Theory of Elasticity

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Lecture – 42
Boundary Value Problems in Elasticity (Contd.)

Hello everyone, this is the last class of this week what we have done so far is we have done the formulation for torsion and that formulation has been demonstrated through some examples. Now in all the examples the cross section of the shaft which is under torsion that cross section was assumed to be simply connected. Now, but in many situation it may so happen you are the shaft that which is undergoing torsion that the cross section is not simply connect it may multiply connected domain.

For instance if you take a circular pipe and which is under torsion the if we take the cross section then the cross section is no longer simply connected is a multiply connected domain. So, what we will see today if your cross section is multiply connected then what modification in the formulation that we have already done what modification in that formulation is required.

And of course, after that modification we will have that that modification will be demonstrated through one example. So, today's topic is the torsion formulation for an of course, with an example for multiply connected cross section you see, if you recall when we discuss about this is the shaft which is subjected to torsion and the domain is multiply connected.

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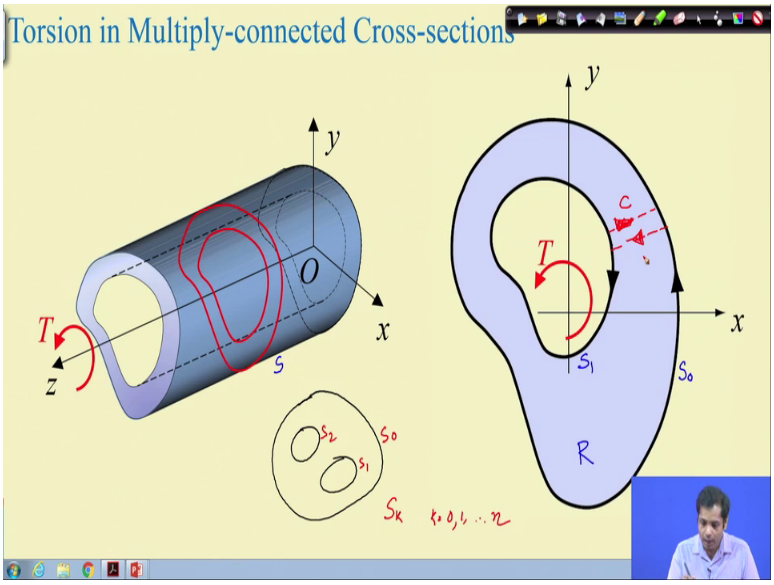

Now if I take any cross section in the domain any arbitrary cross section here, and that cross section is essentially this. Now we want to do and this cross section is subjected to this is subjected to torsion. Now what we want to do is, we want to find out what is the relation between this torsion and the angle of twist what would be the shear stress distribution what would be the out of plane displace out of plane deformation that the same thing that we have done for simply connected domain you see.

If you recall in some of the earlier classes when we were discussing about equilibrium compatibility that time it was mention. The compatibility is a compatibility guaranteed your single value displacement field, if the domain is simply connected for multiply connected domain the compatibility condition is a necessary condition, but may not be sufficient condition.

And if you recall when we talk about circular the formulation in polar coordinate we gave an example, where the compatibility condition is satisfied with the still the single valued ness the uniqueness of the displacement field could not be obtained. Now so the modification required for multiply connected domain in torsion formulation lies in these difference, so let us see how it is to be done.

Now, very similar to our usual notion that, we are using; this is the cross section and this cross section is represented as R this is this is R and then this is the lateral surfaces this is S. Now since we do not have just one lateral surfaces we have outer surfaces and we have inner surfaces in this example we have just one inner surface, but we can have more than one as well. So, let us say these outer surface is so in simply connected domain we did not have to bother about this S was just the lateral surface, but here, so is the outer surface.

And say S 1 is the inner surface S 1 because it is just one internal loop we have if we have an if we have domain like this. Suppose if you take a domain which has two which has this another to another hole then we can have then this will be so this will be S 1 and this will be S 2 and so on ok. So, we can have if it is kth k number of loops we have then it is generated as Sk, where k is equal to 0 to 0 1 to n.

So, S 0 stands for outer surface and all other the all other integers in this is non stands for the inner loops. Now the we will see in the formulation we have to do a contour integration the contour integration will be discussed in slightly details, in the next week when you talk about complex variable approach and there we will see for the time being you take it for granted there you will see then if your domain is come multiply connected domain then you can cut the domain.

And make it because if you see if you start from this point and if you start from this point without holding your pen you cannot come to the inner loop. So, there is no continuity so what you can do is you can make it continuous by cutting this plane cutting this entire domain and this if this is the cut surface. Then what you can do is suppose you move from this you move from this side, and then you move from this side and then go this side and then you move from this side go this and then go this side.

So, you can complete the entire loop and suppose this is C this is (Refer Time: 05:45) C. The application of this the usefulness of this concept will be discussed in the in next week when you talk about complex variable approach. Now so let us see what is what is the formulation.

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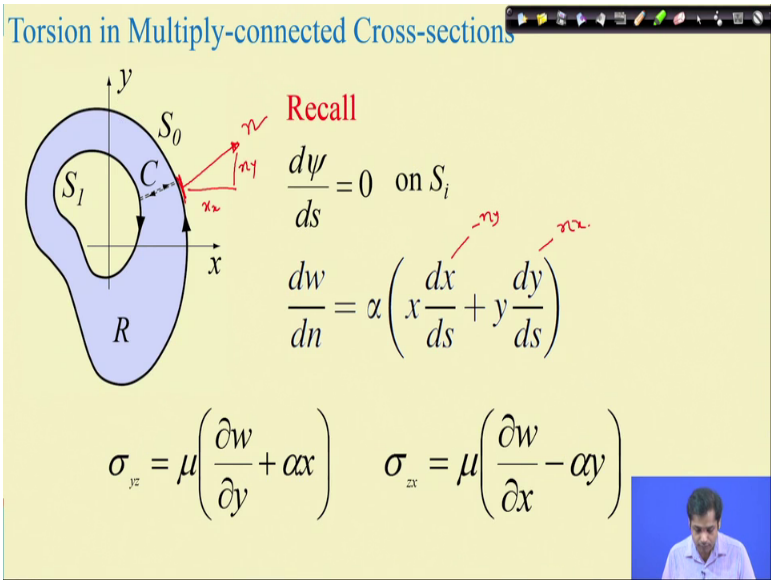

Now, so move on so this is this just now that we discuss this is the domain R and this is C we can cut it you can make it simply connected by cutting it we can make the boundary continuous so, that the contour integration, integration over the entire boundary can be obtained ok. So, next is if you recall these are the things when we talk about the this equation we derived for derived earlier that psi has to be constant on the boundary.

And for simply connected domain that we assume that is that constant value is 0. Now then we also have in the last class we discussed when we in the displacement formulation that the, if w is the outer plane displacement that outer plane displacement is related to related to the angle of twist. And this unit normal as this were if you recall this is essentially your this will be ny this will be minus this is minus ny and this is nx. So, if you take any small domain here any small domain and this is your outward normal n then this is nx and ny nx and ny.

So, this nx and ny related to dx ds as this that we discussed in the last class, so this is how the displacement is related and then we also have if you recall the relation between the stress and the displacement as this. So, what we are going to do is we are going to use these expressions along with the additional expression which is for multiply connectedness of this cross section, which ensures a unique or single valued displacement field.

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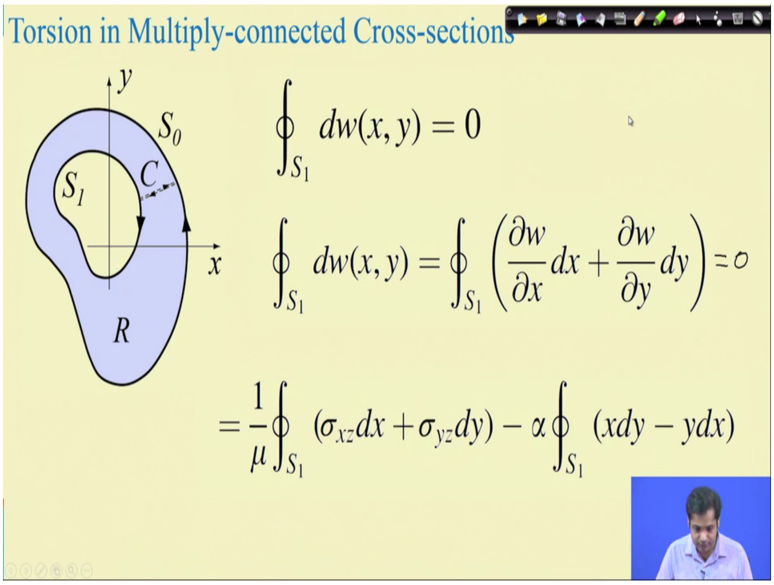

Now so you see the single valued displacement field on this domain if w is the outer plane displacement then can be obtained by this condition, for the time being this as I said this contour integration for the time being you can take it for granted this will be discussed in the next class complex variable approach.

So, we can show that that single valued ness of the displacement field can be guaranteed if the displacement satisfies this equation. Now if this is the equation we have then if we go if we go to the previous slide and see how the and write these expression of w substitute this expression of w here then what we have is this. Now so and then this is on S 1 for the time being for demonstration of the example demonstration of the governing equation or the other the deriving the formal governing equation.

We assume it is just a one loop we have that is why the integration is over S 1 only then we will see if we have several such loops then what will be the necessary modification in that ok. So, if this is the condition we have the, this has to be equal to 0. So, this has to be equal to 0 this has to be equal to 0 right, now then what this is if we again if we substitute the previous expression that is how the displacement is related to stresses these two expression.

If we substitute in these in these expression then the expression what we have is this. Now let us see it has two part one is this one the first one this is one and this one. Let us see let us take two parts separately this has to be equal to 0 right that is the condition. Now let us take two parts separately and then try to simplify them.

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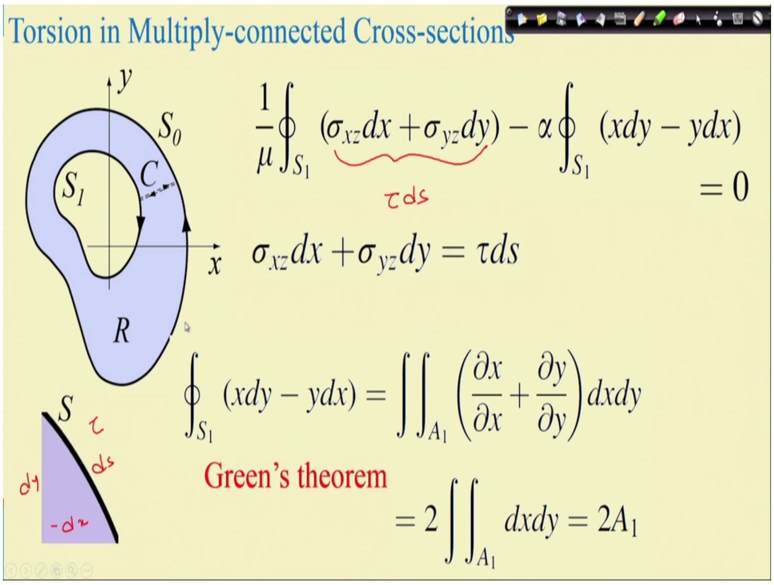

So, the expression what we have is, this is the expression we have right. So, let us take the first part you see the take any small distance a small arc on this surface S and then this is as we already discussed that this will be your this will be dx and this is this is minus dx and this is dy and this is ds. Now then sigma xz dx sigma xz into this and sigma yz into dy if we see the equilibrium on the surface, if we have shear stress and that shear stress is tau.

The tau I am using tau because that is the notation we used for torsion when you in the previous classes are also in the strength of material when you use torsion it is the symbol that we use ok. But when you remember when we a component wise component wise if we you have to write the stress tensor, then the off diagonal terms are represented as sigma ij that is the notation we are using in this course, but you can use tau as well.

So, this is the shear stress on the tau then simply by applying the equilibrium condition we can show that this is essentially tau into ds. So, now if it is this is the first term is tau into ds so the first term is tau into ds, let us see the second term if we take the second term the second term is this. Now if we apply greens theorem greens theorem we discussed towards the beginning of this course and also application of greens theorem, we also discussed in some of the earlier classes.

Now if we apply greens theorem which essentially gives you your projecting integration from projecting integration from area to surface area to boundary or the volume to the surfaces from one space to other another space. So, if we apply greens theorem then this is essentially this and what is A 1? A 1 is the area enclosed by S 1.

So, now this is this now look at this expression and these expressions simply we have this is equal to dx dy and 2 A 1 A 1 d integration, integration of dx dy over A 1 is essentially the area itself and therefore, this becomes 2 A 1. So, the first expression is tau ds and the second expression we have first expression is tau ds and the second expression we have 2 A 1. Now let us substitute these two expression in these in this equation and the final equation that we get is the final equation we get because this has to be equal to 0 and then this is the equation we get.

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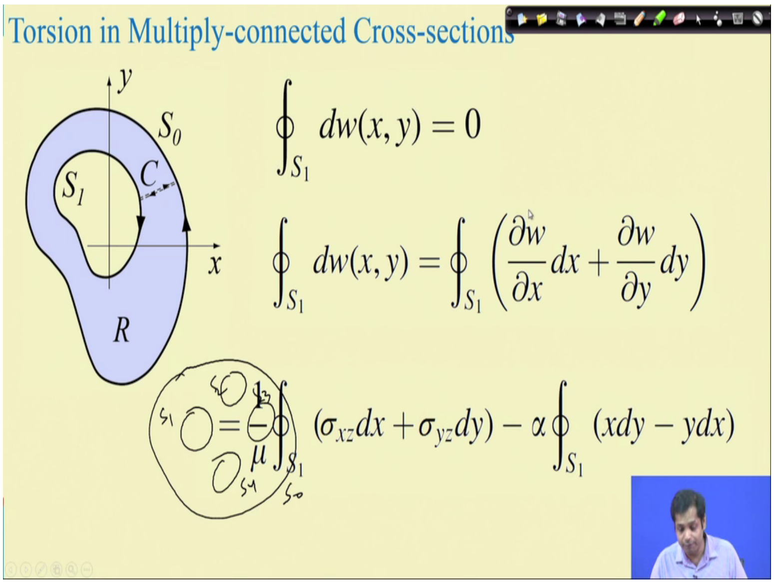

Now suppose if we have two such loops for instance if we have an express if you have a domain like this if you have a domain; if you have a domain like this and you have one loop two loop like this several loops we have.

So, this is S 1 S 2 S 3 S 4 and so on and this is S 0 or so if we have such loops then the additional things will be in this equation is this equation has to be satisfied over every loop, this equation has to be satisfied if we go this equation has to be satisfied over every loop for every loop.

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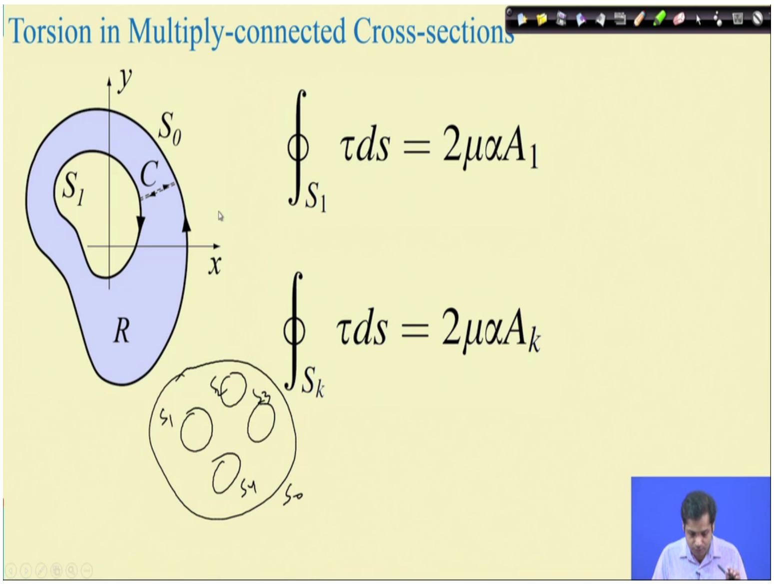

And then finally if we substitute that the expression we get is this so for Sk kth point it will be this. So, now, for every loop this condition needs to be satisfied in this exam in this figure the first figure is k is just one, now what we have once we have the expression.

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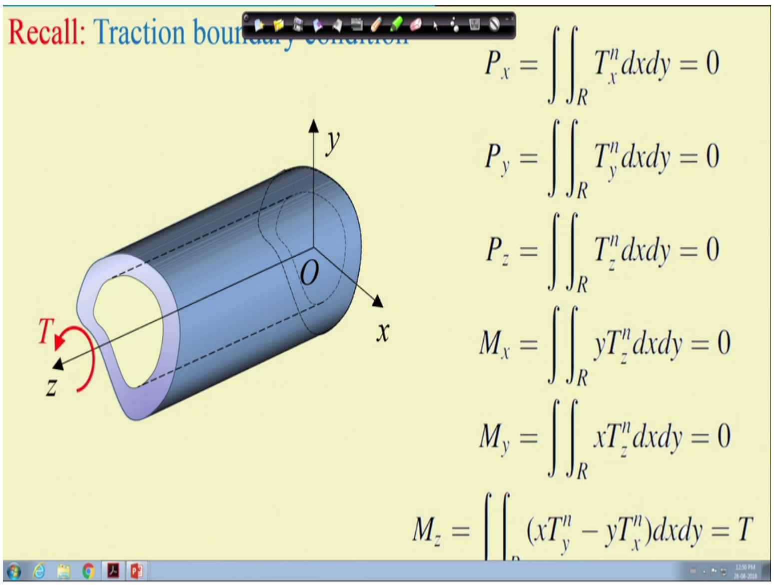

Now, let us see what are the boundary condition so this is a one expression additional expression that needs to be satisfied this expression directly comes from the from the condition that we imposed to guarantee single valued displacement field.

So, this condition in this case is the condition associated because of the multiple connected (Refer Time: 14:29) of the cross section we did not have this condition in case of simply connected domain additional condition. Now so let lets now the boundary conditions these are the if you recall these are the traction boundary conditions all the lateral surfaces the traction boundary conditions are all the lateral, lateral surfaces these are the traction boundary all these surfaces these are the traction boundary condition.

And the finally, the boundary conditions what we have Mz at the end this is the if you apply the torsion and that torsion will be equal to this, all will be equal to 0 if we take psi vanishing at the boundary that we have already shown.

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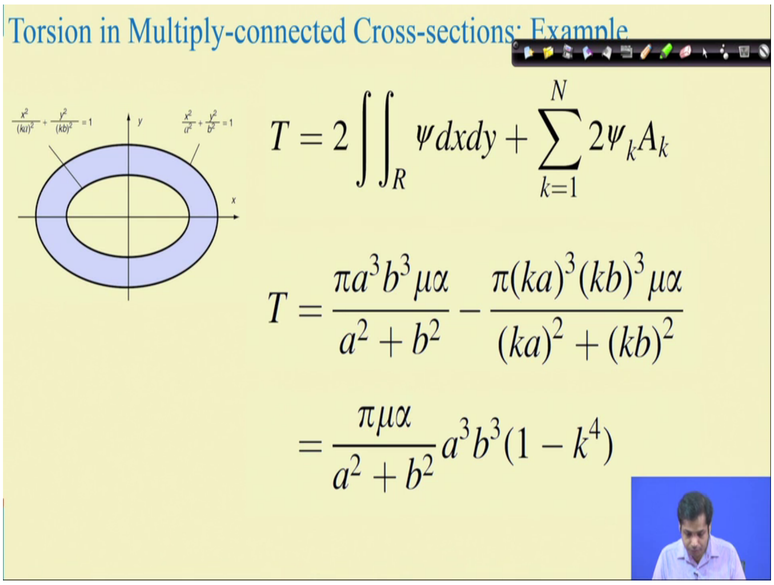

So, if I substitute expression of tau y and this the Ty and Tx the same way that we did in the previous classes. If you substitute in this expression and then that final expression will be like this and recall the additional term we have here is this one. This is the additional term we have because of the multiply connectedness of the domain this is the additional term and these has to be done over the all loops that you have in the cross section. Now well demonstrate this expression through an example so this is the torsion expression for torsion how the torsion is related to stress function psi.

Now, what you have to do is once we now the let us now demonstrate that through one example, but the procedure will be again same we start with an assumption of psi, but when we assume that psi the condition is the psi d psi ds has to be equal to 0. So, psi has to be constant psi we can take as 0, so that it vanishes at the outer boundary that is the same way we did it for simply connected domain. Once we have expression for psi then we substitute the condition required, in this case the simply connected this multiple connectedness condition we can find out the constant associated with the psi such that these conditions are satisfied let us see that.

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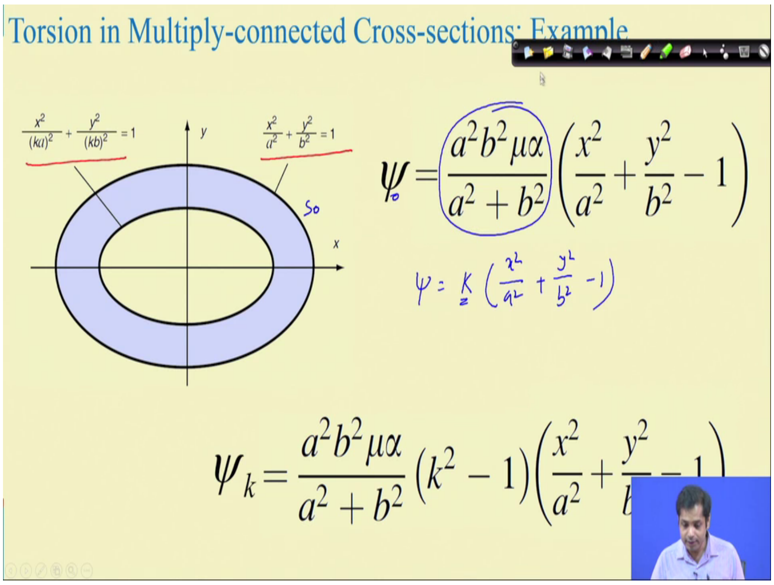

Now let us take one example like this example is taken because we already know the solution of elliptical cross section. But now in this case is an hollow elliptical cross section we will see at the end compare the solution of this problem and the solution of elliptical cross section.

Now, the outer surface is this outer surface is this and the inner surface is this when k is depending on the k you can you can increase or decrease the size of this hole. Now so now so let us see let us start with assumption of psi psi is taken as if you recall in the case of solid circular solid elliptical cross section psi was taken as actually psi was taken as like this psi was K into x square by a square plus y square by b square minus 1.

And then k we determined based on the conditions and the, this k if you recall it was the k it was the constant for psi. Now if I have to do the same thing for this is for the outer surface this is for S o or you can take is psi o. Now if I have to do it for the inner loop then that expression will be equal to equal to this.

Now if you look at it is very similar if you substitute just k is equal to if you substitute k is equal to 0 here, then it becomes your psi this will be very similar to this. Now so this is for psi o and psi k we have now let us see what next now next if you recall this is the condition that we have and we you I leave it to you that is you take if you take this as psi and psi k please check whether these conditions are satisfied or not this condition is satisfied or not and you will check.

In all the boundary condition along with all the boundary conditions these all the traction boundary conditions and also these conditions whether they satisfy or not if you take psi and psi k has this form. Now the next is this is the relation between torsion and the stress functions, now if we substitute the expression of psi and psi k in this relation and then do the exercise that the integration over r then we have then expression like this.

Now, and then this expression finally, become this if you recall if you substitute k is equal to 0 here we should be getting the expression for solid elliptical shaft we will see that at the end. So, this is for torsion now once we have the torsion then we can then we also we now we have the stress function from this stress function we can find out the stresses sigma xz and sigma yz.

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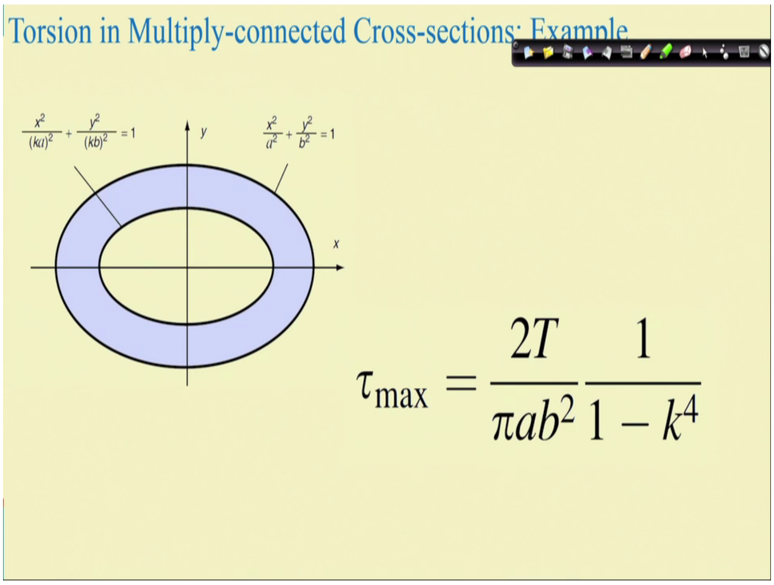

And form stress we can find out what is the shear stress on the boundary or the distribution of shear stresses. And then in the same way once we have the expression of shear stress, we can found out what is the maximum shear stress at what point is stress will be maximum by differentiating shear stress with respect to the coordinate axis and determine the coordinate value at which the shear stress will be maximum.

And substituting that we get the shear stress like this it will be maximum in the outer at the surface outer surface and this is the maximum value. Now you look at so this is the expression for this problem.

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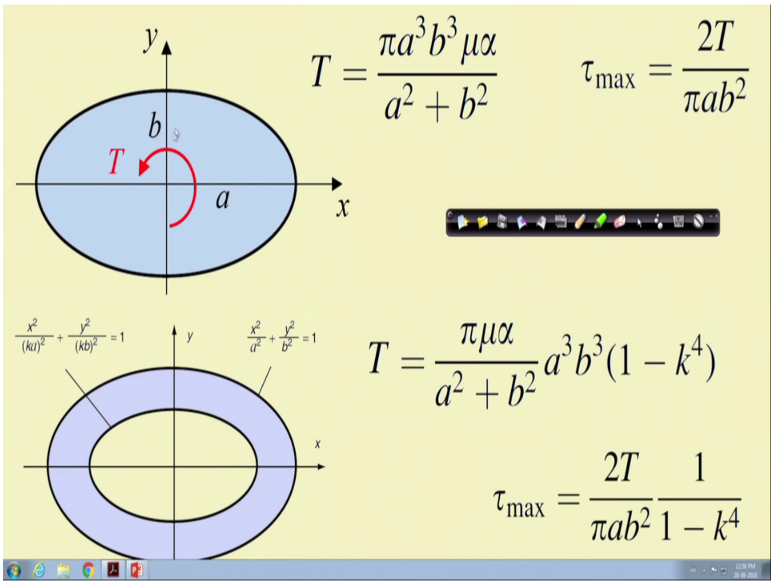

Now let us compare the results of results of this is the results of this is the result of solid cross section, when your solid shaft this is the result of hollow shaft that just now we have seen.

Now, you see if you substitute K is equal to 0 in this case we get this value. So, it is very similar to the very similar to the solution, but with an additional term K which defines the size of this hole. Now if you have please note here do not do not get confused with this K they do not get confused with this K and the K used here K used K use is an expression here K is essentially a dummy index which gives you a summation over the number of holes and K is these K defines what is the size of this inner hole.

So, please this you better you can take it you can instead of K you can take is n small n and this is n and n this is summation over number of holes and this defines the size of a given hole now. So, this is the final expression we have and you can compare this now this is the same way you can.

Now, once we know the expression of shear stress then what we have to do I that exercise I leave it to you once we get the expression of shear stress, then go back to the previous the first slide. We can substitute the shear stress here in the first equation in the first equation this is same way we did it for in this equation number one we can substitute shear stress integrated get w with an additional constant not constant it is a function of x.

And then use the second equation to evaluate that function and from that we get an expression for w. Now I leave it you find out the expression of w these examples are solved if you take any elasticity book these of the examples are solution of these are the examples are given in the book and the steps are also given. Now I leave it to you do this exercise get the solution of w and then compare the solution of w for in hollow elliptical cylinder hollow elliptical shaft with a solid elliptical shaft and then see how are the what are the differences.

Now depending on the cross section, you can apply this similar equations for different problems, but for different problems your psi will be different, but the equations that we derived there equations will be same steps will be exactly same. Now with this I conclude these weeks lecture next week what we see is next week we will see another way of another way of solving these boundary value problems in elasticity.

And that way is using complex variable approach, so what we do in next week is we will derive the all these expression that we derived. So, far we will derive all these expressions equilibrium compatibility navies equation all this expression we derived in terms of complex variables. And then see how those equations written in terms of complex variable can be solved we also see that we also compared that method not the solution that the steps involved in that method.

And the and the steps we have already we have done in all these methods we will see that using complex variable method many problems which are not that easy to solve using the method, so far we have discussed. It will be easier to address them to find the solution of those kind of domain those kind of problem using complex variable method then see you in the next week.

Thank you.