 I could have kept it del 2 as well, but let us write it in detail. And then we have sigma x; sigma x is this, sigma y is this then we have del 2 phi del x del y 2 plus del 2 phi del x 2 and then plus 2 V; this is the first term right and this becomes what? This becomes that del 2 del 2 del x 2 plus del 2 del y 2 of again del 2 del x 2 plus del 2; you can check it del y 2 phi plus del 2 del x 2 plus del 2 del y 2 of V.

So, this is essentially del 2 and then this is again del 2 of phi and plus there will be 2 here there will be 2 V; so, this becomes 2 del 2 del 2 V ok. So, now; so, this is the first term ok; similarly if we substitute, substitute this, substitute these expression this velocity potential expression for velocity potential means b what is the expression for b y and what is the expression for b x if we recall that b x was b x was minus del V del x and b y was minus del V del y.

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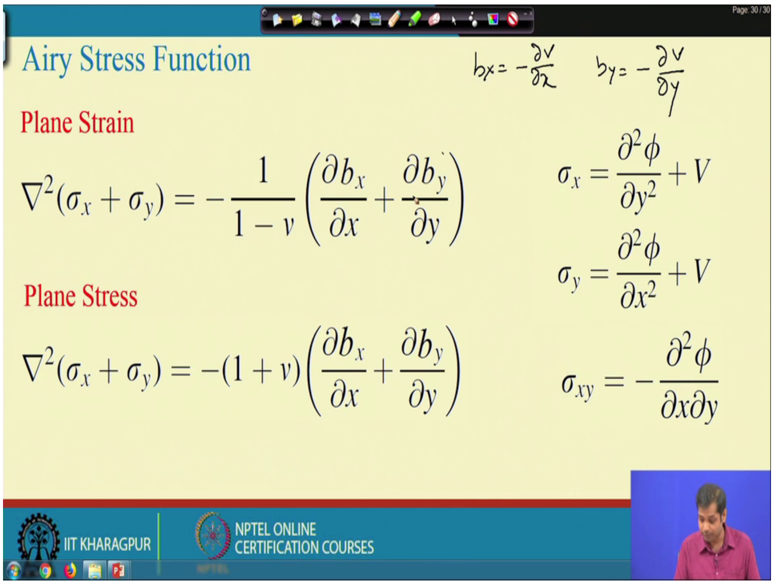

If we substitute that into this equation; into right hand side of this equation and then do the manipulation put. Then what we have finally, we have this is the equation we have finally, this.

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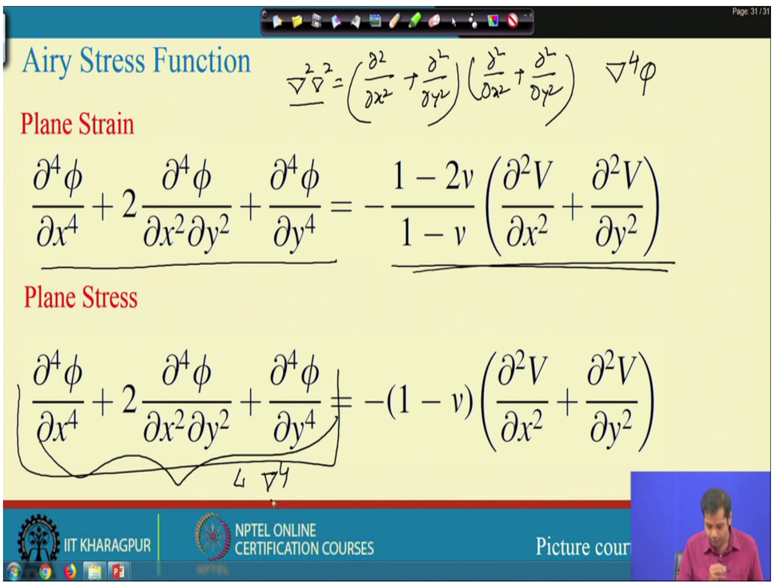

What is this term? If you this term was if you recall this term is essentially the entire term is del 2 del 2 which was del 2 del x 2 plus del 2 del y 2 square or del 2 del x 2 plus del 2 del y 2 ok.

And then this is the; this is the right hand side that exercise we have not done here, but you can do that the same way. So, in both the cases plane strain and plane strain problem; the left hand side remain same and then the right hand side is different depending on the plane stress and plane strain problem. Now, these entire thing which is del 2 del 2 this can be written as del 4 phi; this del 4 operator is the this operator is del 4 operator del 4 operator ok.

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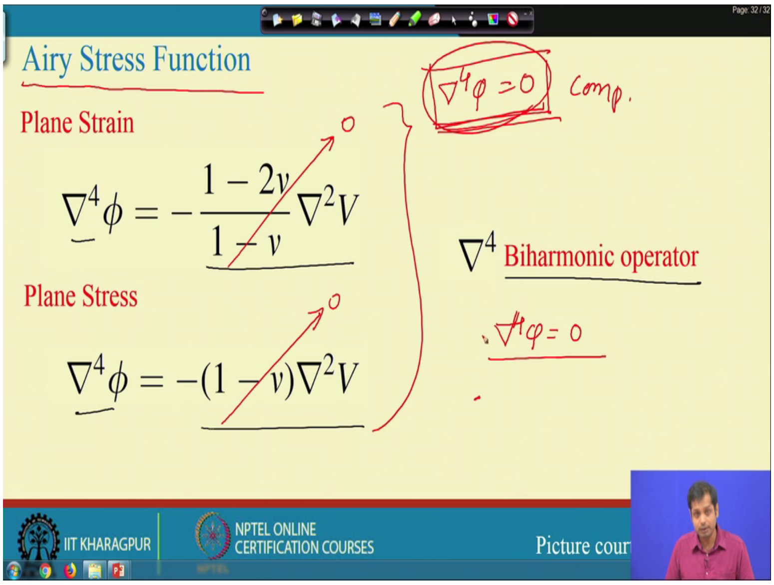

So, the essentially then equation becomes in terms of operator we have if we have to write essentially the equation become this. Your del 4 del 4 phi is equal to this and del 4 phi is equal to this and del 4 is called biharmonic operator.

Now, again if V is equal to if there is no body force then this term become 0, this term become 0 and this term become 0 and then essentially we are left with del 4 phi is equal to 0. Whether it is plane stress or plane strain problem in both the cases this is equal to 0, this equation is called biharmonic equation.

Now this equation gives tells us a very important very interesting story and probably if you recall in one of the classes we discussed that as well. You see before coming to that point what we have done here is we have what advantage we have here is see essentially we had 3 unknowns sigma x x sigma y y and sigma x y and the 3 equations equilibrium equations 2 and one compatibility equation.

Now if we have to solve this 3 equation we have to if we have to directly solve this equation, directly find out these all the unknown we have to solve 3 equations. Now here what we did is we combined 3 equations and now we have just one equation and therefore, we have just one unknown, this unknown is phi. And the how the phi is defined? This phi is defined in such a way the relation between phi and the stresses that defined in such a way that equilibrium condition is automatically satisfied. And then del 4 phi is essentially what? Del 4 phi is essentially the compatibility condition right; it is the, if phi satisfy this equation then the compatibility condition automatic; compatibility condition is also satisfied.

By construction phi satisfies equilibrium condition in addition to that if phi satisfy this equilibrium this biharmonic equation means compatibility condition is also satisfied and phi is this solution phi carries the information about the solution of this problem. Now for us now this phi is called Airy's stress function ok.

Now what we do is instead of solving these equilibrium equations and compatibility equation; we solve this we have just one unknown, but again there is no free lunch, if we have to reduce the number of unknown we have done it, but at the cost of increasing the order of differential equation. Here you see now we have to deal with fourth order differential equation and we will see that how it can be, how it is to be solved.

Now, now from onwards our equations governing equation is this is our governing equation, del 4 phi is equal to 0; if the body force is 0 otherwise the governing equation would be this and these equation will be subjected to some boundary conditions. We will solve this equation, find out phi; once we have phi satisfy this equation and all the boundary conditions, then we know the relation between phi and stresses.

From the relation we can determine the stresses, once we have the stresses we know the relation constitutive relation which is the relation between stress and strain from that constitutive relation we can find out strains. Once we have strain which is the derivative of displacement by integrating strain we can have displacement and then we have the complete solution of the problem ok.

Now here one important point we you observed before also when we solve this right hand side is 0 irrespective of the plane stress and plane strain; if the body force is 0. And another important point is the solution of these solution does not the phi does not carry any information about the material property. It means that and phi is directly measure of the stress field it means that if there is no body force, in absence of body force the distribution of this distribution of stresses they do not depend on the material property.

But went from stress we get the strain then the material constitutive property constitutive relation will come into picture. It means that if we take 2 identical sample identical sample means geometry and boundary conditions same, but one sample is be pair of steel and another one is say be pair of some other materials say aluminium or glass or wood. And if we apply if both are having same boundary condition same dimension and the same external loading stress loading or the traction boundary condition; then in both the cases the distribution of the stress the stress field will remain same; whether it is steel wood or any other material the stress will remain same.

And that is the important that is a very important property that we use in photo elasticity. And the detail of that photo elasticity how to exploit this equation in experimental in stress analysis that we discuss towards the end of this one of the lecture towards the end of this course.

Now what we do is once we have this; next is this is next what we do is once next class we start with this equation and then see how to solve this equation what are the different techniques or that we already discussed that we will be using power series method. We will we will take an example and demonstrate how these airy stress function can be used to solve the boundary value problems in elasticity ok. So, next class we will be demonstrating this through one example see you in the next class.

Thank you.