Theory of Elasticity
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Lecture – 03
Introduction to Tensor (Contd.)

Welcome this is third lecture of module 1 ah. So, in this lecture we will review tensor algebra and we will understand some of the operations required to understand this course ah properly the tensor operations. So, in the last class, we have introduced the tensors and what is zero'th order tensor, what is first order tensor, what is second order tensor and probably I have also told what is the 4th order tensor. So, for instance we have also seen the third order tensor; that is the Levi Civita symbol or permutation symbol. So, in this lecture we will review some of the tensor algebra which we have already seen in the last class. So, we will see it in a little more detail.

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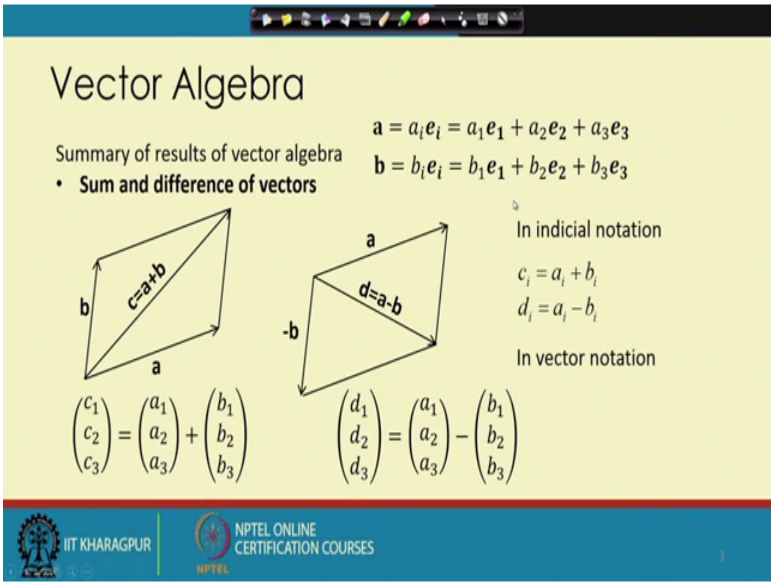

So, we will start with a very simple thing So we will start with vector algebra. So, as we have seen the, we have introduced the indicial notations ah; so any two vector a and b can be written in this form and then these the geometric interpretation of vector sum and vector subtraction is this; so which is a which is very well known to us from our basic vector knowledge.

So, in a component form we know how to write it. So c 1 c 2 c 3 if it is the sum, then it is component wise sum, a 1 a 2 a 3 is added with b 1 b 2 b 3, so a 1 plus b 1 a 2 plus b 2 and a 3 plus b 3. So, similarly this subtraction is also the sum of negative of b 1 with a 1 ah with a; so, a 1 minus b 1 a 2 minus b 2 a 3 minus b 3 and so on.

So, any initial notation, this is written in a very compact form, this is c i is ai plus bi, di is ai minus bi. So, this notation implied that this is the sum of two vectors a and b. So, ah this we know it from our last lecture.

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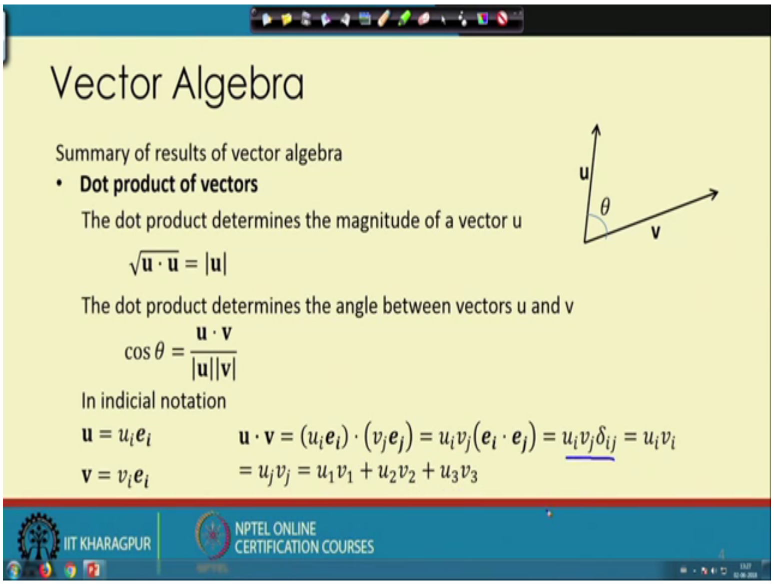

Now, similarly we have introduced the dot product. So, geometric interpretation of dot product is all of us know, if the angle between two vectors are theta, then dot product is nothing, but the magnitude of u magnitude of v and then cos of theta. So, in a ah essentially the dot product determines the magnitude of the vector. So, then angle between the vector can also be determined from the dot product, because we know the expression for dot product. So but if you do vector in a initial notation, the dot product in indicial notation that also we have seen in the last class; so which is ui v uj vj ah.

So, now, ah here to do the dot product we need to take the dot product between two; this is vector e i and e j which is we have seen the delta i j, because ei dot e j will be 1, only when i equals to j; so otherwise it is 0.

So, if i not equals to j then ah it is 0, this quantity will be 0. So, I can simply write this is ui vi or uj v j when i and j are same. So, this if we expand it, so u 1 v 1 plus c 2 v 2 3 b 3, so this becomes easy in writing with the indicial notation.

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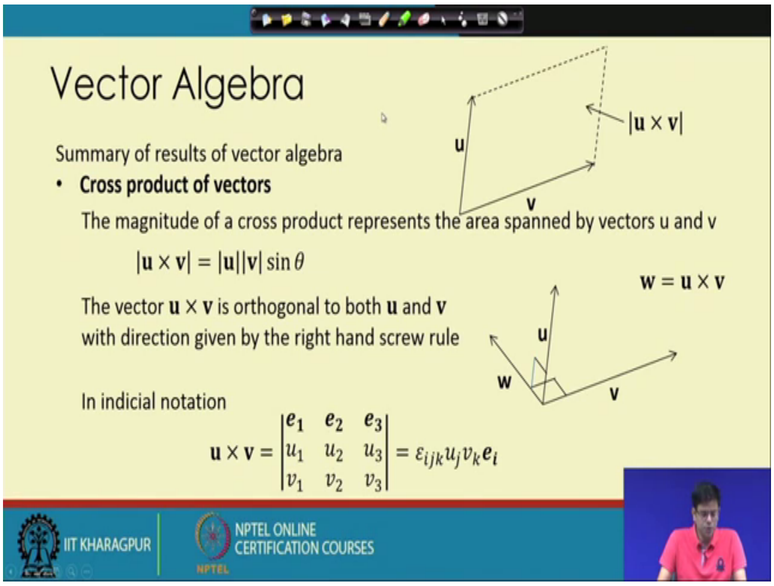

Now, if we ah similarly introduced cross product. So, cross product of a vector also you know which represents the this area. So, ah u cross v is mod of u plus mod of v or the magnitude of u or magnitude of v is then sine theta and this is, this is a vector. So, u and v is orthogonal to both u and v and the direction is given by the right hand corkscrews.

So, this is very well established result from our basic vector knowledge. So, if you write u cross v, so this is the determinant of this u cross v and if we write it in terms of Levi Civita or ah permutation symbol, so if epsilon ij k is u j v k e i. So, this also we have seen in the previous lecture how to represent it through a; in a compact from through a Levi Civita symbol.

So, now, this epsilon i j k or the epsilon is a third order tensor which has 27 component we have seen. So, among 27 components 3 components are plus 1, 3 components are minus 1 and other components are 0 that we have seen in the last class. So, now once we able to write it in a very compact from our proofs and our ah representation will be in a very compact from through the indicial notation.

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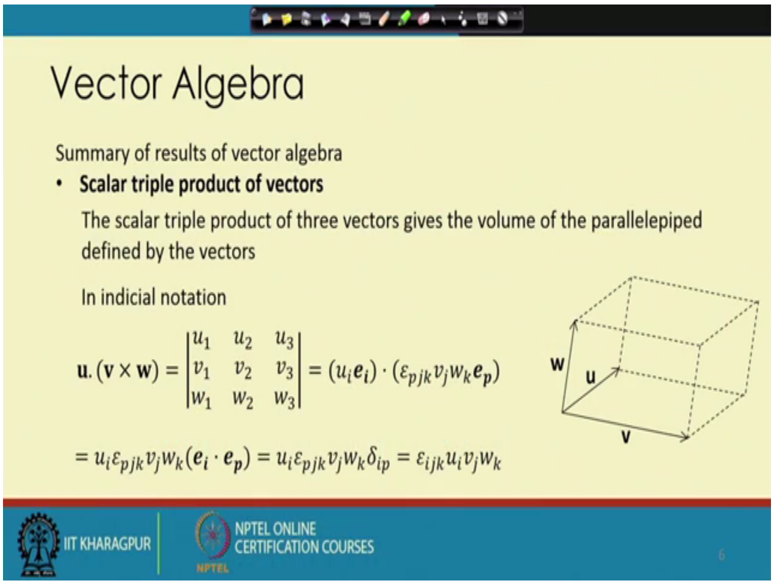

So, another thing we have seen in the last class which is also the scalar triple product. So, it is very well known, the physical meaning of it, it represents the volume of the parallelepiped. So, this is already from our knowledge of basic vector algebra. So, if we want to do, this will be dot; so this if we want to do dot product between two vectors that expression we know and if dot 1 of the dot product is expressed in terms of a cross product of other two vectors; that is v cross w, then this becomes u dot v cross w.

So, now if we write it in a proper manner, so, the u is represented ui e i and then any two vector v and w is represented again with the third order tensor epsilon. So, which is epsilon p jk v jw k ep ep, and so if we now, these alls are scalars. So, these alls are scalars. So, we can take it out and then do the dot product operation between ei and ep. So, if we do this dot product operation. So, as we know from our relations, these ei and ep will be 1 only when i and p becomes equal.

So, this leads to delta i p and then i can confidently change the symbol, because delta i p I can write it 1 when i equals to p. So, I can just simply substitute here i j k and then vj ah wk. So, in this p I am assuming it is equals to i. So, then only this expression will value the; otherwise this delta i p, when i and p are different will be 0. So, finally, this quantity will be 0, so this will be non-zero only when i equals to p. So, I can substitute conveniently ap equals to i. So, this leads to a compact notation and this we have seen also in the last class.

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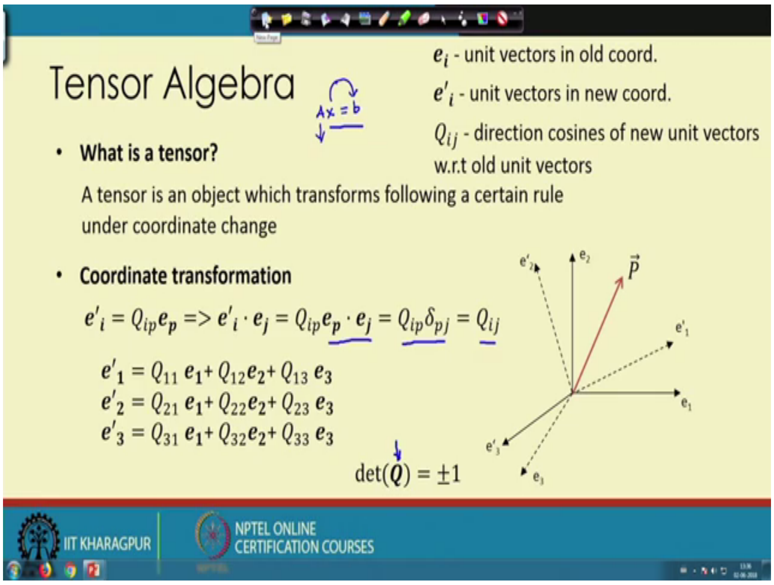

So, now we have also introduced what is tensors. So, here I will give you another definition of a tensor. The tensor is an object which transform following a certain rule under coordinate change. So, if you have seen our basic definition of tensors, the tensors is a linear transformation which transforms a some quantities to other, same to another quantity.

For instance the second order tensor transform a vector space to another vector space. Similarly we can, for instance the 4th order tensor; the 4th order tensor transforms a second order tensor to another second order tensor. So, just like we have introduced matrixes matrices or the square matrices or the matrix vector equations A x equals to b. So, if we just write A x equals to b. So, this is a second order tensor which transforms these vector x to vector b. So, if x vector belongs to a vector space and b belongs to a another vector space, then we can say this vector a transforms one vector space to another vector space.

So, we can view it ah. If you remember from our the solid mechanics knowledge that coordinate transformation ah which we have also briefly discussed in the last class that when we transform from one coordinate to another coordinate, then we need to multiply a rotation matrix. So, how to calculate this rotation matrix also we know because rotation matrix is composed of direction cosines between two coordinate system.

For instance this is a coordinate transformation a example, if e 1 e 2 and e 3 are initial coordinate system or the old coordinate system and then if we rotate it arbitrarily with, and the new coordinate system becomes e 1 prime e 2 prime e 3 prime and if I; there is a vector p, then how should i represent p; that is the object. So and how the p will be in terms of e 1 e 2 e 3 system and e 1 prime e 2 and e 3 prime system.

So, now, ah we need to define first how these e 1 and e 1 prime is related. So, if e 1 and e 1 prime from our basic definition of a tensor, we assume that u 1 prime is transformed through a or ep transform to a tensor e i p through this . Now if we just pre multiply with or post multiply or take the dot product with the ej, then we can simply prove that ei prime dot ej, it is Q ij. The proof is very simple, because this will be delta p j and then delta Qi p delta p j will be when p equals to j, then Qi 0.

So, similarly we can write that e 1 prime is Q 1 1 e 1 Q 1 2 e 2 and Q 1 3 e 3 and so on. So, this defines our the component of rotation matrix. We will see through an example. So, but interestingly one must notice that on and one can prove from these, that these determinant of this matrix Q which is Q i j, component wise it is Q i j is actually the plus minus 1. So, this kind of transformation is known as the orthogonal transformation. So, this matrix q belongs to orthogonal set of orthogonal matrices. So, this is an orthogonal transformation. So, it is better to remember this thing and we have also seen this earlier in our coordinate transformation part.

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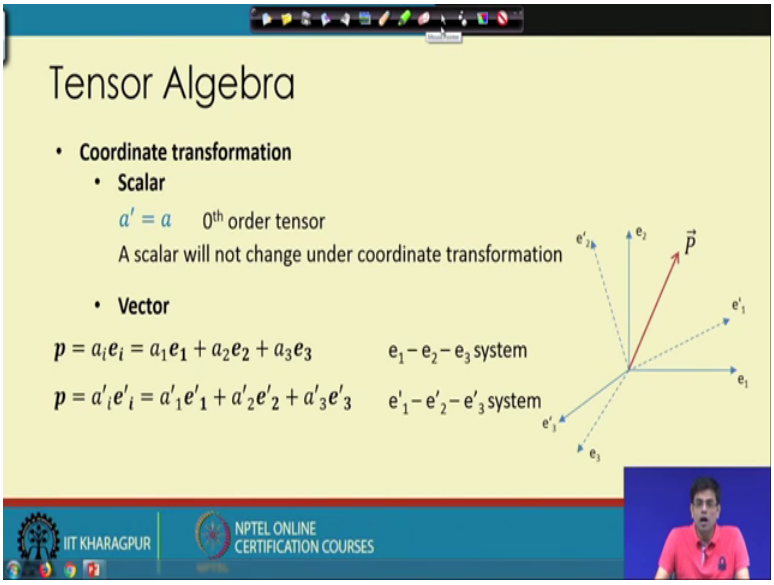

Now, our we will just proceed very simple thing, if I want to transform a scalar. So, scalar is a quantity which has only magnitude, there is no direction. So, scalar can be transformed to any coordinate system and which does not affect.

So, this is that is why we see the scalar is a 0'th order tensor. So, scalar, we will not change under the quadrant translation. Now simply the vector, vector if we just write it in this two different coordinate system ai e i and ai prime e i prime, so it will be like this. So, this and how ei and ei prime is related that we know from the previous slide. So, we can just conveniently transform the vectors from one coordinate system to another coordinate system.

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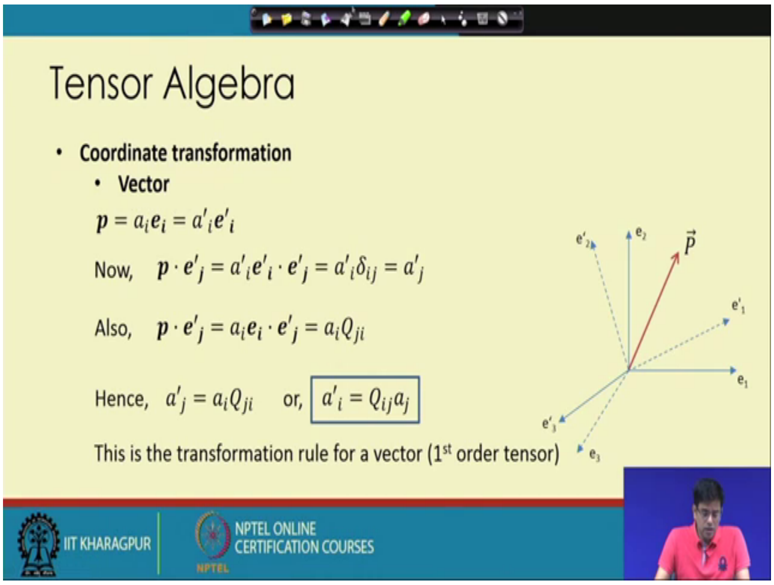

Now, if we ah really want to derive that, then it is very simple p i can write a i e i and p also we can write in ai prime ei prime. So, if we take dot product and then this proof is very simple; actually if we take dot product with the ej prime, then it will be a ij a prime j and then we can relate ei and ej prime with the Q ji, and so a and the component wise, how it will look? It will be derived from the, it will be obtained from the Q ij components; so that we have seen in the previous class.

So, finally, you see these components of rotation matrix rotation matrix is multiplied only once here because vector is a one direction of quantity; so it has one direction. So, that is why we say it is a first order tensor.

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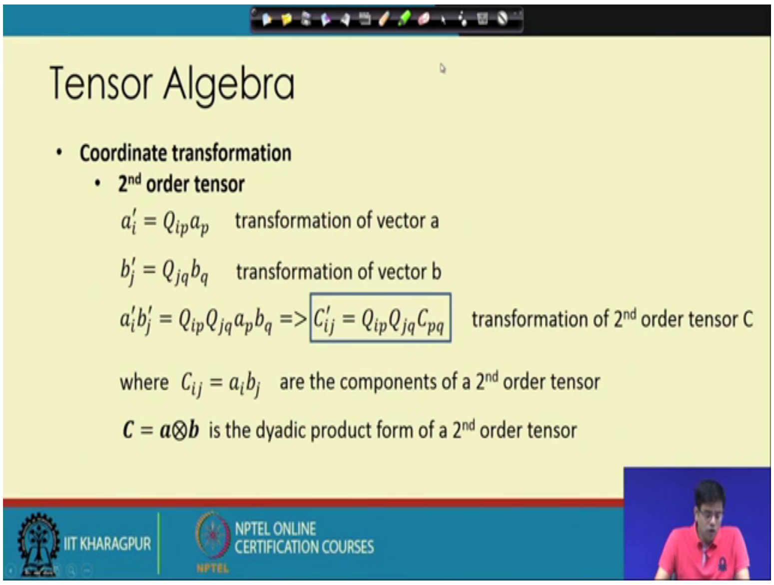

So, similarly if I, if want to transform a tensor, how it will look? So, tensor transformation, we will see it little more ah detail. So, if I want to write ai prime with the ah ap quantity; that is prime coordinate system and the non prime coordinate system, it is related with that. So, similarly b prime b bj prime is related to this. This we have learnt from the previous slide. So, now if I want to transform the tensor obtained by these two vectors, then i a i prime i a a a i prime and b j prime, I can just simply multiply these quantities here and we have to keep in mind that this is two times.

So, now if I write this if I say these ai prime and bj prime C ij prime, then this looks like this. So, this is a transformation formula for the second order tensor. Now if you see carefully this when it becomes a tensor, so it has two, two times it is multiplied with the component of the orthogonal tensor Q.

Now, this Cij how it looks? So, that also we have defined in the last class. So, it is the direct product which is ai bj. Now this is in a tensorial notation, this is in the indicial notation, so it is known as the dyadic protocol or sometimes it is also known as the tensor product. So, now we know how to transform a tensor to a another tensor under coordinate transformation. So, the important thing to remember here that since the second order tensor have two directional ah part; so it has two direction. For instance the stress tensor, stress tensor represent how you always write sigma xy sigma xx like this.

So, two subscript you always write; one subscript is for the direction of normal, another subscript is for the direction on which the stress is working, so it has a two direction. So, since it has a two direction, it will be two times the orthogonal component of the orthogonal will be, orthogonal matrix will be multiplied.; so this as we should remember.

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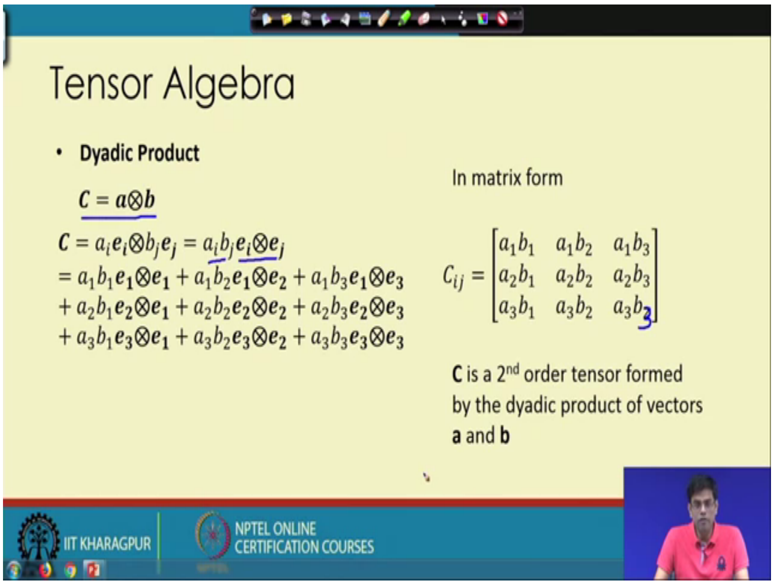

Now if we the dyadic product that we have introduced, we can write it very proper manner. So, this dyadic product we can write it the vector a a i e i tensor product or dyadic product b j e j. So, finally, these ai and bj it is the scalar, so I take it out and then the tensor product is between these two unit vectors. So, if I expand it ah; this is an initial notation, this is in tensorial notation; so if I just expand it so u 1 tensor product e 1 and then so on right.

So, if i write it in a matrix form in this quantities, the components of C; so C ij will look like this, so a 1 b 1 to a 3 b 2 right. So, so i think this will be a 3 b 3 right. So, so this C is a second order tensor from by the dyadic product of vectors of a and b. So, how these. So, what do we have learned ah? we have learned how to transform scalars, how to transform vectors and how to transform second order tensor.

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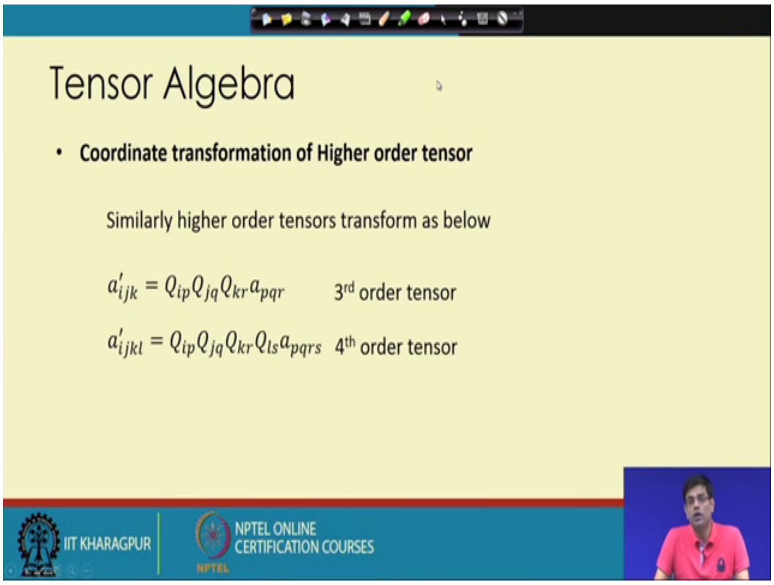

Now, we need to understand, how the higher order tensors ah transforms, because we will not use much of the higher order tensor in this course, but it is better to know how to transform the higher our sensor tensors. So, similarly for a third order tensor, the components of Q will be multiplied three times, because it has three directional part and for the 4th order tensor, it will have the 4 directional part. So, in this way we can transform efficiently the third order and 4th order tensors.

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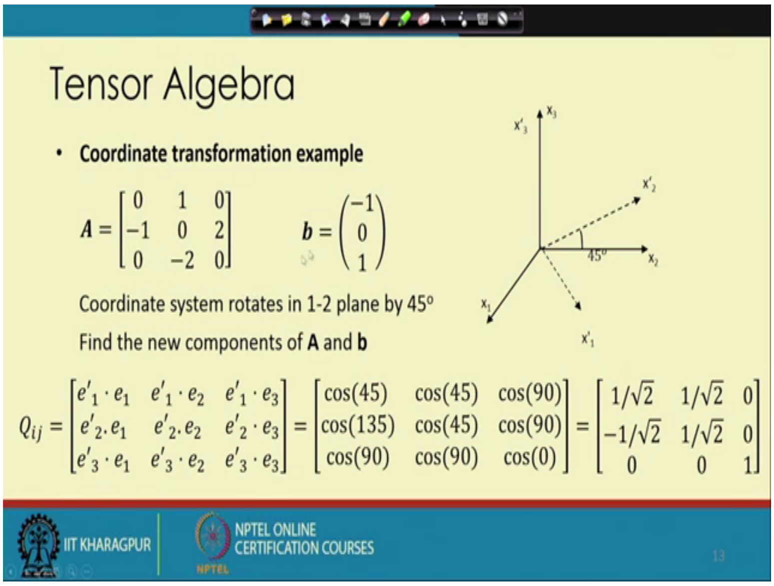

So, let us see what type of coordinate transformation through an example. For instance if the A is a second order tensor and b is a vector we want to transform A. So, first let us rotate this coordinate system x 1 x 2 and x 3 with ah in the xy x 1 x 2 plane, we rotate it

45 degree. So, now, first our first job is to find out the components of Q and components of Q if I write it in this form, in a matrix form so e 1 prime dot e 1; so which is actually the x 1 and x 1 and the cause of angle between x 1 and x 1 prime which is 45 degrees, so of course 45.

So, similarly e 2 prime dot e 1; that means, the x 1 ah x 2 x 2 prime and x x 1, so which will be again 35 degree, because this is 90 minus 45 ah. So, this is 45 and then this 90 will be added, so 135 degree, so 45 plus 135 degree, so the 90 plus 45 degree, so this becomes 135 degree. So, similarly you see this becomes cos 135 degree. So, now if we see that what is the e 3 prime dot, e 3 which will be, because e 3 prime and ah e 3 is essentially here x 3, the angle between x 3 and x 3 prime.

So, since we have rotated 1 2 plane, so x 3 it does not change, x 3 and x 3 prime coincides, so this angle between this is 0. So, actually this is a 2 D rotation. So, one important thing here just for your information that if you do a 3 D arbitrary rotation, this is not commutative. So, this information ah we can keep in mind. So this becomes our rotation matrix.

Now once we find out the rotation matrix our all quantities we can transform from x 1 x 2 x 3 system to x 1 prime x 2 x 2 prime x 3 prime coordinate system. So how it looks like?

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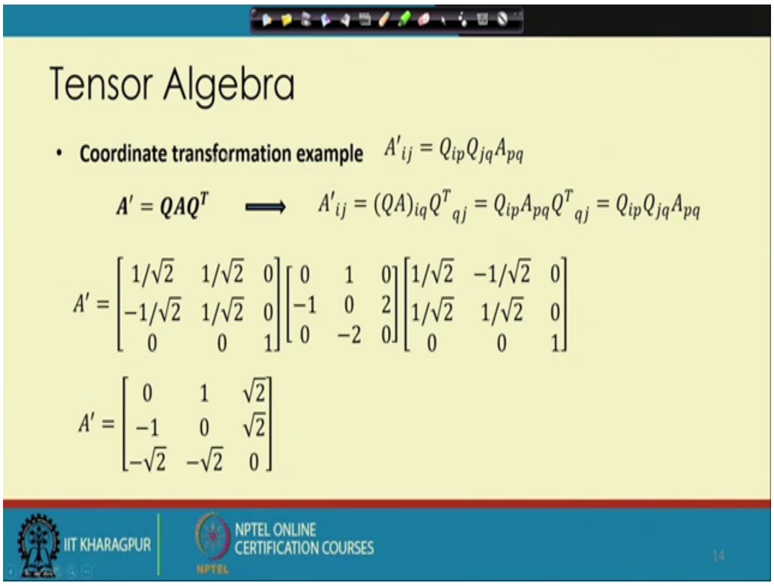

So, now this is the tensor; so Ai prime ij is Q i p Q j q A p q. Now this if we write in a tensorial form, so it looks like this. So we can also prove that. So, if I take the 'th i j'th component of a prime, so which I ah which I can write it in this form Q A i q that is we take these tensors first and right its i i q th component and transpose of q that is qjs component right.

So, if i, now again write it write Q and A in tensorial form in the initial form, so I introduced p in this in this index; so Qi p and Apq and then q q j. So, if I write it now you see this is Qi p Qj q Ap q, so which is similar to this. So, the initial form is this and the tensorial form is this. So, this tensorial form is important, because ah we do not have to keep in mind in this indicial form, but we should always able to ah given a tensorial form we should always able to write the indicial form or given it initial form, we should always able to write the tensorial form.

So, now, finally, once we know this form then A prime I can just simply write Q A and Q transpose and this A prime becomes this. So, this is a example of how we can transform the coordinates ah. We, how we can transform a tensor in a ah rotation of the coordinate system.

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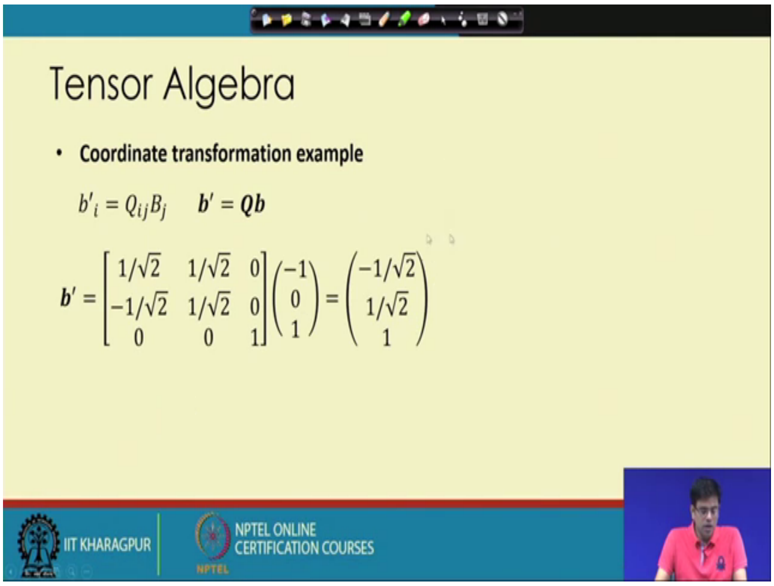

Now, similarly the vector; vector is again b, i prime, we can write Q and b. This is the very well known result. So, we just multiply with the rotation matrix and this becomes the, ah vector in the prime coordinate system.

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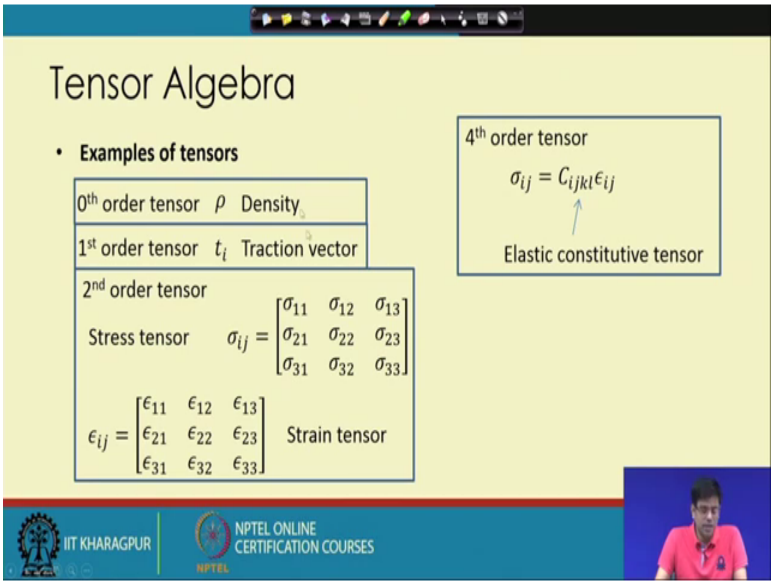

So, now some examples of the tensor; so what we have learned is the, scalar is a 0'th order tensor, wso which I have already, discussed in the last class. So, density is a scalar. So, it has no direction, 1st order tensor traction vector. So, traction sometimes, we probably those who have done solid mechanics fraction, probably they have hard. So, traction vector or velocity, these are the 1st order tensor, because these are the vectors.

So, 2nd order tensor, 2nd order tensor is trace is a 2nd order; tensor strain is a 2nd order tensor, because it has two substrate. So, the in the description also we write it in, two subscript. So, this is very important to know. So, stress strain is a 2nd order tensor. Similarly 4'th order tensor those who have, I think all of us probably have heard this Hooke's law. So, this Hooke's law when we use the, elasticity matrix, or the constitutive matrix, this is actually a 4th order tensor. So, we will see it in detail how this 4th order tensor comes and how these we can write these 4th order tensor in a matrix form. This we will see it in a later part, but later part of this course, but it is better to know right. Now, that these constitutive matrix, is a 4th order tensor.

So, you see, here also we can verify the definition of the tensor. So, this C is actually, the, this C a is a 4th order tensor which transform a 2nd order tensor strain epsilon to, another 2nd order tensor which is a stress. So, in a bitter way which transforms the 4th order transfer, transforms a 2nd order tensorial space to a another, second order tensorial space. So, in this way we can define any order of tensor.

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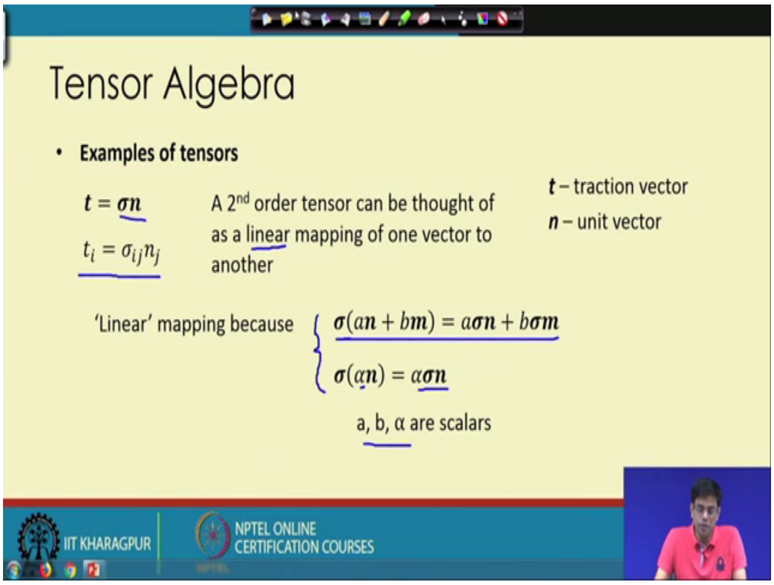

So, now if we, see examples of tensors, for instance; the tractions, the traction vector, we can write the Cauchy's law of, if you have gone through the solid mechanics. So, it is sigma n. So, n is the normal. So, if I write it in a, indicial notation ti is sigma ij nj.

Now,. So, as I was discussing, in the last class that a second order tensor can be thought of a linear mapping from 1 vector to another vector. So, they essentially it is a linear transformation. So, why this word linear means the linear, means the basic definition of a, it comes from the basic definition of a linear function. So, if I now, use two arbitrary scalar a and b and two arbitrary, vector n and m and if I post multiply with this tensor sigma, then the my result should be a sigma n plus b sigma n. So, if this property is satisfied and if I multiply this alpha with n and then, pre multiplied alpha with the n and then those that vector if I, post multiply with the sigma then it becomes equal, equal to the alpha sigma n.

So, if these two properties are satisfied then, we say this, tensor is a linear tensor, linear, function or linear, tensor. So, this linear mapping we say ah. So, this transfer actually the transformation is linear. So,, the reason for linearity is these two property. So, it satisfies these two property.

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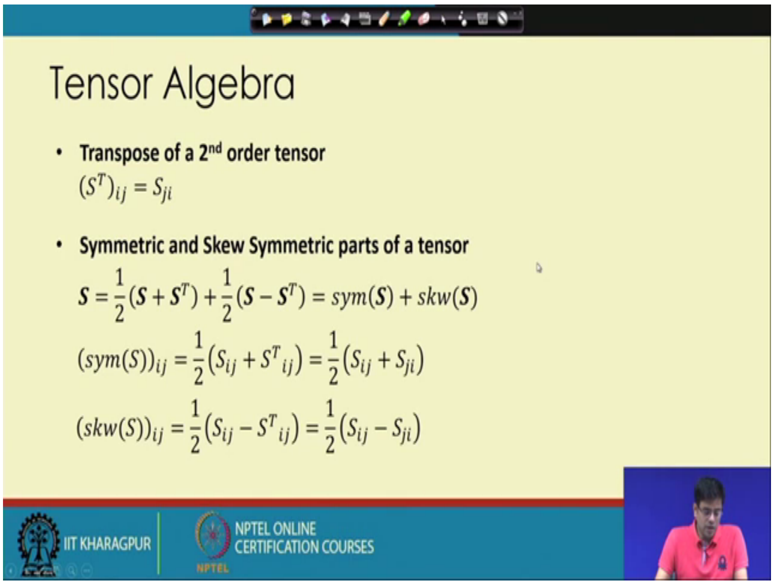

Now, once we know what is the tensor and it's transformation then, we can understand the transpose of a tensor. So, transpose of a tensor, second order tensor is very similar to transpose of a matrix, square matrix. So, Sij is actually, Sij transpose is actually Sji. So, in the last class, we have also seen what is the, skew and symmetric, skew symmetric and symmetric part of a tensor, second order tensor. As we have discussed in the last class that any matrix can be written in terms of a summation of a, a symmetric tensor and skew symmetric tensor. So, which can be written is that, in this way; if in this expression if I, add half of a transpose and deduct half of it S transpose and rearrange it in properly. So, this is the symmetric part of the tensor and this is the skew symmetric part of the tensor. So, this in a, initial form looks like this that we also, you have seen in the last class.

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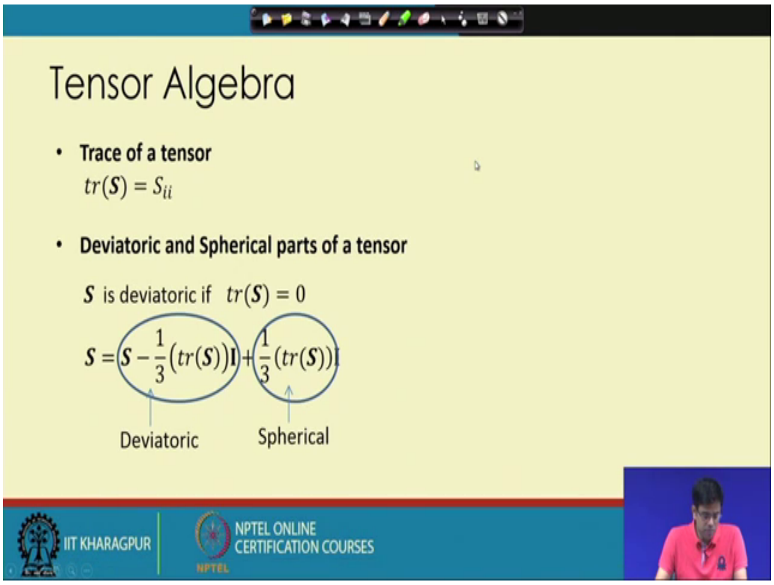

So, now, we can have an example, before that there is a trace of a tensor. Trace of a, tensor is also we have seen which is the summation of a all, diagonal element or the summation of a diagonal components of the matrix, second order tensor S. So, which is trace of S is a Sii in indicial notation another important component is that deviatoric component of a tensor and spherical or volumetric component of the tensor.

So, if S is a deviatoric component, the definition is S is deviatoric if trace of S is 0. So, what we do is the essentially, we deduct one third of trace of S and i with the S and then, add this one third of trace of S. So, this components becomes deviatoric and this components becomes spherical. So, these we will see, through an example also.

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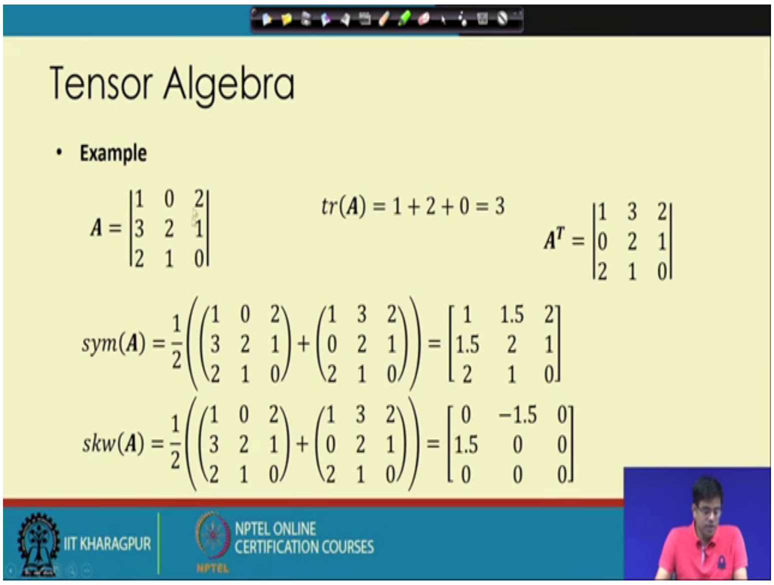

So, suppose this is A met 10 second order tensor A and then trace of A is trace of A is this, which is the A of II. So, the 1 2 and 0 which is 3 and then symmetric part of the tensor; we can write it S or, A half of A plus A transpose. So, A transpose is these. So, we can just add it. So, this is a symmetric part of the tensor,. If you see why it is symmetric, the off diagonal elements are same.

So, skew symmetric part of the tensor. So, this has to be the negative, this is minus actually so, this is minus. So, if I do this, half of A transpose minus A, sorry A minus A transpose then even see that this is A symmetric, part of the tensor. Why it is symmetric? Because off diagonal elements are negative to this, that is A ij or the skew of A ij skew of A ij is, Aij equals to minus of A j i and all the diagonal, component is actually, the 0 all the diagonals are 0. So, this is an example of a, tensor. We can write it in a skew and skew symmetric and symmetric form.

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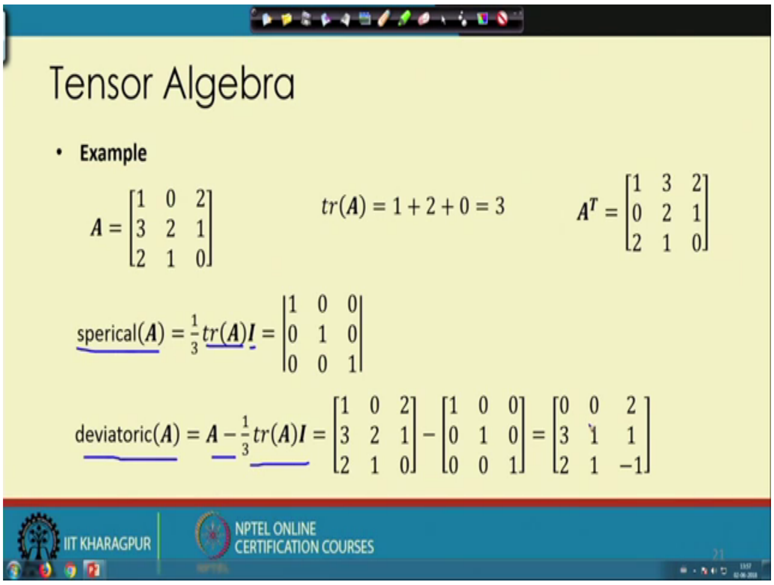

Now, what is the spherical component of the tensor ? So, if I write you we know what is stress. So, one third of tress of A into I, see trace is a scalar quantity and spherical part of the tensor is a tensorial compo quantity.

So, we have to multiply with the identity tensor and which becomes an identity tensor. Now, deviatoric part, deviatoric part is essentially a minus one third trace of A I. So, if we just deduct this component, this becomes the deviatoric component of the tensor A. Now, if you look carefully, the trace of this deviatoric component is 0. So, it is not necessarily all the component, all the diagonal component of the tensor is 0, but summation that is the trace of deviatoric of A is 0 1 minus 1 is 0 actually. So, trace of deviatoric of A is 0. So, this, we can remember ah.

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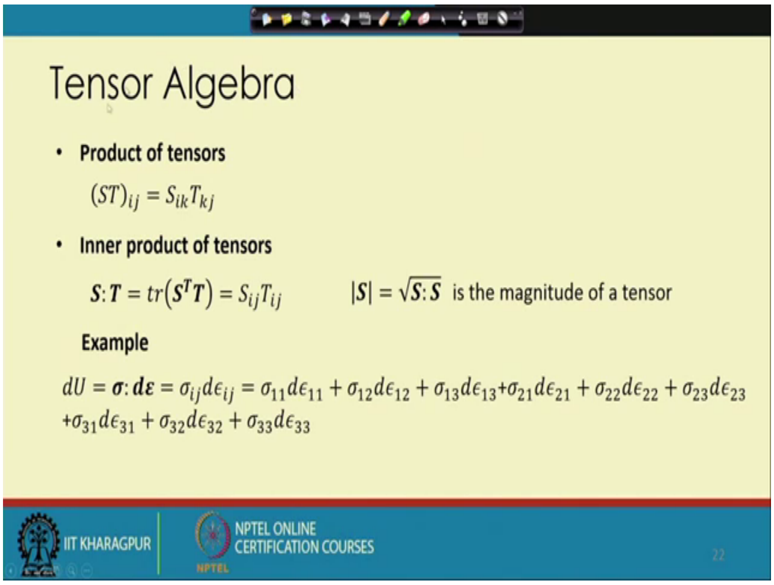

Now, in the important product for the tensor, We will be using most is the product of tensor that is if we multiply two tensor, how it will look? So, we have seen if we multiply two vectors, what will happen like dot product cross product and the dyadic product or the tensor product. Now if we multiply product of these two, two tensor S and T and then ij, then it is simply S of ik T of ki. So, this is known as the inner product of the tensor. So, this can be written in the tensorial format in this from. So, this is the contraction sign or sometimes, this is known as the contraction operator. So, S contraction T is essentially trace of S transpose E S transpose T;so which I can write in a proper manner Sij Tij.

Now, the magnitude of the tensor for instance the magnitude of the matrix we, if we write the, it is the inner product of 2 2 matrix and root over of that like the vector. So, S contraction S and root over of that. So, if you look carefully since trace is a scalar quantity trace is a scalar quantity, this quantity is actually the scalar.

So, now for example, if you have seen in the solid mechanics, this energy expression, then sigma contraction d of epsilon which is essentially sigma ij d of epsilon ij and if we write it in this; if we expand it the indicial notation, then it becomes a scalar coordinate. So, the important thing here is the inner product of a tensor is a scalar quantity. So,, we need this, in our relation. So, we will be using this quantity often. So, it is so, the new product of two tensor. We have learned in this class is the inner product of two tensor which is trace. So, trace operator is, in that sense it is very important operate ok. So, here we stop today and in the next class or next lecture, we will start with the tensor calculus.