**Theory of Elasticity**
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**Lecture - 34**
**Problems in Flexure (Contd.)**

Welcome we are discussing boundary value problems for flexure. We are following the stress function approaches. So, in the last class what we have discussed is the different types of stress function possible. For instance, quadratic and the fourth order stress function and these stress functions for a different it has a different constants. So, by tweaking or by efficiently managing these constants or in explicit way that if some constants we take nonzero then, we can create the different stress state in the body.

So, at the end of the last class what we have discussed is the how to obtain a solution of a beam which is having a point load through the Theory of Elasticity approach.

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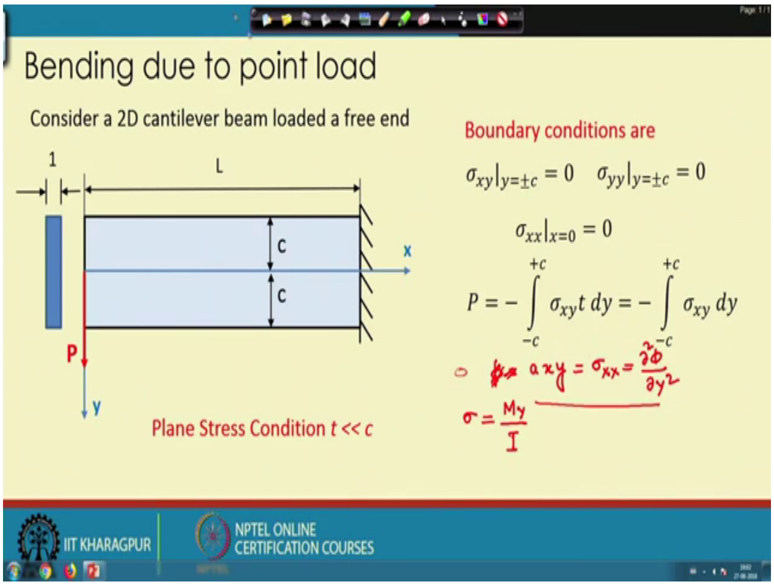


So, in this beam problem if you look carefully that we are using a 2D cantilever beam loaded at free end and this free end the resultant force is P. And we are following plane stress condition because we take c is much greater than or 2 c is essentially much greater than t. So, now, the boundary conditions as we have discussed in the last class, boundary conditions since we have following stress function based approach the boundary condition has to be in terms of stresses. So, these are the boundary conditions, we need to use it.

Now, here we have done it with a simple stress function which is motivated from our knowledge of strength of material if you remember that stress function which is pi is we have taken as phi we have taken as a xy. So, this axy is motivated from the our knowledge in the bending moment or the sigma fracture formula that we have learned sigma equals to M y by I. So, this leads to sorry this is not pi this is essentially the sigma. So, sigma xx which is essentially del square phi by del y square.

So, this is the motivation if we remember. Now this motivation is through the strength of material approach that we have following we have known till date. Now this can be this can be also if we in a some problem where the strength of material approach, we do not know the solution by strength of material approach. So, what should be the stress function and what can I assume as a stress function? This can also be done. So, let us see this problem. If we do not want to use our concept of strength of materials, how this problem can be solved?

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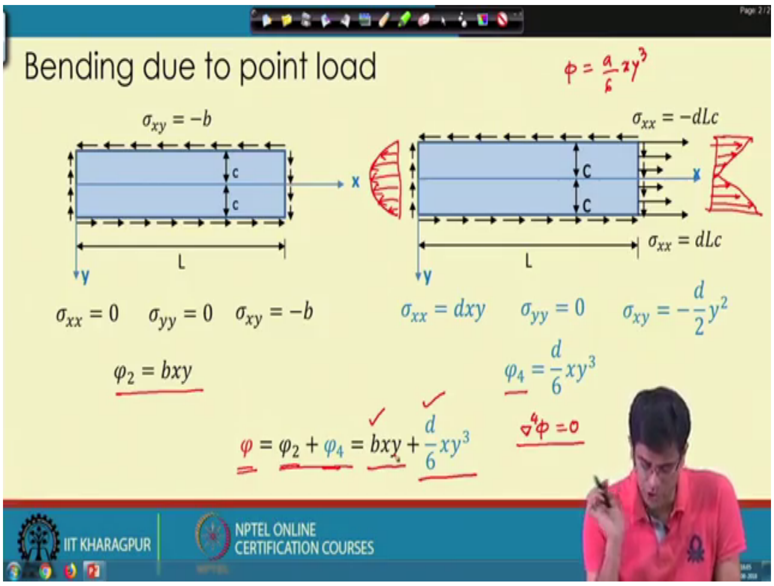


So, for instance, if you remember in the last class that we were discussing about a quadratic polynomial phi 2 of bxy which represents the pure shear case. And there is phi 4, the fourth order polynomial which is represent this stress state. So, which is essentially this sigma xx is of this form and sigma yy is 0 and sigma xy is a quadratic form or it represents a parabola. So, if you have remembered this thing, we have discussed it in the last class. So, now the basic idea is we know these stress functions and which we have found out. Now, if we superpose or superimpose these two stress function and describe that is our phi, then also it satisfy the bi harmonic equations.

For instance if I add 5 2 and 5 4 and then see that phi 2 and phi 4 is my phi, then this phi is satisfying this equation. So, if you remember carefully this biharmonic equation is essentially a linear differential equation. So, the linearity property allows us to superimpose the two different phi. And since the individual phi; this phi here 2 satisfies bi harmonic equation. This phi 4 also satisfies biharmonic equations and then some of these two phi 2 and phi 4 will also satisfy the bi harmonic equation. This is the property of the linear; linear linearity property.

Now, if we use this is as phi which will leads to my phi is b xy and d by 6 xy cube, Now which leads to two constant b and d. So, earlier if we remember that we were only doing with phi or the we had only 1 constant and which is if you remember which is a by 6 xy cube which was our phi. Now this phi and this phi, what is the difference is that? We just add another constant or another term in that phi and which is coming from the pure shear expression of the quadratic stress function. Now let us see, if we use this how it will look like.

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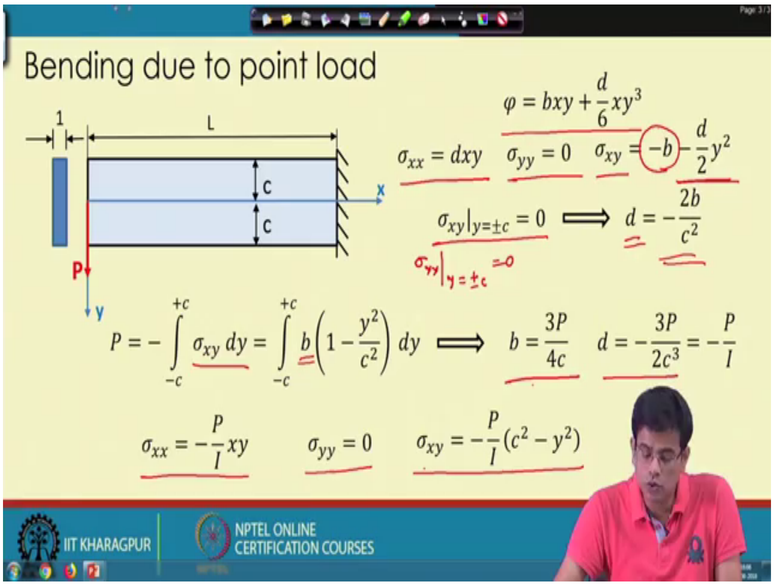

So, this is my phi and so, my stresses will be sigma xx will be dxy sigma yy will be 0. See these are very similar to the previous way of solving. This thing and the only difference lies in the sigma xy where the b minus b term is added with this term. So, now if we, if you carefully look, these already satisfy the forces that is sigma yy at sigma yy at y equals to plus minus c equals to 0. This already satisfy this phi so, which is sigma yy is essentially 0.

So, now by the other thing that sigma xy that is there is no shearing force at the top and bottom layer bottom fiber of the beam. So, this becomes gives us d is essentially minus 2 by c square. So, how if this you substitute y equals to c and then equate 2 to zero. Now the third condition that is minus c 2 plus c minus of sigma xy dy which is equals to resultant force P. Now this if we write in terms of b because we substitute d here and which is 2 b by c square.

And then plus and then use this put it back in this expression and then integrity, we find b is this. Now, once b is this, we can also calculate d from this expression and then finally, invoking the movement of inertia which is two-third c cube two third c cube, then we can write d is this. Now if you look the stresses in terms of these constants, then stresses is this sigma yy is 0 and sigma xy is 0 which is pretty similar with the previous case. So; that means, there are several ways to find out the solution of this problem within the stress function approach. So, you can even choose the stress function intuitively from your knowledge or some superposition of the two stress function depending on the stresses that will occur in the body. So, this is another approach we have seen here.

So, now; so, what we have discussed in the last class also that this sigma xx sigma yy and sigma xy is very much same with the our knowledge of stress stresses for the beam. So, which we have approached via strength of material approach and where we have considered as a one dimensional beam with a cross section and all those things. But here it is a truly two dimensional approach whereby which with certain modification on the stress functions, we got this thing which is similar to the beam of the stress strength of material. Now, our objective here now is to find out the displacements. What will be the displacements if we have these stresses?

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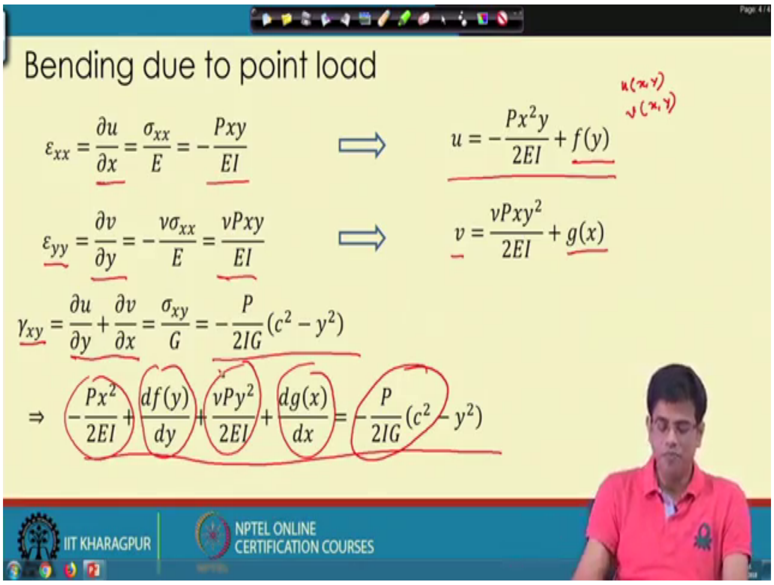


So, now and finding out displacements is very easy because you know the stresses very well. And if you know the stresses, then you have to invoke the material behaviour which will essentially here the isotropic Hooke's law here; isotropic material we are considering. So, which contains only poisons ratio and Young's modulus. So, epsilon xx we know which is del u del x and so, since sigma yy is 0. So, that is sigma xx by E. So, we know that from our Hooke's knowledge and sigma xx also we know which is minus pxy by ei. So, now if we solve this differential equation, then we get a form of u. So, which is minus Px square y by 2 here y plus fy. You see this integration is with respect to x and remember u is function of x and y and v is also function of x and y.

So, both of these are function of xy. So, the integration constant when you are differentiating with respect to x, then this will be a function of y. Similarly epsilon yy which is del v del y which is since sigma yy is 0. So, sigma yy by e term arm cancels. So, minus mu is the Poisson's ratio sigma xx by e and which is nu pxy by ey. Now, again by integrating this, I can get the v where the integration constant will be function of x.

Now if we substitute u and v in the shear strength which is gamma xy is del u del y plus del v del x which is sigma xy by g the shear modulus. The relation between shear modulus and Young's modulus and Poisson's ratio is we know which is e by 2 into 1 plus nu. So, the sigma xy also we know from our previous calculation and if we substitute this sigma xy, then the right hand side will be looks like this.

Now, if we substitute this u and v that means del u del y and del u del v del x here. So, we get a expression or we get a equation of this form. Now look carefully, these some terms in these equations are purely x, some terms in these equations are purely y and some terms are constant. For instance, this term is a constant term and this term is essentially purely a x dependent term and this is a y dependent term. This is also x dependent term and this is y dependent term.

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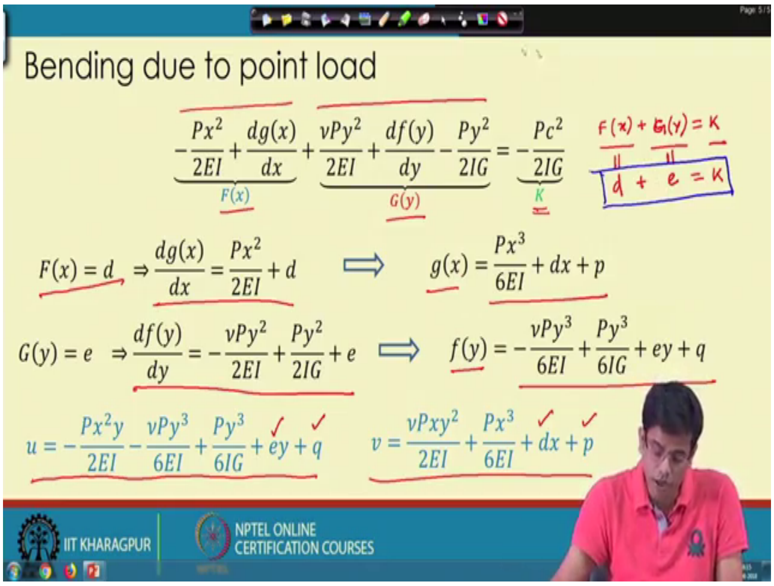


So, now if we write it in a proper way, so let us see how these things is. So, these all these two terms are x dependent which I am saying F x and this is a the function of x and this is y which is a function of y Gy and this is a constant which is K. Now you see if I write it here that F of x the function of x plus function of y sorry function of y Gy equals to some constant K.

So, now this is a function of x this is a function of y and this is constant. Now if it has to satisfy for all xy, then each of these has to be constant because otherwise it will not match with the K. So, let us assume that function of K function of x is constant which is d another constant and function of y which is e. Now, if we assume that function of x is z, function of x is d, then this term will be constant. Now if so, this represents this; now again this is the differential equation, Now this in differential equation, we can if we integrate it; then we get this g x which is a one variable function of x. So, this becomes this now similarly G y which is e essentially, we can write this as this as a differential equation.

Now, this differential equation again if we integrate; so, we get f y which is of this. Now finally, my u and v which I have obtained from the previous slides can be written in this form. So, this u and v just we plugged g x and f y in the u v. So, here you see that there are 4 constant one is e and q and then d and p. These four constants, we need to find out, now from these constants from where we will find it out. The first we need to realize one thing that d plus e equals to k this is another equation we have. So, these equation we should not forget because these there are 4 constants. So, we need to have a 4 equation at least.

So, to find out the complete displacements. So, d plus equals to k so, now, that reduces one of our constants. So, which is e or d, I can express in terms of the other. So, now, we need 3 more constant 3 more equations. So, what are those equations? Now this will come from the rigid body modes to prevent the rigid body modes. So, in a two dimensional problem what are the rigid body moves, we need to address for instance in a two dimensional problem.

We have 3 such thing that is x displacement y displacement and rotation about the xy plane. So, these 3 boundary conditions will give us the three required constant to find out the displacements properly let us see.

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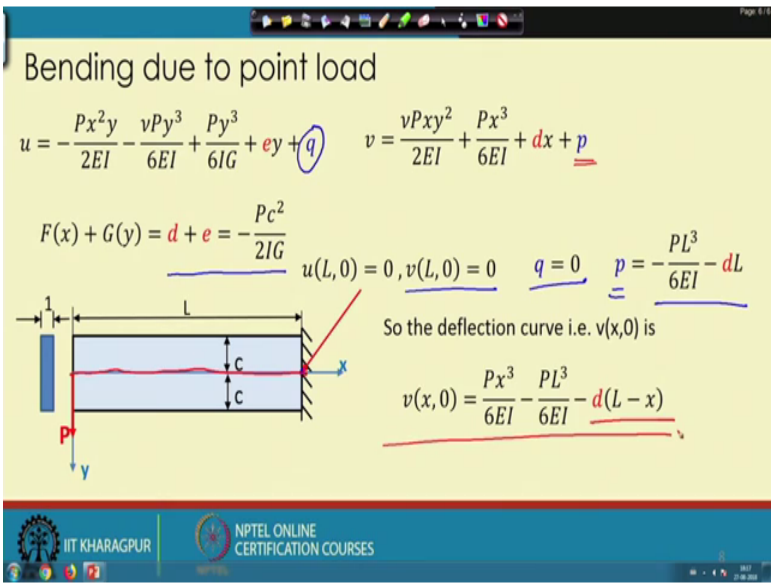

So now, here; now see, this is our constants which we are discussing in this; so, e and q. So, this is my one equation that we need to use.

So, first constant that we want to, first boundary condition that we want to use is that at this point that u at x equals to l and y equals to 0 v at x equals to l and y equals to 0 is 0; that means, this point is a fixed point. So, at this point if I use this boundary condition straight way from the first equation, I can say this q term goes to 0 right. Now, if I put this boundary condition here, then I can get p in terms of d right. So, now, the deflection curve which all of you have familiar with or the sometimes we also call it elastic equation of an elastic line or something like that.

So, this curve essentially this line which is at y equals to 0. We can find out which is v if I put y then P x is cube 3 6 E I minus PL cube by 6 E I. So, and in terms of d so, because again we did not solve this yet. So, P is in terms of d. So, that is why constant d remains in the elastic curve. Now you see to find out this elastic curve or the deflection curve, we need to find out d. So, what should be the d? Now finding out d, we can use at this point that our original knowledge or strength of material or another condition.

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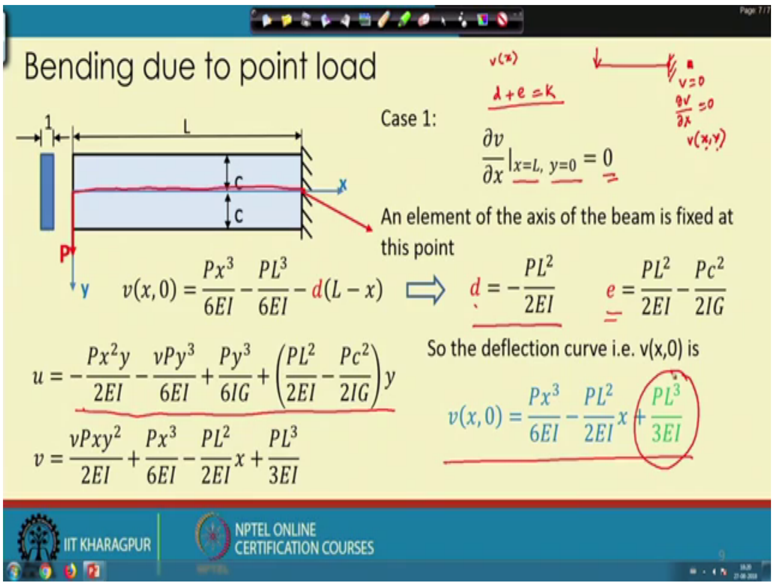


Let us see how to find out d here. So, let us see for the first case where if you remember that cantilever beam we are talking about. Then this is the what was the this thing at this point u in the v equals to 0 and del v del x equals to 0. Now what happened in the strength of material case is that v was a function of x only right. So, del v del x is 0 at this point; that means, slope at this point is 0.

But here, what is now is the problem is that our v is actually function of x, y because we are considering the body as a two dimensional body. Now this v x, y so; that means, x equals to l, y equals to 0 we have to consider. So, that x equals to l y equals to 0 del v del x is 0. So, if we consider those, what does this means? That means, at this point any element in the x axis will give me in the or the beam axis is fixed at the at this point. That means the rotation along the axis of the beam is fixed. So, this actually gives me this; so, now, if I use this condition, then I can find out d which is this. Now once I find out d and if you remember our another condition was d plus e equals to k which k is essentially minus Pc square by 2 I G. So, this equation if we substitute d, then we get the E.

Now, once we find out d and e, we can find out the complete solution in terms of all constant known. So, and this solution is if you look carefully this is u solution means along x axis what is the displacement and this is the v solution. Now here, if you look carefully that if I put y equals to 0; that means this line is my deflection curve. So, that deflection curve is at y equals to 0 which is this. So, this we obtained some of you have must have obtained from the in the strength of material or integrating the differential equation that is beam differential equation b to the power four y by dx to the power 4 or the moment curvature relation.

So, now here if you remember what was the displacement for this get, it is PL; PL cube by 3I. So, if you see that if I put x equals to 0 here; so, these terms actually gives us the PL cube by 3 I the original strength of material which we have derived from our strength of material knowledge. Now in addition to that it also gives you that u v so; that means, through this stress function approach, what we have achieved is that more accurate expression if the beam is two dimensional. If we consider the beam is two dimensional what will be the displacements in x direction, what will be the displacement in the y direction, we have achieved more.

For instance, if the cross sectional displacement if I want to find out somewhere here the cross section which is at a distance y from something then this expression will directly give me the cross sectional displacement of the beam. So, this can also be extended for the three dimensional problem, but again the expression will be very lengthy and complicated thing.

So, idea is basic idea is to know the things the procedure and how we can approach it; so, the for this kind of problem. So, this so, this solution of this displacement even though it matches the stresses are matches with the our strength of material case, strength of material knowledge, but we did not have any way to find out those u displacement and v displacement at these points right or we did not do that at that time. So, here the theory of elasticity, knowledge of theory of elasticity allows us to analyze it more accurately or to find out the displacements at the cross sections and in the x direction.

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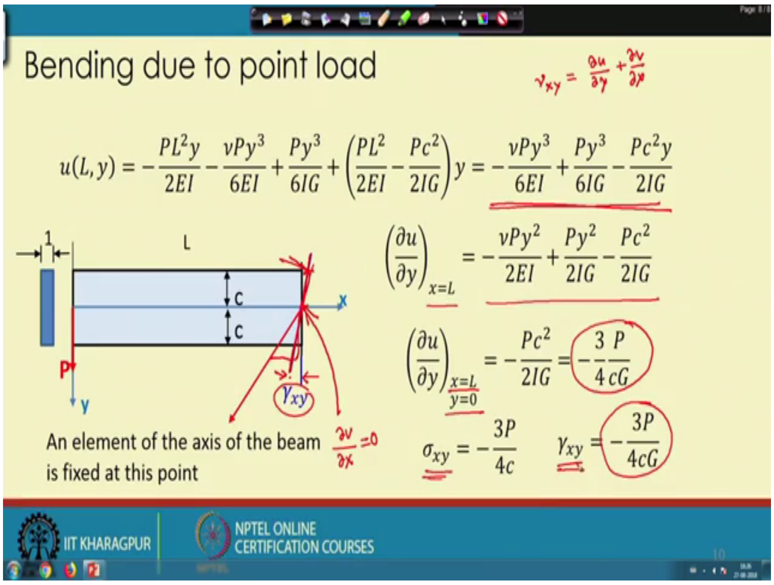


Now another important fact let us see. If we use these displacements and to find out the what happens to the stresses in the beam. So, you see this is our displacement u L, y means along the Y.

So, if we now find out if we now do some manipulation, then we can see that our u displacement is 0; that means, at x equals to L at this point; x equals to L and we are checking the what is the variation of the displacement. So, now, this is my displacement along this line. Now the del u del y at x equals to L which is this. So, that means, the change of this u displacement along the vertical line of the or the fixed end of the beam. So, if we now you see what is this displacement, from where we obtained this displacement we obtained this displacement that here what we have assumed that del v del x is essentially 0 at this point. So, at this point this del v del x is essentially 0 that we have assumed so, through this we have obtained this these displacements.

And now if we see what happens to the vertical fibers. So, that is del u del y at x equals to y is this and so, that means, it gives me this line which is actually the if the beam is originally this. So, this gives me a line so, a deformation which the tangent to this is this line.

So, this tangent is essentially representing this the value the angle here this angle this angle is essentially the gamma xy or the 3P by 4cG. So, which is essentially how we can obtain means how the vertical fiber and the fixed end will rotate. So, this rotation can be obtained if you find at x equals to L and y equals to 0. Find out this quantity which is essentially minus Pc square by 2IG and 2IG um that I can be substituted at is two-third c cube.

And if we do that, then this will be my this quantity. Now sigma xy if you remember at this point which is del or the gamma xy if you remember that del u del y plus; that means, gamma xy is I am talking about this point. So, del u del y plus del v del x. So, del v del x we have taken 0. So, now, del u del y will be my gamma xy. So, del u del y is essentially 3P by 4cG and the shear stress is 3P by 4c minus 3P by. So, that is the counter that is the clockwise rotation of the fiber; the vertical fiber. At this end, so this gives me additional information how the shearing effect happened in the beam and this can be found out at any cross sections.

So, what we have seen here is that if we take the usual strength of material approach and then find out what the displacements theory of Elasticity approach gives us little more information about the body.

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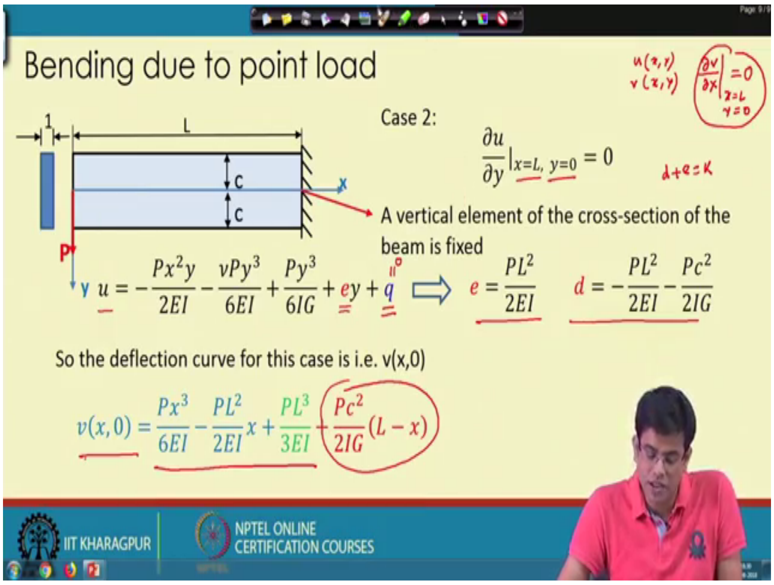


Now, instead of that if you remember that we had of u here is a function of xy and v as a function of xy. So, in the earlier case 1, the in plane rotation; how we have prevented in plane rotation? We prevented by introducing the concept introducing the condition the del v del x at x equals to L y equals to 0 is 0 that we have invoked.

So, which is essentially coming which is very similar to the strength of material approach for a cantilever beam at the fixed end. But instead of that if I want to restrict the vertical fiber in the element, a vertical element on the fiber; if I want to restrict, then this condition; that means, del u del y at x equals to L and y equals to L. If I fix this so; that means, a vertical element of the cross section of the beam is fixed at this point. So, earlier we are fixing the horizontal element now here I am fixing the vertical element. So, now, if I fix this thing in this beam, then my again I can find out the constants which is essentially the u.

So, if I take the derivative of u with respect to y, then this term goes off and this term already we know which is 0. So, this term goes off. So, we can find out e and e is this. Now once we get the e; so, e plus d plus e equals to k which is we know from earlier equation. So, we can substitute e and find out d.

Now, if you see if I now want to find out the elastic curve which is in addition to this original elastic curve for this condition, we have one more term here. So, you see this term which is not very similar to the strength of material approach; that means, the this L minus x Pc square by 2IG L minus x term is additionally we creating the displacements.

That means, that if I change if I instead of restricting the horizontal behaviour or a horizontal rotation instead of at this point. If I prevent the element of the vertical line at this point to rotate, then our deflected curve is essentially different. So, these actually known as effect of shearing on the deflection.

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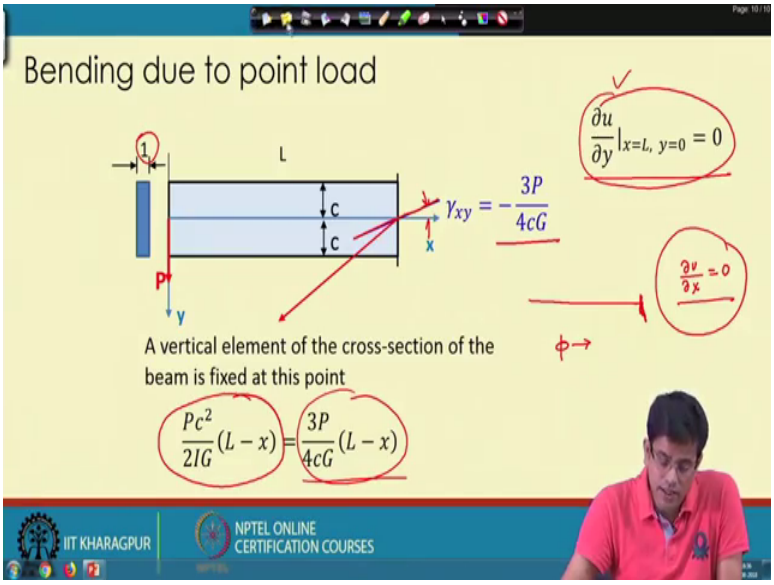


So, now we will see what this means. So, here actually in the same way if we have seen this as the deflected curve, now this deflected curve with this a boundary condition. The deflected curve again it rotates in this way. So, this again the rotation from this which is which can be written as the minus 3P by 4cG the same way we have derived it for the previous case.

Now, this extra term is actually can also be written in this form which is very simple because if I now substitute the I, then I get this term. So, you see the main distinction between two boundary conditions. One of these is so, at this point this is the elastic line at this point. If I want to fix del u del v by del x equals to 0 then I get something which is very similar to the Strength of material thing.

That means, if I want to fix an element at this point in the axis of the beam, then I get a type of deflection. And then if I fix instead of these, if I fix the horizontal element at this point which is governed by this equation, I get a different deflected curve; that means, and this difference extra thing is this quantity. So, this is known as the effect of shearing on the deflection.

So, clearly this strength of material this theory of elasticity approach gives us more flexibility, more opportunity to find out or to see the how the deformation on a body really happens. So, it is in depth we can analyze the structure for instance that we could not have u displacement in the strength of materials.

We could not find it out the because theory does not allow us to do that, but theory of elasticity gives you the opportunity of how you really can do this and this is a stress function approach we have discuss we have discussed. So, in an action what we have learnt so, we finally, summarize it. So, what we have learn we just took one beam a simple cantilever beam for which the strength of we have the every knowledge of strength of material that is what are the stresses, what are the deflected curve, equation of the deflated curve everything we know.

And then we assume some stress function phi which I have shown you how to do it for two from an intuitive sense and how to use the different stress patterns to superimpose and to get the required shape function stress functions. So, once we solve this stress, once we find out the stress functions; stress functions will certainly will have some constants. And our job is to find out those constants through the stress boundary conditions.

So, with this stress boundary conditions, we find out the stresses correctly. Now once we get the stresses what we need to find out? We need to find out displacements. So, we invoke the material laws or the constitutive laws into it and then we find out the displacements from it. So, for instance in this case, the epsilon xx epsilon yy gamma xy the shear strength so, we know these how these are related with the displacements.

So, then again we have a differential equation in terms of u which is first order differential equation. So, this first order differential equation we can integrate, then again we have some constants. So, those constants again we have to find out from the displacement boundary conditions and the slope boundary conditions.

So, for the slope boundary condition and displacement boundary condition is essentially to prevent the rigid body rigid body motion of the beam so, that it does not rotate arbitrarily. So, rotate and translate. So, these to prevent this rigid body motion, we need to substitute those displacement and slope boundary condition. In the slope boundary condition, we see we can use different slope boundary condition. For instance, we can use this or we can use this.

So, if we use different boundary condition naturally, we will have a different deflected curve and so, that means, we can find out the deflection in u as well the displacement u and v in the both the direction because our problem is essentially a two dimensional problem here. So, this can be certainly extended to three dimensional problem and the stress function will be three dimensional case for a three dimensional three variables xyz 3 variables. So, polynomials of three variables we have to consider. So, number of constraints will; obviously, increase.

Now, here we have used the unique thickness or essentially the thickness is very less compared to the depth of the beam. So, this restriction can also be hah removed and we can analyze the beam as a three dimensional problem, but yes the expression will be much more complicated and one can surely try those things. So, with this I stop here today. So, we will meet in the next class.

Thank you.