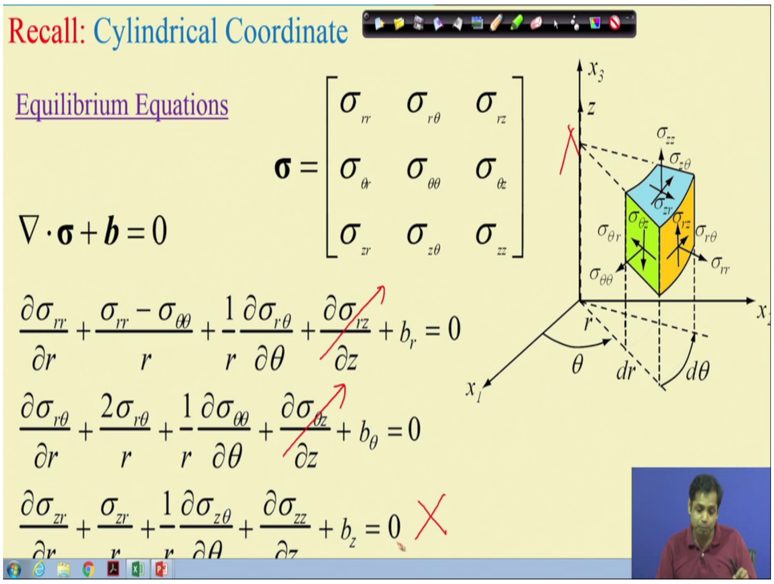
Theory of Elasticity
Prof. Amit Shaw
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 31
Solution of Boundary Value Problems (Contd.)

Hello everyone, this is the fifth week of this week, a fifth class of this week. You see in the last class we demonstrated how airy stress function can be used to solve elasticity problem in Cartesian coordinate system; we will be doing similar exercise, but in polar coordinates system. We will just write down the equations in polar coordinates system today and the next class we will demonstrate those equations through few examples ok. So, today airy stress function in polar coordinate system.

(Refer Slide Time: 00:50)

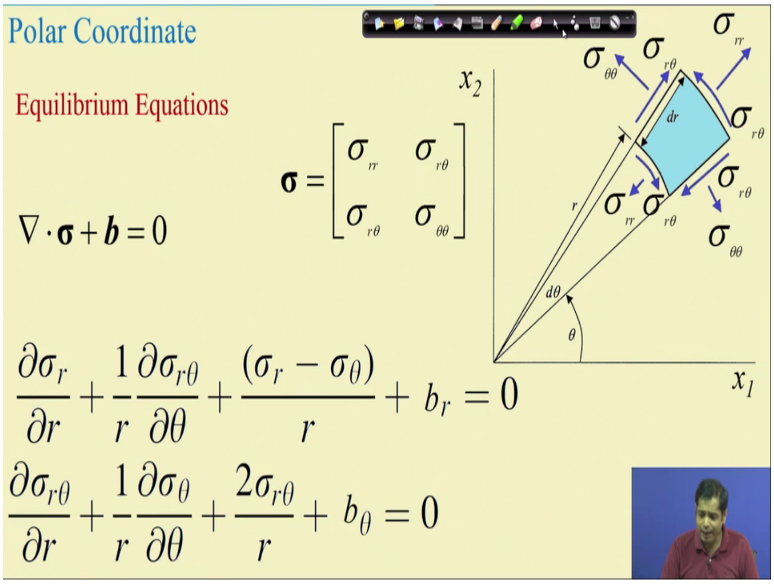


Now, if you recall in cylindrical coordinate system these are the components of stresses, we have 3 directions which are which are r x r direction, in a radial direction, then in a theta direction and then along z direction and these are the different components of stresses in different directions.

Irrespective of the coordinate system we choose the basic the governing equation the dell dot sigma plus b body force is equal to 0, that will remain same. Depending on the coordinates system we choose, the expression of del and expression of and the components of stresses and component of b will be different. Now, if you substitute this component of sigma and corresponding component of del then this is the governing equation in cylindrical coordinates system; we derived it in previous classes one of the previous classes. Where br b theta and bz are the body forces in different direction r theta in z directions.

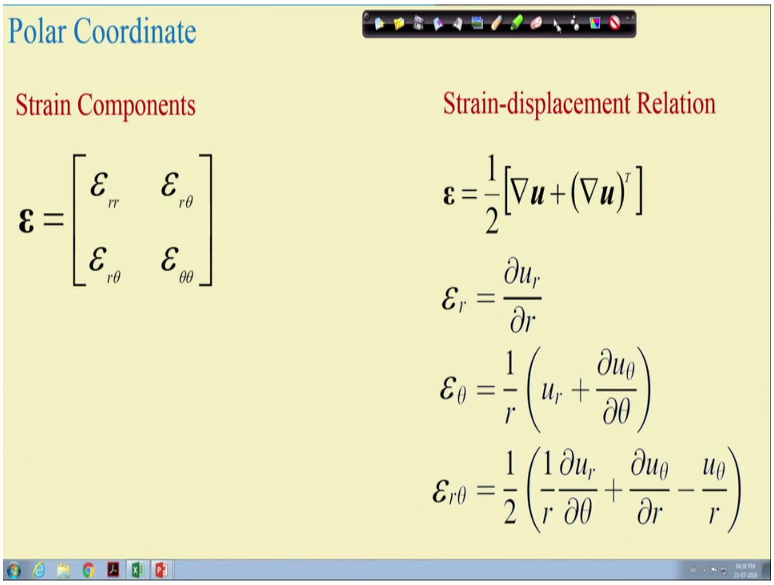
Now, in order to get the polar coordinate system, what we have to do is we have just polar coordinate system mean then the problem is problem is a plane problem. So, the direction z direction we have to remove the z direction; so, all the components of stresses that we have in z direction that will vanish. So, this will be the we would not be having this component, we would not be having this component, and the entire equation will not have because now, that that particular that particular direction does not exist. So, then if I have to get the corresponding equation in polar coordinate system then the equation becomes, this is the polar coordinates system, in the components these are the component of stresses.

(Refer Slide Time: 02:36)



Now at this point these are the component of stresses which are independent. Because depending on plane stress or plane strain problem we may have stress in z direction or may not have stress in z direction. But, that stress for plane for plane strain problem you may have stress in z direction, but that stress may be obtained by the other component of stresses. So, these 3 independent stress components now, corresponding equation becomes this. All the terms associated to z direction, now they straight away we can remove them; so, this in the governing equation in polar coordinate system.

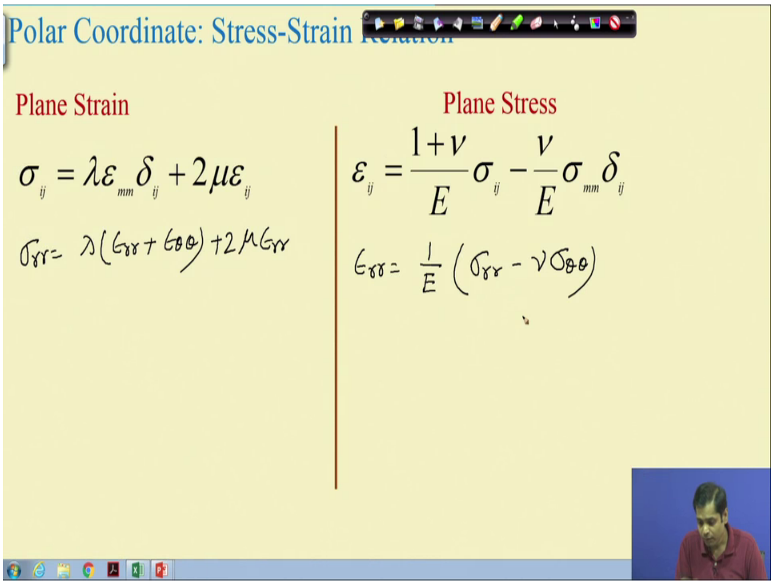
(Refer Slide Time: 03:22)



Now, next is strain components now, similarly these are the 3 strain components sigma r epsilon rr epsilon theta theta and epsilon r theta, again depending on whether it is plane strain on plane stress problem we may have strain in the z direction. For instance for plane stress problem, we can have strain in z direction, but in that case that case that strain component can be obtained by these or the by the other strain components.

Now again irrespective of the definition, irrespective of the co-ordinate system, these definition of the, these relation will remain same this is the strain relation, these are the gradient of displacement field. Now, if I write these express this relation in polar coordinates system, then we have these are the 3 different strain components ok. Now epsilon rr, epsilon zz epsilon theta theta epsilon epsilon z r theta it is epsilon rr, epsilon theta theta and epsilon r theta ok. Now next is so, we have the stress components, we are strain components, we also have the governing equations in polar coordinates system.

(Refer Slide Time: 04:35)

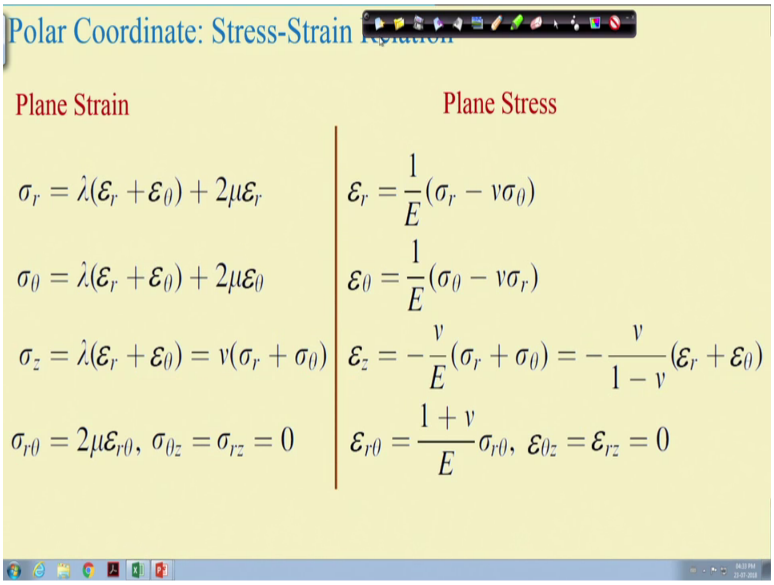


The next is let us see what are the different what are the stress strain relation in plane in different formulation plane strain and plain stress formulations. If you recall this is the relation between stress and strain. So, sigma r becomes in this case if you recall here sigma r, sigma r is equal to or for instance; let us write down the expression here we also have the expression.

So, this is the expression for sigma ij is a stress component as a function of strain. Now suppose if you want to find out strain in sigma r in r 0 vector the sigma rr will be similarly lamda, lamda into epsilon rr plus epsilon theta theta and then plus 2 mu epsilon rr. So, this will be the sigma r components, similarly we can substitute rn theta in place of i and j we get other stress components as well.

Then similarly strain is this. So, similarly if you have to get this say for instance strain in epsilon epsilon r, epsilon rr it will be 1 by E into sigma rr minus nu into sigma theta theta. Similarly we can substitute different components in place of i and j we get the other strain components as well.

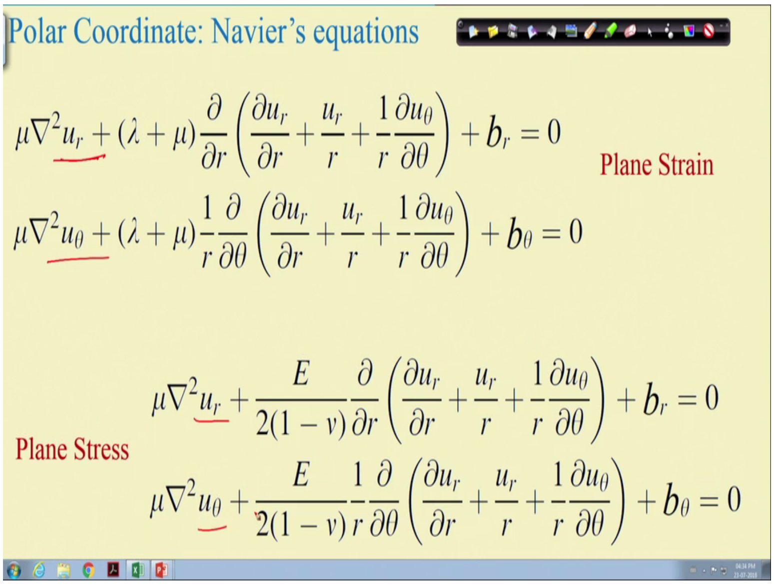
(Refer Slide Time: 06:10)



Now, if you do that so, these will be the different stress and stress components in plane strain. Please recall this has been discussed many times, but again for plane stress problems, the stresses in a particular direction is 0 all these stresses are 0. And for plane strain problem sorry for plane strain problems strain in particular direction all the strain along z direction is equal to 0, but it does not mean that all the stresses will be 0, we may have stresses, we may have sigma z as a function of sigma r and sigma sigma theta.

Similarly, for plane stress problem, the stresses all the component of stresses in z direction is equal to 0, but it does not mean the strains are also 0 we can have the epsilon z, but that epsilon z can be obtained by knowing epsilon rr and epsilon theta theta. Please note that it is whenever I, whenever it is written epsilon theta it essentially means epsilon theta theta, and epsilon r means epsilon rr ok that is the notation we use ok. So, we now know: what is this stress strain relation for stress strain relation in polar coordinate system.

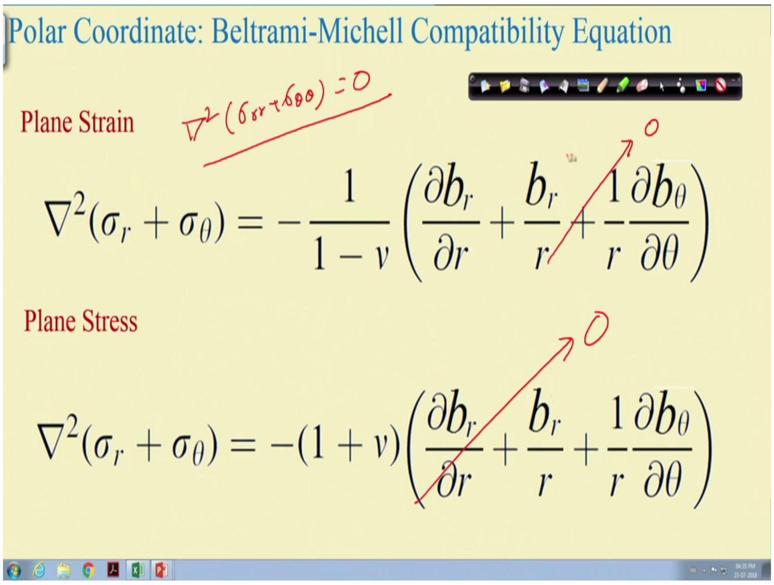
(Refer Slide Time: 07:24)



Now let us write the other equations Navier's equation which is essentially the displacement formulation. So, we write everything in terms of in the equilibrium equation we write everything in terms of displacement. And if you do the similar to the Cartesian coordinate system we do the same exercise, we get the corresponding equations in plane strain and plane stress. This is epsilon in r direction this is theta direction r direction and theta direction.

Now so, this is for plane stress and plane strain the as far as the derivation of this concern it is exactly same, as for Cartesian coordinate system only difference will be the operated del are now, different and the components of stress and strains are different, rest of the things is just the substitution of the rest of the steps are same.

(Refer Slide Time: 08:25)

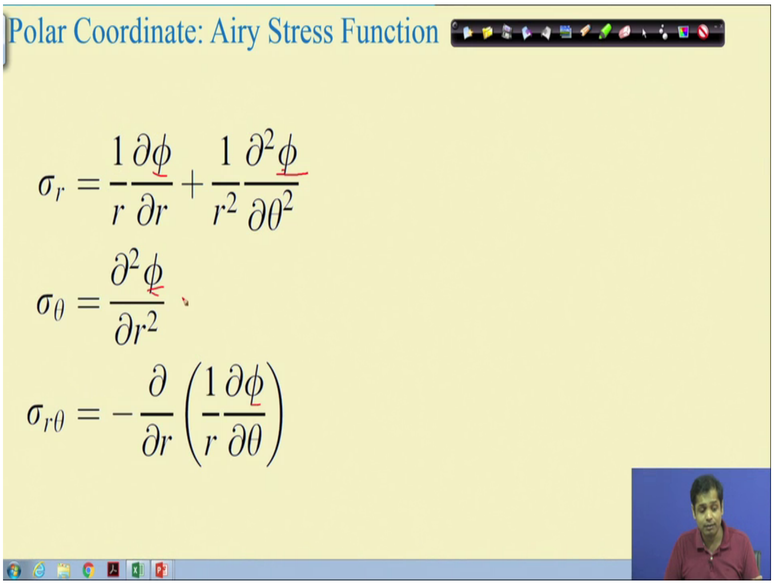


Now once we have this Navier's equation, then also let us write these governing equation in terms of displacement, which compatibility equation and this is the corresponding equation in terms of displacement ok. Now, you see again in if the body force is 0, then this part will this part vanishes this part vanishes then what happens? Your equation becomes del 2 del 2 sigma rr plus sigma theta theta that is equal to 0 it is again exactly same as exactly same as Cartesian coordinate system ok.

And for plain strain and plane stress both are same next. So, we have written the equations in terms of stresses and let us now let us now see how to solve these equations ok. Please note that there are many solution methodology, but the solution methodology that we are discussing we have been discussing in this course is stress based formulation and solution by power series method, there what we did is we combine all these relations we define a potential was a stress function airy stress function.

And, which the airy stress functions are defined in such a way that the governing equations are automatically satisfied and then the rest is the compatibility equation then we substitute all these definition all these expression for airy stress function in the compatibility equation and get the by harmonic equation. That is what we did it in Cartesian coordinate system and we will be doing that same thing exactly for polar coordinates system as well.

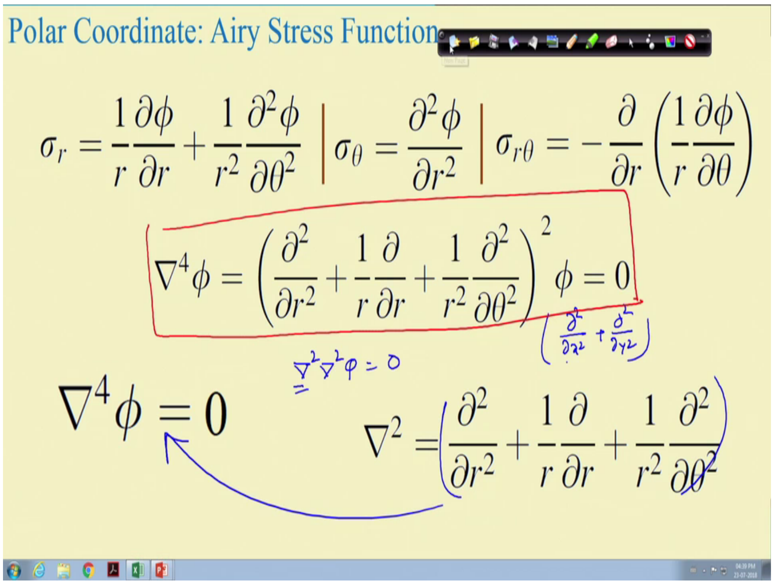
(Refer Slide Time: 10:07)



Now these are the definition of airy stress function, phi is the airy stress function Now once you can do that exercise you substitute this all this component of stresses in the equilibrium equation and then the check whether the equilibrium equations are satisfied or not. As the problem generally we have been considering there is no body force that is why in this definition airy stress function the body force term is not included.

But if you have body force term in your formulation then you have to include your body include that body force term in the a in the in these definition. Now, once we have airy stress function then the rest is you have to substitute this you recall this is the compatibility equation, this is the now in that expression we have to substitute sigma r and sigma theta.

(Refer Slide Time: 11:03)

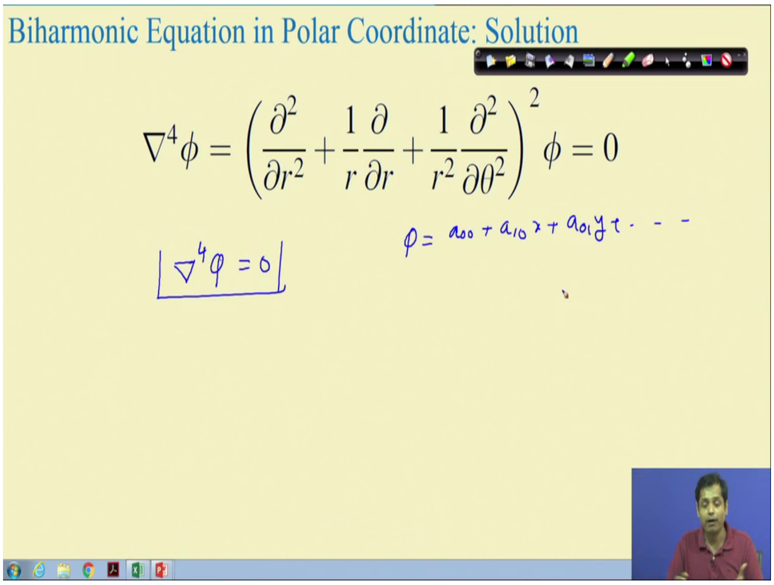


And if you do that substitute that this definition of sigma r sigma theta theta and sigma r theta then we have by harmonic equation like this, here we assume the body force term is 0. So, which is again essentially del 4 phi is equal to 0, del 4 phi is equal to 0 del 4 phi is essentially if you recall del 4 phi is del 2, del 2 phi is equal to 0 where del 2 operator in del 2 operator in Cartesian coordinate system was del 2 del x 2 plus del 2 del y 2 if you recall, but now in polar coordinates system, that operator becomes this.

So, if we substitute this operator in this expression the entire operator in this expression. So, we get this. So, this is our final equation; it is the same equation that we had in Cartesian coordinate system, but now the operators are different because the coordinate system is different. Now, the rest is once we have the governing equation, this is the governing equation the solution of this equation is phi which gives us the stress field.

So, once we know the stress field by the by Wi-Fi then you have to apply stress strain relation to find out strains. Once we have strain then apply strain displacement relation to find out displacement and in the entire process wherever we come across the constant for integration, we have to use different boundary conditions to find out those constants that will be demonstrating through one example shortly.

(Refer Slide Time: 13:00)

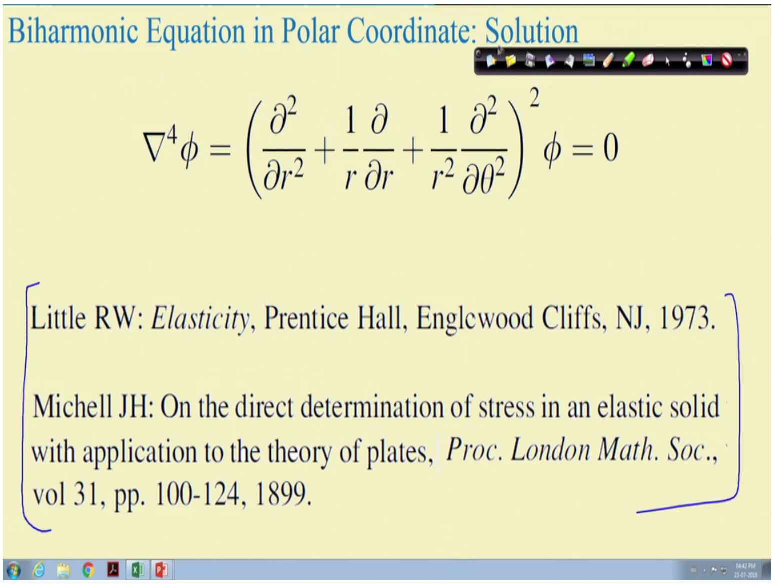


Now, you see. So, this is this equation you recall in the Cartesian coordinate system, what we did is del 4 phi was del 4 phi 0 even del 4 phi is equal to del 4 right del 4 phi is equal to 0, we assumed that phi as polynomial right; suppose for instance it is a 0 0 plus a 1 0 x plus a 0 1 y a and so on. So, we assume the phi is a polynomial, and then all the polynomials which are less than third order the order is less than up to cubic polynomial that to cubic polynomial automatically satisfy these equation, because this equation is fourth order, but if we have a polynomial which is.

So, if we if for problem where we assume the expression of phi as up to cubic polynomial, we really did not have to explicitly show that if that expression of phi that assumption of phi satisfy this equation, because it is it satisfied order is less than the a order in the equation. Now, if we have if we assume phi which is more than third order, then we had to satisfy compatible these equation and we substitute phi in that expression, get some relation among these constant, and that relation we can use to find out the along with the boundary conditions to find out the constants ok. That is how we apply the airy stress function in Cartesian coordinate system.

Now, in here it is we have it is just not you will see here we have r term and theta term and therefore, unlike the Cartesian coordinate system, here in expression of phi we may have say may have some expression of r as a polynomial all some expression of theta as trigonometric function let us see that now.

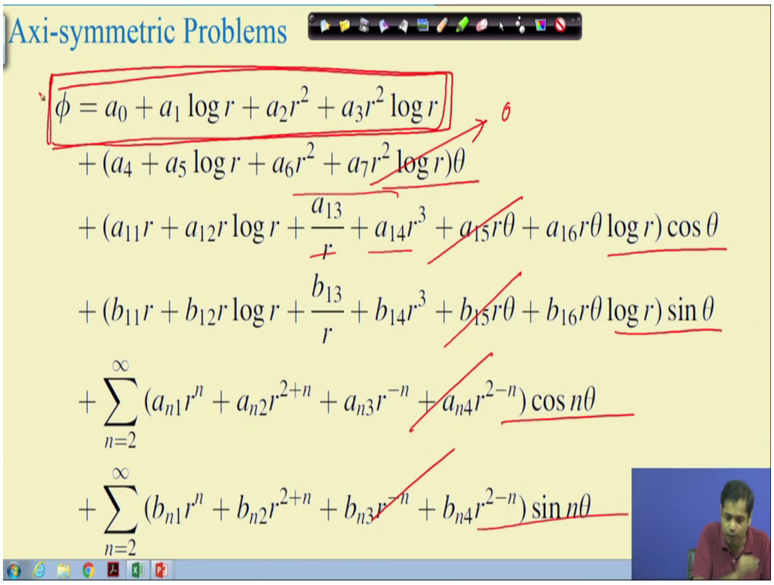
(Refer Slide Time: 15:18)



So, this is the general expression of this is the now what we have we can find out general expression of phi from this equation? How to get that? Suppose you assume that you have you can apply we have a 2 coordinates one is r and theta and phi is a function of r and theta, we can assume that separation of variable can be applicable. And, then we can have we can assume that phi is a combination of some function of r and some function of theta, and then substitute that in these equation and find out the relation between those functions. That exercise will not do in this class, if you see any book on elasticity that exercise is given and the detailed solution of phi is given in this 2 in this to article.

So, how to find out the solution that steps that approach is given in the book, and further details of that solution further detail of relation between different constants and everything that is given in these 2 articles. So those who are interested to get the insight into the how the solution is obtained for the general solution of this equation is obtained, you can please refer to this 2 article. Now the general solution of phi based on the exercise given in this in this 2 article the phi can have a general expression like this.

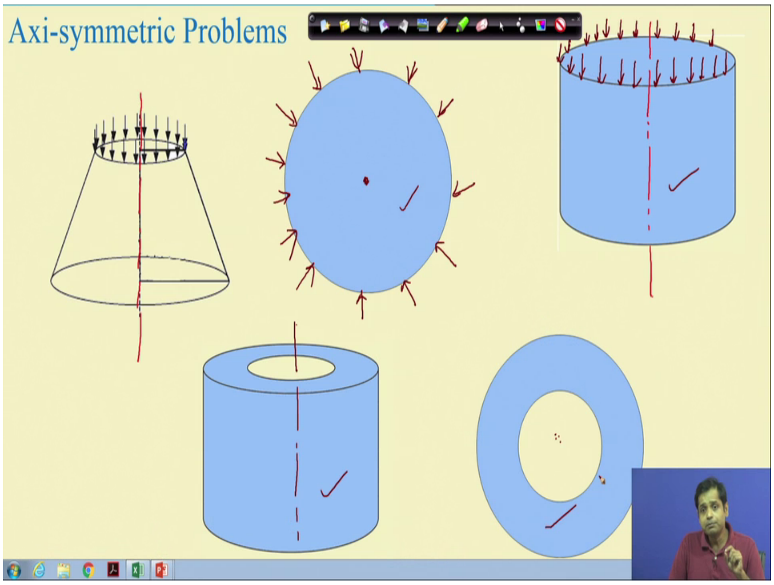
(Refer Slide Time: 16:49).



So, these expression is a general solution of the by harmonic equation, that just now we had we have seen the previous slide. Now please look at this expression these are infinite series this is an general solution, but when we apply this when we apply this method to a particular problem, we would not we would not take all the terms we will take we will take a finite number of terms from this general expression we will we will show you that. Now here the approach will be like this.

Approach very similar to the problem in Cartesian coordinate system the first thing what we did is we once the problem description problem is described once we have the problem description before us, the first step in the solution we did is we assume some expression of phi and in that case that expression was assumed as a polynomial. Now here also the first step is assumption we have to assume some expression of phi, but then here the phi is assumed some phi is phi is assumed by taking some finite number of terms from this general expression. Now, for instance suppose before actually we apply these to a more general problem, let us let us see a very let us let us slightly simplify the problem for the demonstration.

(Refer Slide Time: 18:36)



Suppose we have a problem which is axi symmetric. Axi symmetric problem means means the problem the geometry, boundary conditions, loading everything is symmetric with respect to an axis. For instance in this case if you take this the entire problem, the entire the geometry and these all these loading is symmetric with respect to this axis. So, this is called axi symmetric axis right.

Similarly, if you take this this problem everything is with symmetric with respect to an axis if you have a loading if you were loading something like this for instance, all these everywhere us. When we say in an axi symmetric problem it is just note the geometry, the geometry boundary conditions loading everything has to be symmetric with respect to this axis. In this case it is a plane problem, in this case the axi symmetric axis is the axis is the point is this and the axis passing through this normal to this plane is axi symmetric axis. Now it may be applied to some external loading for instance it is applied to some problem loading like this uniform pressure everywhere. So, these are some axi symmetric problem ok

Similarly, this is in this case the this is your axi symmetric axis this case the axi symmetric point is along is a normal to this normal to the plane of this figure. Now, one important point to be noted here we will come to that point shortly, if you recall we discuss some time back, we when we actually describe different kinds of continuum, we define 2 such continuum one is the connected domain or one is simply connected domain and one is which are not simply connected domain multiply connected domain. For instant these are all simply connected domain whereas, this domain and this domain are multiply connected domain. And if you recall at that time you made a statement, but we did not prove that we made a statement that the compatibility condition and the equilibrium why we needed compatibility condition? To get to have an unique displacement field right.

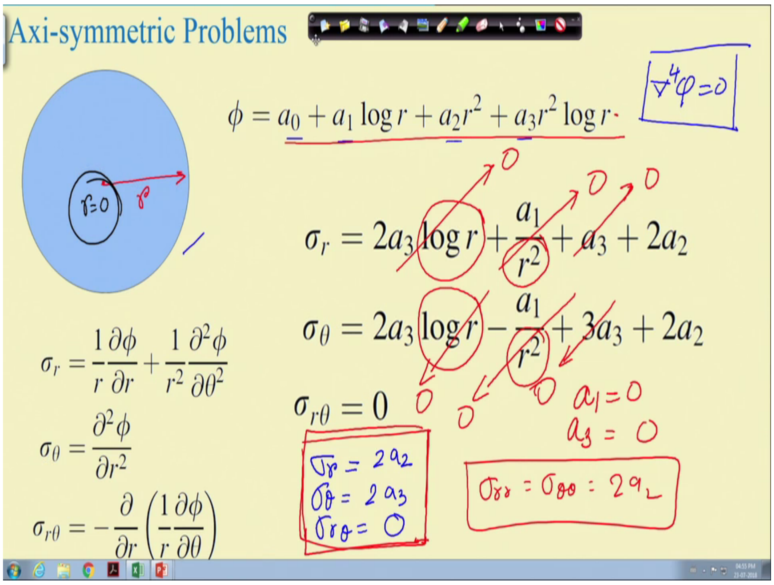
Because, the displacements need to be compatible with each other if the compatibility condition is not satisfied, then the solution may not be unique. So, compatible condition helps us to gave to arrive that in unique solution right displacement unique displacement field. At that point we made a statement that this is may not be guaranteed if the domain is not simply connected domain ok. We will shortly see through an example through one of such example if the domain is not simply connected even the compatibility condition is satisfied, but the uniqueness in the displacement field is not guaranteed it is a necessary condition, but not sufficient condition.

Now if now suppose we are we have to find a solution of a problem of axi symmetric right. Now in an axi symmetric problem we know that everything is symmetric with respect to an axis, and therefore, now the as I just now discussed we have to when we start a solution the first step is to get an approximation of phi assumes of phi. Now, when we assumes of phi it is an infinite series we cannot deal with infinite number of terms, we have to get the finite number of terms right there we have to use some information about the problem. What are the information we have about the boundary condition what are the information you have of the nature of the problem? By intuition if we can if we can assess some deformation or some deforming stress deformation characteristic of the problem, that may also go in that information may also go in while choosing phi.

For instance in this case if our problem is axi symmetric problem, then all the term which has which has which is which is dependent on theta, for instance this term depends on theta these term depends on theta these term depends on theta these term depends on theta, these term depends on theta, if the problem is axi symmetric, we can straight away we can remove this term. And we are left with only this term only 1 2 3 4 only 4 constants we have if the problem is axi symmetric problem. But the problem is not axi symmetric problem, then we cannot make we have to we have to we have to take the theta dependent term as well.

Now, let us see suppose now this is my phi we will also see that the sum of the sum of the term also we may take for an axi symmetric problem, but very simplified case we straight away we remove all the term associated all the theta dependent term and then take only this term ok. Now, let us see what happens next.

(Refer Slide Time: 24:25).



If we take only this term right and then what we have? We have a relation of sigma r, sigma theta sigma r theta and phi. So, straight away we can find out we can we can. So, we take this. So, we can substitute this phi from the expression into this expression and get the expression for sigma r, sigma theta and sigma r theta. And if you do that exercise we have an expression like this.

Now, here is an important point. One exercise that you have to do is check whether this phi this phi when we say this phi, satisfy by harmonic equation or not whether this phi satisfy del 4 phi is equal to 0 whether these satisfied or not ok. You have to first you have to check these first if you find that these phi satisfies this equation, then it is a valid stress function. So, from this phi we can obtain this stresses and this, whatever constants we have this constant, we can we can obtain this constant by applying the boundary conditions or the information about the problem that we have.

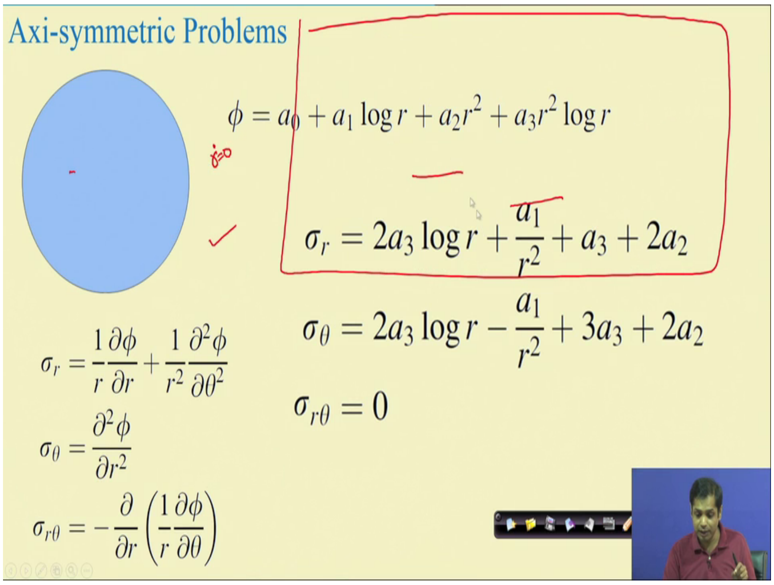
Suppose your domain is this now this is this we have a point here and then this direction is r this direction is r ok. And at this point at this point r is equal to at the axi symmetric point r is equal to 0 ok. When we say that this is my potential this is the stress function and from this stress function we have an expression of stresses like this, this stress function valid over the entire domain and therefore, the stress components that we have from this stress function, this stress component also valid over the entire domain it has to be valid over the entire domain is not it.

Now, but the problem is if it has to be valid over the entire domain and entire domain consist r is equal to 0 as well. Point r is equal to 0 also a point belongs to that domain. If this equation is valid over the domain then this equation has to be valid at r is equal to 0 as well because r is equal to 0 is a part of the domain. But now look at we have one by r square term here and then we also have log r term here right then if we have if this equation has to be valid over the entire domain, the straight away we can find we can say that this has to be 0. A 1 has to be 0 and a 3 has to be 0. So, this part become 0 this part become 0 a 3 is 0 again this part is 0 this part is 0 again a 3 is 0 right.

Now if they are 0, because the domain contains because otherwise that the continuity is not satisfied, we have an infinite the one by r square is not valid displace valid stress field at r is equal to 0, then if we substitute all the term is equal to 0 then what we are left? What are left then sigma r becomes sigma r becomes sigma becomes 2 a 2 and sigma theta becomes 2 a 2 and sigma theta becomes 0, this is a very important observation.

Now this observation says that if you take this as your stress function and your domain is this simply connected domain, then these stress function can give you only constant stress field. Sigma r and sigma theta both are same sigma r r is equal to sigma theta theta is equal to 2 a 2 in this case ok. Now let us this information we get from this ok. Let us go beyond this information let us see further let us go further deeper and then see what other information this gives us.

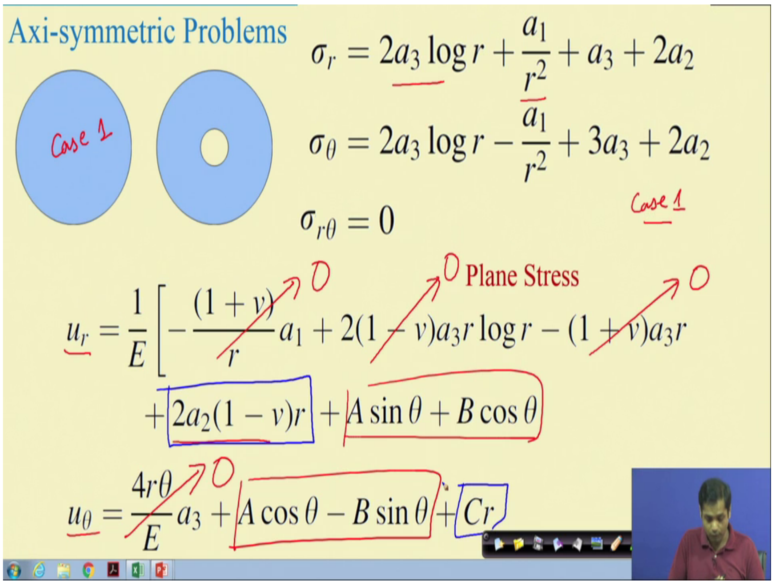
(Refer Slide Time: 29:31)



Now, but you know interestingly if your domain is different, if your domain is like this you take it here. Now, if the domain is not like if the domain is like this, then r is equal to 0 the this is r is equal to 0 and r is equal to 0 is r equal to 0 does not belong to this domain. And therefore, this equation is valid over the domain by since r is equal to 0 does not belong to the domain, and therefore, this term and this term will not give you invalid stress field right.

Because, then in that case all this term are valid term all this terms these terms are we can keep 1 by r square term and log r term if the domain is this. So, this is simply connected domain, but if the domain is not simply connected domain, then we can have the entire thing which is and this is not constant stress field. But if we start with if we take this as our potential this is our potential, and our domain is simply connected these as our stress function and the domain is simply connected domain they these can we can only arrived at constant stress field with this stress function.

(Refer Slide Time: 30:51)

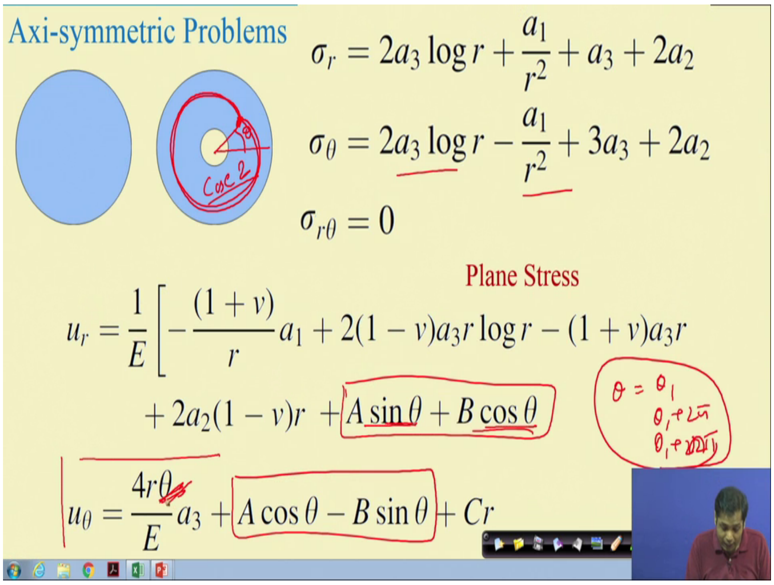


But if the domain is not simply connected then this can give us a different stress field. Now, so once we have this stresses for either this domain of this domain, consider is a plane stress problem and then we know the stress strain relation for plane stress, substitute all these stresses into that relation get the stain component. And once we have the strain component integrate that strain component to get the displacement. And if you do that exercise then this will be the displacement field this is for this is for radial displacement and this is for tangential this is for this is for ur and this is for u theta ok.

Now one more thing now consider 2 cases case 1 case 1 or case 1 case 1 your domain is this. If your case 1 domain is this a 3 and a 1 all are 0 ok. Now in this expression then naturally this term this term become 0 a 1 term become 0 these term become 0 and then these term become 0 we are left with only this term ok. Now if we substitute some boundary conditions and all we can also show that these term will also be 0 the constant term will also this bar A and B the constant term you also be 0 you can choose the boundary condition accordingly.

So, the stress strain component only has these term right the displacement component only have this term and the displacement component only have this term there is no theta term here for case 1. Now, what happen case for case 2? Also case 2 suppose this is case 2 case 2.

(Refer Slide Time: 33:08)

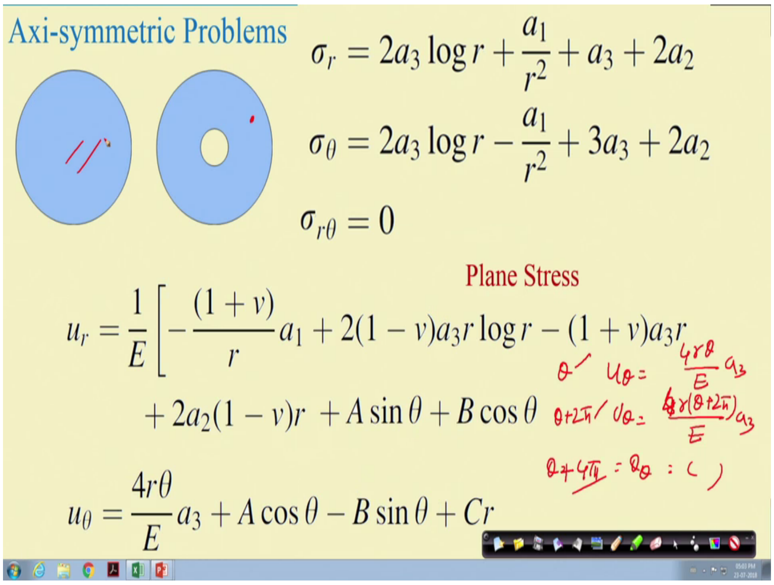


Case 2 we really do not have to make a 1 and a 3 0 because having this term in this in this stress field does not violate the definition of continuous stress field of the stress field is still valid because r is equal to 0 does not belong to this. Now a 3 and a 1 are not 0 if in the case of case 2; we can assume some boundary conditions and make at the most suppose this part becomes 0 or this part have some values or 0 this part this part.

But rest of the things a 1 a 3 a 2 all are non zero we cannot make them 0 by just the condition that we applied for the first case. Now you see look at this u 3 the v it contains theta ok. Now, theta could be anything if you recall that expression the general solution, the expression is obtained is theta theta could be say theta is could be theta is maybe theta may be theta or maybe theta 1 maybe theta 1 plus 2 pi or maybe theta 1 plus 2 n n pi could be anything theta could be anything right. What it means is suppose you start with this is your theta you take a point this is theta, and then again you come to this point your theta becomes theta plus 2 phi, again you come to the same point now the theta become theta plus 4 pi.

Now what is for the different for the same point can be expressed by different different theta right. But, if you substitute different theta here sin theta having sin theta and cos theta term for the same point if you substitute theta plus theta plus 2 n pi, this will remain same, this term will remain, this will not be will not be different, but what will happen is these term 4 r theta this term will be different.

(Refer Slide Time: 35:21)



For instance when your theta is so, u theta becomes u u theta becomes 4 r theta by a 3 sorry you write u theta is equal to 4 r theta by E into a 3 for theta for theta. For theta plus 2 phi u theta becomes 8 r theta sorry theta becomes 4 r theta plus 2 pi by E into a 3. For theta is equal to theta plus 4 pi u theta will be something else. But theta theta plus 2 pi theta plus 4 pi they all correspond to same point refer to the same point is not it? It shows that for the same point we have different u theta means u theta is not unique we have different values of u theta for the same point right.

But that problem we did not have in the case of the first domain because the domain was search that we can straight away make a 1 and a 3 0 a 1 and a 3 0. But for the second problem we cannot make a 1 and a 3 0 and this a 3, these term associated a 3 u u theta u theta term this will create problem. Problem in the sense this will this causes the this call a causes the displacement field this imposes non uniqueness in the displacement field, that is the reason why you see the stress function this that we started with that stress function satisfy by harmonic equation right. And what is by harmonic equation it is as the if we if you are staring by harmonic equation, when then if its satisfy by harmonic equation means it satisfy the equilibrium and compatibility both.

Even after that even after satisfying the compatibility equation for the second case, where the where your domain is now simply connected, we have seen that is we may have non unique displacement field, but in the first case displacement field was unique.

That was the reason at that time we mention that it is a necessary condition, but not sufficient condition it may not be guaranteed. We will come to this point again when we discuss the complex variable approach to solve different problems ok.

So, now so, what we do if we stop here today, next class we will start with this slide the general expression take up some example and then see how that example can be how the stress field in the displacement field can be obtained by stress function method. But, our starting point will be this expression this general expression from phi, from this general expression we take some finite number of term, and then find out the solution find out the values of different constants.

Ok I will stop here today, see you in the next class.

Thank you.