Theory of Elasticity
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Lecture – 41
Boundary Value Problems in Elasticity(Contd.)

Hello everyone, this is the 4th lecture of this week. In the last class what we did is we demonstrated the torsion formulation through one example. We chose example of a example of a shaft with elliptical cross section. We derived the expression for stress, expression for warping. And then when then we saw then when that ellipse becomes a circle then the solution is consistent with the solution that we had in strength of material. And also we saw that the warping of the cross section when it becomes a circle is 0.

Now, let us continue with the same thing will the purpose of today's lecture is 2. First is we will see one more example for example, of a different cross section. And then so far we have done stress based formulation we will see what is the displacement formulation, for torsion problem.

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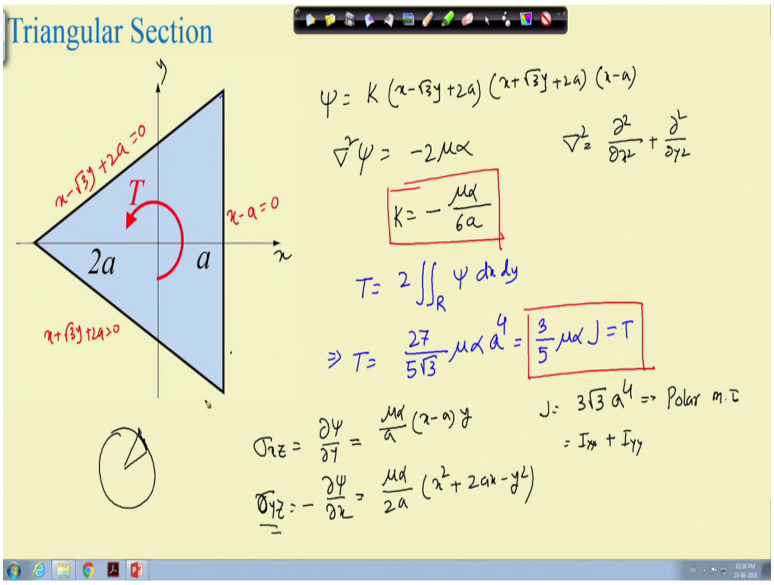

So, let us start with another example; this example is the cross section is a is a equilateral triangle. The as far as steps are concerned it is exactly same that we already discussed in the last class.

It is just we are demonstrating the same steps through different examples and then checking how these warping function is different for different problems, how these stresses are different for different problems, but the essence of the steps are all same ok. So, what we let us start with this example. So, first thing we have to consider the expression we have to assume the expression of psi. And then expression of psi should be such that it vanishes at the boundary. And the expression that we chose in the case of elliptical cross section is the a equation of the ellipse.

The same thing we do here, now since it has 3 components 3 parts. So, these lines these 3 lines have 3 different equations. So, what we can do is; we can for instance what is the equation of this line, if we write the equation of this line. Equation of this line is x is equal to x minus a is equal to 0. And then if we write the equation of the other this line this line it becomes x minus, x minus root 3 y you can check it plus 2 a that is equal to 0. And then equation of this line will be it is x plus root 3 y plus 2 a that is equal to 0. So, this is the equation right.

Now, then we can take psi as psi is equal to some constant the same constant. Then this is equal to this x minus root 3 y plus 2 a and then x plus root 3 y plus 2 a. And then finally, x minus a and you look at this if you express the psi if you substitute the values of x and x and y at the boundary the psi becomes 0. So, this is our assumption of psi.

Now, once we have that psi you have to determine the constant k. Now, what is the equation? The equation was del 2 psi the Poisson's equation that was is equal to minus 2 mu alpha. Mu is the shear modulus and alpha is the angle of twist per unit length. If we do that exercise and del 2 operator is we know that del 2 del x 2 plus del 2 del y 2 right.

Now, if we substitute psi in these expression and then calculate the value of k, the K will be you can do this exercise I am just writing the final expression; K will be minus mu alpha by 6 a. So, this is the expression for K constant K. So, we have the expression now we have the expression of psi as a function of x and y and other material parameters. Now once we have that then you recall we also know that, since psi vanishes at the boundary it satisfy all these boundary conditions all the boundary traction boundary condition of the surfaces.

Now the end boundary condition will be end boundary condition is, where at the end where torsion is being applied. And that relation between the torsion T and the and this and the and the function psi, the stress function psi we know that relation is T is equal to T is equal to 2 into integration over the cross section R psi dx dy.

Now, psi is the expression of psi is known with K is this, and then you can substitute this and we can integrate the entire thing over this triangle. And if we do that exercise that expression will be T is equal to 27. I have the final solution with me 5 root 3 then mu alpha a to the power 4, a is the this is how the a is defined 3 a is the height of this equilateral triangle.

Now, this is essentially becomes 3 by 5, you can write it 3 by 5 mu alpha then j, j is the where this is equal to torsion T. So, this is the equation ok, where j is equal to this j, j is equal to polar moment of inertia that is for rectangular cross section here. This is equal to 3 root 3 a 2 the power 4 that is polar moment of inertia ok, or second moment of inertia.

So, this j is equal to Ixx plus I you know j is equal to Ixx plus Iyy ok. So, this is the relation between torsion and the and the angle of twist. Now, once we have that then let us find out what is the distribution of stresses. Now recall only nonzero stresses are sigma xz and sigma yz; all other stresses are 0, because we are only considering the torsion. Now the nonzero stress components are related to related to stress function that relation is if you recall sigma xz sigma xz, that is equal to del psi del y and then sigma yz, sigma yz is equal to minus del psi del x ok.

Now, if we substitute this relation if we substitute psi from here, and write this expression and then final expression of this will be. So, this will be mu alpha mu alpha by a you can please check x minus a x minus a into y. And this expression will be mu alpha by 2 a. Now x square plus 2 a x minus y square. Now look at let us see whether it is by intuition it is consistent with our intuition or not see what is sigma xz on this plane. If you take on this axis x is equal to a, x is equal to a what would be the this is x axis; this is x axis and this is y axis.

So, on this plane what stress we have on this plane the stresses we have is only sigma yz right. And what is the direction of sigma xz, direction of sigma xz is in this direction means; on this surface if you try to visualize it is a 3 d object. It is on this surface, but in the along the longitudinal direction that is the direction of sigma xz. And that stress will be 0, only the stress we have is in this direction because of the torsion. And which is consistent with this formulation if you substitute x is equal to a here, then we see this is equal to 0.

Similarly, if you take the component of the corresponding stresses on this and on this and then check those stresses will be also be 0. Now this is the expression of stress. Now, once we have the expression of component of the stresses we have to find out what is the maximum stress, what is the expression for tau max. Now recall in the case of elliptical section, we tau max how only once we have the expression of tau, we can differentiate it will exclude the coordinate axis and find out for which value of x and y this stress will be maximum.

Now, one thing we observed if you recall the example that we solved in the in the in the in the previous class. The stresses are maximum at the surface and which is by intuition also you can say. Based on the strength of material you already had the solution of circular shape. And where were these stresses were maximum, it is maximum at the periphery the when at the at the outermost circle right. So, that is on the surface it is the maximum ok.

So, if we have if you have a circular section. So, your stress is this the at this point this is the maximum stress. So, it here also it will be same on this surface only on this plane this stress will be maximum. Now look at since it is symmetric with respect to the longitudinal axis. So, all and all these surfaces your stress will be same the shear stress will be same. If it is same then naturally we can say if we substitute because finding stress on this surface will be easier, we can just substitute x is equal to a. So, finding stress on the surface shear stress due to torsion. On the surface is enough because on other surface is also the corresponding stresses will be same because of symmetry.

And if you do that exercise and then stress will be I just write the final component of the stress.

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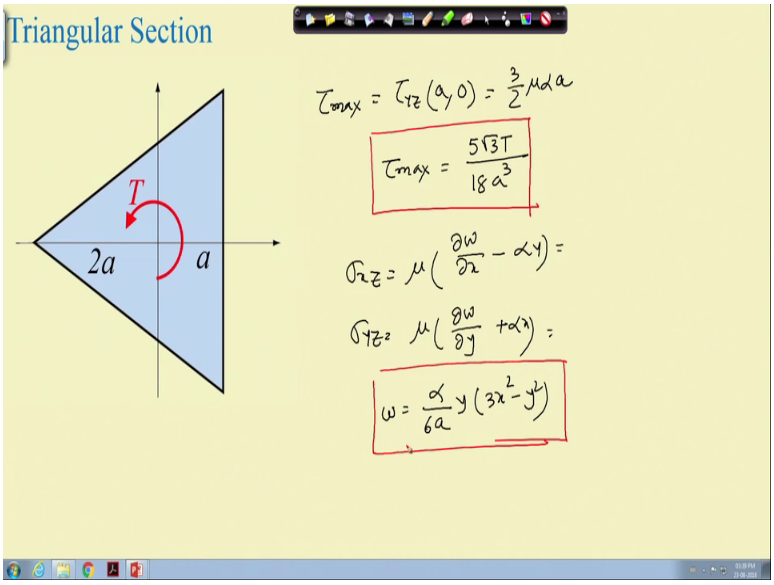

The stress will be tau max, tau max is equal to which is tau yz. You get at x is equal to a and y is equal to 0 and this will be 3 by 2 mu into alpha into a. And if you write in terms of in terms of torsion, tau max will be if you substitute alpha in the previous expression, we know the relation between alpha and T. And if we substitute this relation into this the final expression we get is 5 root 3 into torsion divided by 18 a cube ok. This is the final expression for shear stress.

So, this is how the shear stress and torsion is the torque is related to each other. And the final is the we have to find out what is the expression for expression for w. And w expression you recall how to find out the expression of w, the equation that we use is the strain. The sigma xz if you recall sigma xz was mu into del w del x minus this equation, you already derive in previous classes and sigma yz is equal to mu into del w del y plus alpha x.

Now sigma xz expression for sigma xz is this and expression for sigma yz is this. Now if we substitute expression for sigma xz here and then integrate it get w. And in that integration you will have some function of y because that integration will be with respect to x. And that function of that function unknown function y can be obtained by the second equation this equation. And if you do that exercise and this exercise as I said previous class also these exercise we have done many times, in many examples in this course.

Now, if you do that the final expression of w will be alpha by 6 a y into 3 x square minus y square. This is the final expression for this ok. So, this is the warping w now um. So, you look at the steps as far steps are concerned it is exactly same the same steps we followed in the case of elliptical section. And if it is other section also if as long as you are use using this formulation the stress function form will based on stress function, your steps will be exactly the same.

Now, once we have w let us see how if we if we plot the contour of w on this area and how it looks like, it will be something like this.

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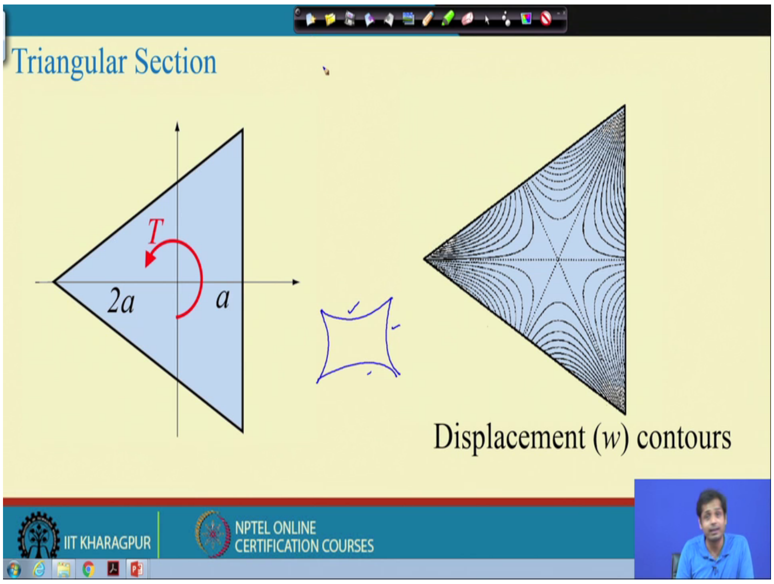

This is the displacement w on this similarly you can plot the contour of tau x contour of tau xz tau yz. Now so this is the displacement control for rectangular for triangular section.

Now, similar approach you can follow for other section as well; for instance, if your sections are your section could be square, it could be rectangle, it could be rectangle with

Different aspects we choose. If you have those kind of section, you your section could be something like this as well; for instance, if your if your section could be something like say something like this. So, if your section is something like this procedure exactly same you follow, you know the equation of all these surfaces. You assume we have to assume size such that when it is substituted on this boundary that value is that value becomes 0. Now, and then rest of the process is exactly the same.

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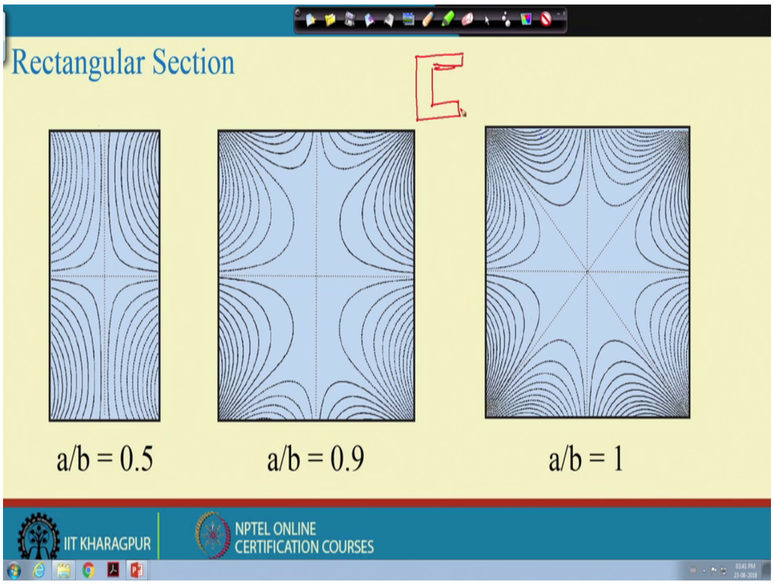

Now, let us see when your circle these are the 3 different cases the last extreme one is the square. And the first one is the rectangular section with aspect ratio this with a by b a 0.5, when a is the smaller dimension and b is a larger dimension. Here it is close to point close to 1 it is 1 and you can see how this distribution of the warping w they take different forms. So, if you have different cross section.

Now, you can you can have a cross section like this as well, suppose you can have a cross section like this like a channel section and this is subjected to torsion. And same way you can apply to get the distribution of this warping or the w, or to get the distribution of stresses on the area. So, this so what we have done so far is; we have we have we have derived the formulation for a torsion and that formulation we derived is based on stresses.

So, essentially these are the stress formulation. And then we also demonstrated that formulation through some examples. And other examples other problems you can follow the similar approach. Now, let us now formulate for the completeness of the our discussion, let us see if we have to derive equation in terms of displacement the same way that we did in case of (Refer Time: 15:50); in case of in case in the previous classes, when we discuss plane stress plane strain problem we discuss formulation based on stresses and based on displacement both.

The displacement based formulation essentially was a Navier's equation. So, let us let us derive the similar things similar equivalent equation for torsion.

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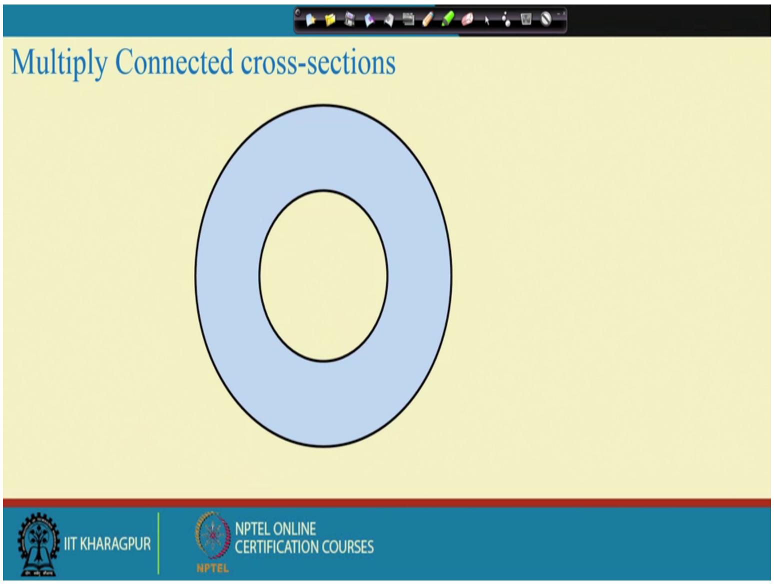

Now, so next is our second part of today's lecture is before that; you see all the examples that we have solved so far or not solved, but gave some gave some example or the final expression. All these cases your domain the cross section is simply connected domain.

But you may have a domain cross section which is not simply connected, multiply connected. As I was discussing in the last class suppose, you have a pipe and which is that pipe is subjected to torsion. Then what happens to the distribution of stresses? What happens to the warping? Whether at all warping takes place or not, if takes place and how to determine those how to compute those values so that is an important problem?

So, next class we will address this problem, how to if your cross section is something like this then how to deal with this cross section, but today let us derive the equation for displacement ok; displacement based formulation ok. So, displacement based formulation if we, if we recall our once again means; everything we have to write in terms of displacements. You recall when we when we actually for plane stress plane strain problem when we derive it when we derive Navier's equation. We use the equilibrium equation and in the equilibrium equation we substitute the stress strain relation. And then strain displacement relation and the final expression we get in terms of displacement.

We did not use explicitly compatibility equation because that was not necessary because your writing equation directly in terms of displacements. So, it is the same for this as well we will be using only equilibrium equation and the stress strain relation, and strain displacement relation right.

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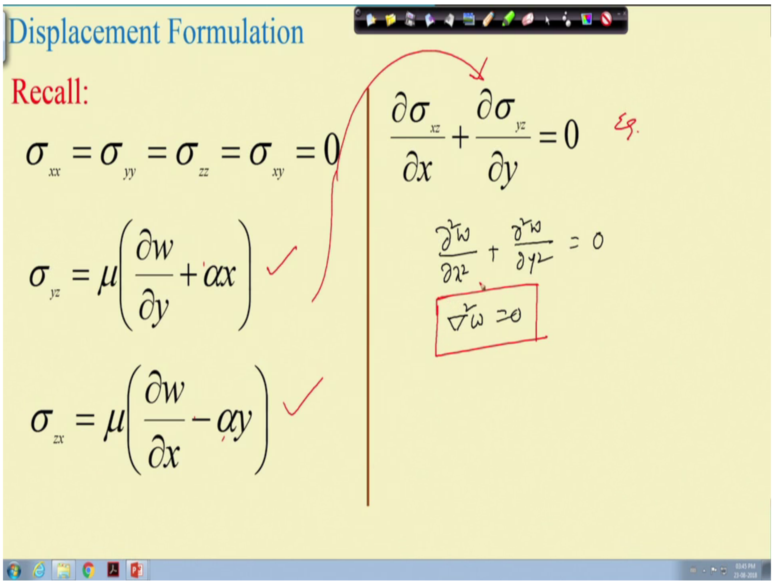

Now, if you recall these are the relations that we have this is equilibrium equation. This is the equilibrium equation and this is anyway, this is we started with these assumptions that all other stress components are 0 except sigma yz and sigma xz.

And sigma yz and sigma xz is related to displacement, as this now this relation this relation and this is the relation essentially stress strain and strain displacement relation both are included. So now, next what we have to do is we have to substitute these 2 expression into equilibrium equation. And if you do that we get an expression in terms of only displacement. And if you do that then your expression becomes this will ok. So, so if we do that then this becomes, your final expression becomes del 2 you can check del 2 w del x 2 plus del 2 w del y 2 that is equal to 0 ok.

So, this is Laplace equation del 2 w is equal to 0. So, if you write torsion the same thing in terms of displacement, the governing equation become Laplace equation, but recall when you wrote when we when we had written these expression in terms of stresses; that equation became the final expression was in terms of phase function, the final expression was Poisson's equation ok. So, once we have this then let us if you recall ok.

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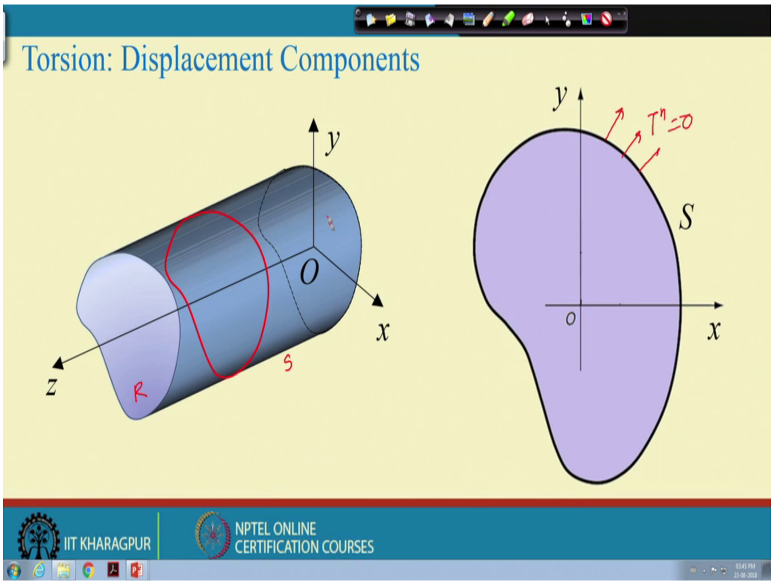

Then what we have is so, so this is the surface of this surface of this, this is the surface, this is let us use different color this is the surface and this is R right and this is also ok.

So, on the surface your tractions are 0 anyway, always these tractions the T n the it is anyway 0. So, all these surface is a stress free surfaces and the traction boundary condition needs to be applied here. And the traction boundary condition becomes if you recall the same.

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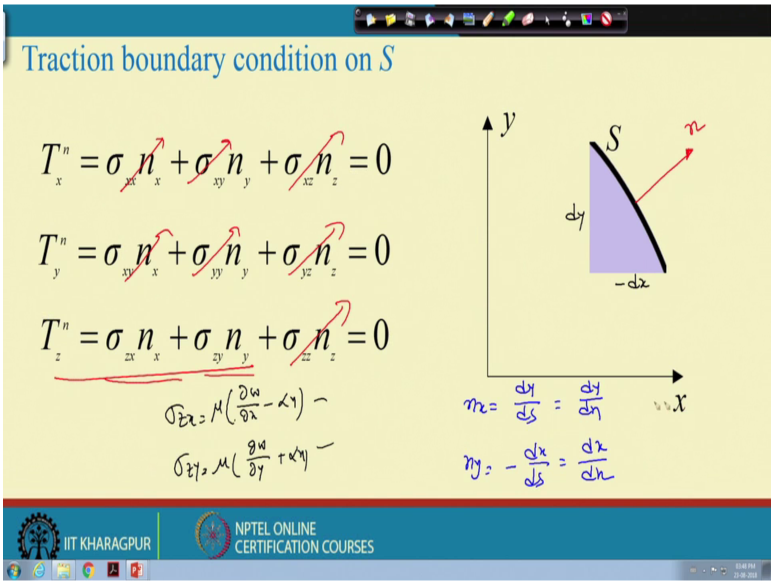

These are this is the way we derive we have the traction, and on this surface all these traction boundary all these tractions are 0.

But since all these components these components vanishes, these components all component vanishes because either nz is equal to 0, or sigma z z is equal to 0, only nonzero components you have is this so your traction essentially becomes this right. Now if you recall what was the expression for nx and if this is minus dx, and this is dy, and this is your n and this horizontal component is nx and ny. It is bit components in x on y direction is nx and ny, then the nx was the nx was dy by ds and ny was minus dx by ds.

Now, this is s and it is normal is n. So, if you write this expression in terms of it is normal then this becomes dy by d n, and this becomes dx by d n ok. So, n is the outward normal this. Now so if I substitute this expression here and we also know that is sigma z and sigma in some of these in some places we already wrote it that expression of expression of sigma z ok. So, expression of zx just now we had the expression of zx, zx is equal to del w del x minus alpha y, that is and then there is a mu here. And then sigma zy or yz mu del w del y plus alpha x.

Now, if we substitute this and this into this equation, the final equation what we get is this, we get an equation like this.

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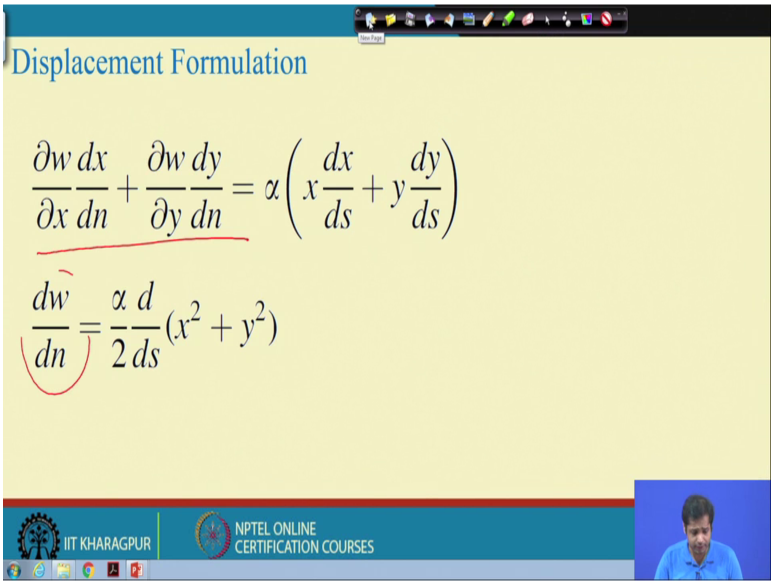

Now so what we have now note that before that let us write let us let us see that. So, you get an expression like this and then finally, what you do is you substitute this if we substitute this dw dn. This if we if we if we take the derivative with respect to s outside and then this we can be this part right hand side part can be written as, this right hand side part can be written as dw dn. And if we take from left hand side dw dn, right hand side if I take d ds outside then we have then equation like this ok.

So, this is important now then finally, we have to apply the boundary condition.

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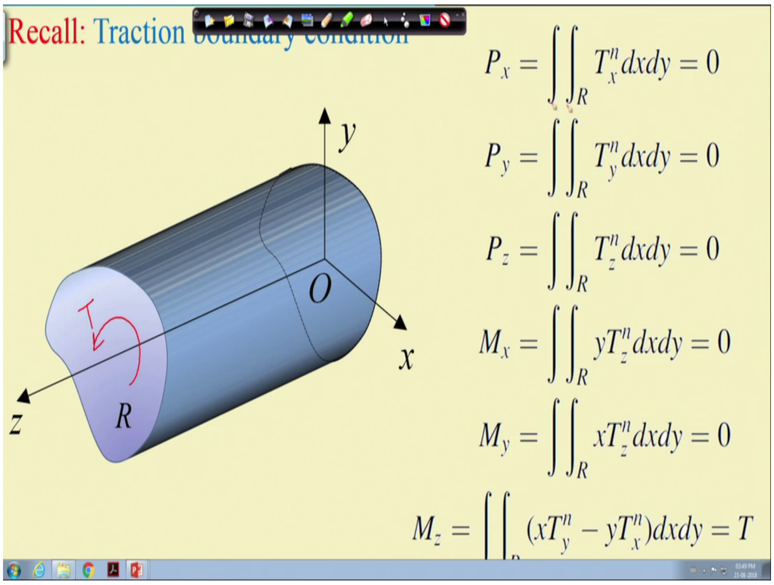

The boundary condition is these are the boundary condition if you recall. These are the traction boundary condition now these traction boundary conditions; all these boundary condition we can show that these on the surface this homogeneous boundary condition is satisfied. At the end this has to be related with the applied torque. So, if this is the applied torque this is the torque applied Y So, this should be equal to T.

Now if I substitute x, if I substitute expression of Ty and Tn the expression just now we have and in this then the expression final expression we have is this.

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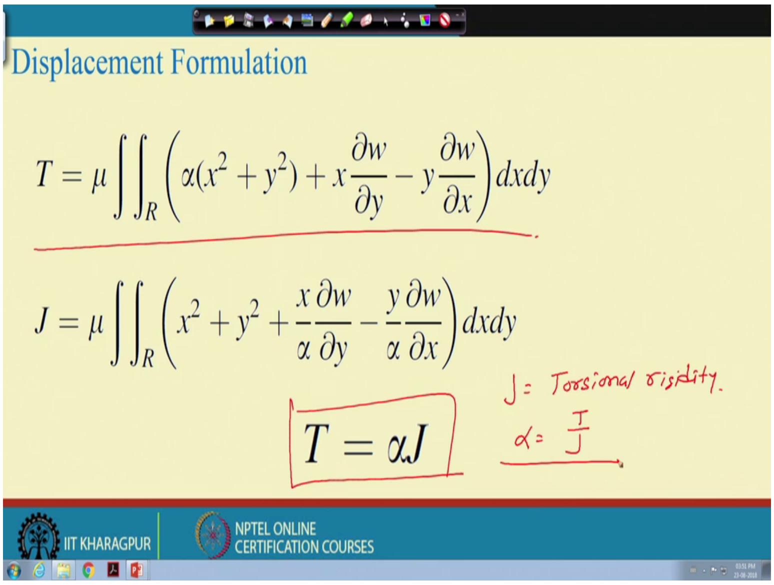

So, since we have done this exercise for stress formulation I am not doing it explicitly here. You can just it is substitution of the expression and get the final form. This is the expression of T that you can get, now in that expression if that expression again can be rewritten as this, but before rewritten as this if we say that J is equal to this from this expression from the entire expression. This expression if we suck if we which we define j as this and substitute j here. The final expression we get is T is equal to alpha into J. J is called torsional rigidity J is called torsional rigidity

You see why it is called why it is called rigidity; essentially what you see alpha is equal to alpha is equal to you have T by J right? What is alpha? Alpha is alpha is the twist angle of twist per unit length. Now if we take and this if we apply torque if you if you apply more torque the angle of 2 J increases, but for a given torque how, how alpha is related to J. Alpha is J is more, then alpha is less means; for a for a if the body is more rigid then your angle of twist will be less which is very obvious. And if the body is less rigid flexible your angle of twist is more, that is why J is called rigidity and since it is related to the angle of twist related with torsion, this gives you this is called torsional rigidity ok.

So, this is the final form of this is these are the formulation form, this is the formulation for the same problem, but written in terms of displacement. Now let us let us we will stop here today. What we do next class is as I just said this displacement based formulation is discussed just for the completeness. So, that you know there is a formulation you can do. You can formulate this in terms of displacement as well like you like we have done it in previous cases.

So, this is the formulation for that. Now, we will not be doing any you will not be demonstrating this formulation through any example um. In our next class is as I just now said we will see the how to address a multiple if for a problem, where your cross section is not simply connected, whether we can use the same set of equations or we have to modify the equations. If at all we have to modify the equation how the modification has to be done. That is the subject of a next class which will be the last class of this week ok. Stop here today see you in the next class.

Thank you.